Constant Function Market Makers
Pushing Uniswap & friends to do more with lower fees

Guillermo Angeris and Tarun Chitra

May 2, 2020
Outline

History

Examples of CFMMs

The trading set

An analysis of CFMMs

Acknowledgements
Trading assets

- Often, we need a way of trading assets
- (Normally) easy: make an offer to buy or sell
- In the traditional setting, this led to order books
  - Buyers post ‘bids’ of *maximum price* they are willing to buy for
  - Sellers post ‘asks’ of *minimum price* they are willing to sell for
  - Trades occur when a buyer is willing to pay more than minimum ask price
Disadvantages

- A trusted party keeps a record of outstanding bids and asks
  Linear space requirement

- When the highest bidder bids more than the lowest asker [...] 
  Price may update *slowly*, esp. with a small number of agents

- Exchanges usually add subsidies to ensure there is liquidity
Alternatives to the tyranny of the order book

- Question: can we replace human market makers with algorithmic market makers?

- Yes! — Automated Market Makers (AMMs)
  - Bounded loss: can give explicit bounds on the maximum loss
  - Liquidity sensitivity: fixed-size trade moves prices less in thick markets
  - Small storage requirement: accepting or rejecting trade depends only on the current reserves
Automated Market Makers

Savage ’71, Hanson ’02

▶ **Idea:** use a (simple) formula to determine asset price

▶ *Liquidity providers* pool their assets (say A and B) into *reserves*

▶ Price set too low: agents purchase reserves at current price

▶ Price set too high: agents sell to reserves at current price
Automated Market Makers
Savage '71, Hanson '02

- **Idea:** use a (simple) formula to determine asset price

- *Liquidity providers* pool their assets (say $A$ and $B$) into *reserves*

- Price set too low: agents purchase reserves at current price

- Price set too high: agents sell to reserves at current price

- Using this idea, set price based on assets remaining in reserves

- *e.g.*, if too much of asset $A$ remains, compared to asset $B$, decrease the price of $A$
Automated Market Maker examples

- Simplest example: fixed asset price at all reserve amounts
  *i.e.*, a flat line

- Another example: reported price is ratio of two asset reserves
  This curve is Uniswap!
A brief history of AMMs

- Savage ’71, Hanson ’02: Logarithmic Market Scoring Rules can be used for AMMs in prediction markets
- Chen, Pennock ’07: axiomatic formulations exist for AMMs for more markets
- Othman, Sandholm ’10: liquidity-sensitive AMMs exist that don’t have unbounded loss
- Othman, Sandholm ’11: constant utility AMMs for prediction markets
- Buterin, Koppelmann, ’16: extension of AMMs to constant product markets, applications in exchanges
Outline

History

Examples of CFMMs

The trading set

An analysis of CFMMs

Acknowledgements
Uniswap (and constant product markets)

- Constant product markets (e.g., Uniswap) is the family of curves whose reserves $R_\alpha, R_\beta$ must always satisfy:

$$R_\alpha R_\beta = k,$$

for some constant $k$ (no fees), we will call this the trading function.

- In this case, we will assume that $\alpha$ and $\beta$ are coins, though they can be any asset.

- To satisfy this equation, the marginal price of asset $\beta$ with respect to $\alpha$ is always

$$m_u = \frac{R_\beta}{R_\alpha}$$
Uniswap market function

- If we plot the set of reserves that Uniswap can have, we get a hyperbola

![Graph showing a hyperbolic relationship between $R_o$ and $R_a$.]
Uniswap market function

So, traders can exchange coin $\alpha$ and $\beta$ with the reserves, so long as the resulting reserves remain on the curve.
Analysis of market functions

- Uniswap is easy to analyze: trading function is simple
- What about more complicated cases? (e.g., Balancer, Curve, etc.)
- While we could work out derivatives for prices, how do LP returns work? Are they good or bad for arbitrageurs?
- Additionally, many trading functions give the same trades!
Outline

History

Examples of CFMMs

The trading set

An analysis of CFMMs

Acknowledgements
Filling in the graph

► Weird idea: what if we ‘fill in’ the graph?

► In other words, what if reserves above and to the right of the graph were also possible?
Uniswap trading set

The new graph (or set) will look like
Uniswap trading set

Since we’ve only added points, all of the reserves that were feasible before are still feasible.
Uniswap trading set

But, no rational trader will ever pick a point *inside* of the set!
Uniswap trading set

We will call this set the trading set, \( T \subseteq \mathbb{R}_+^2 \), for short
The trading set

- The trading set $T$ is often easier to deal with than the actual trading function and is the same for rational agents.

- In particular, many trading functions yield the same set!

- Yet, the trading set is convex in practical scenarios (and all known CFMMs).

- Convex $\implies$ (a) easy to optimize and (b) easy to analyze.
Outline

History

Examples of CFMMs

The trading set

An analysis of CFMMs

Acknowledgements
Constant function market makers

- Surprisingly, all CFMMs of this form have the same properties
- Marginal prices are simple to compute
- It is immediate that arbitrage is (computationally) easy, implying that prices match external ones
- Can compute liquidity provider returns, often in closed form
- All of this follows from basic convex analysis!
Marginal price of a CFMM at given reserves is proportional to the *supporting hyperplane* of the set $T$ at these reserves.
Liquidity provider portfolio value

- The liquidity provider portfolio value is exactly given by
  \[ \inf_{r \in T} p^T r, \]
  where \( p \) is the vector of prices with \( p_i \geq 0 \) the price of coin \( i \)

- This is the negative of the support function of the set \( T \)

- Almost always easy to evaluate for convex sets \( T \)

- First way of analytically computing liquidity provider returns
  of complicated CFMMs (e.g., Balancer with \( n \) coins)
Why convexity?

► Sure, it’s easy, but why?

An analysis of CFMMs
Why convexity?

Note that arbitrageurs won’t care! Marginal price is the same.
Why convexity?

So we might as well fill it in
Extensions and questions

- We can extend this to include fees (*path deficiency*), but is generally more complicated since $T$ changes after each trade.

- Of course, all of this holds for $n$ coins, not just two.

- Questions still remain... manipulation price? Lower bounds?
CFMM properties (summary)

▶ Main idea: recast all numerical properties of CFMMs as geometric properties of convex sets

▶ Can use this to talk about marginal prices, the arbitrage problem, liquidity provider returns, etc

▶ Second implication: it is possible to optimize over the set of CFMMs!

▶ We can make specific CFMMs for specific applications, including fees, curve shapes, etc.

▶ And many more possibilities
Outline

History

Examples of CFMMs

The trading set

An analysis of CFMMs

Acknowledgements
Acknowledgements

- Tim Roughgarden (Columbia)
- John Morrow (Gauntlet)
- Shane Barratt (Stanford)