THE TRIANGULATION CONJECTURE

CIPRIAN MANOLESCU

In topology, a basic building block for spaces is the n-simplex. A 0-simplex is a point, a 1-simplex is a closed interval, a 2-simplex is a triangle, and a 3-simplex is a tetrahedron. In general, an n-simplex is the convex hull of $n+1$ vertices in n-dimensional space. One constructs more complicated spaces by gluing together several simplices along their faces, and a space constructed in this fashion is called a simplicial complex. For example, the surface of a cube can be built out of twelve triangles—two for each face. Topologically, the cube is indistinguishable from a sphere (and also from a tetrahedron, or from an octahedron), since all these surfaces can be deformed into each other without tearing them apart; we say that they are homeomorphic.

Apart from simplicial complexes, manifolds form another fundamental class of spaces studied in topology. An n-dimensional manifold is a space that looks locally like the n-dimensional Euclidean space. Manifolds are ubiquitous in many parts of mathematics; for instance, they can appear as spaces of solutions to systems of polynomial equations, or to systems of differential equations. However, knowing that a space is a manifold does not tell us much about its global structure. To study the properties of a manifold, it is helpful to triangulate it, that is, to construct a homeomorphism to a simplicial complex. For example, the surface of a sphere is a two-dimensional manifold, and it admits a triangulation with twelve triangles, in the form of the cube. (Of course, it also admits many other triangulations.) A triangulation yields a combinatorial description for the manifold. Furthermore, if we have two manifolds and we try to tell them apart, the first thing to do is to check if their topological invariants (such as their homology groups) are the same. If we are able to triangulate the manifolds, it is straightforward to compute their homology groups in terms of the two triangulations.

The triangulation conjecture—first formulated by Kneser in 1924—claimed that every manifold was triangulable. The conjecture turned out to be false in general, although it is true for manifolds of dimension up to 3, and also for all differentiable manifolds (those that are "smoothly" like Euclidean space, so that one can do calculus on them). In Kneser's time, it was already known that every two-dimensional surface is triangulable, due to the work of Radó. The case of differentiable manifolds was settled in the 1940's by Cairns and Whitehead. In 1952, Moise showed that any three-dimensional manifold is differentiable, and thus triangulable.

Much of the later progress towards settling the conjecture was done by people associated with UCLA, at various points in time. In 1968, Rob Kirby, then a professor at UCLA, discovered the so-called torus trick, a technique that enabled him to find (in joint work with Laurence Siebenmann) the first example of a manifold that does not admit a piecewise linear structure. A piecewise linear structure, also called a combinatorial triangulation, is the kind of triangulation in which the manifold structure is evident—technically, a triangulation in which the link of every vertex is a sphere. Most of the triangulations of a manifold that one can think of are of this type. The simplest way to construct a non-combinatorial triangulation is to first triangulate a non-trivial homology sphere (a manifold with the same homology groups as the sphere, but not a sphere), and then to take its double suspension. One then needs to appeal to the Double Suspension Theorem, proved in the 1970's by Bob Edwards (also at UCLA) and J. W. Cannon, to see that the resulting space is a manifold (in fact, a sphere).

The work of Kirby and Siebenmann showed that there exist manifolds without piecewise linear structures in any dimension greater than 4. Dimension four is very special in topology, and new techniques were needed in that case. In the early 1980's, Michael Freedman revolutionized four-dimensional topology, and in particular gave an example of a four-manifold (the E_8 manifold) that has no differentiable or piecewise linear structures.

The first counterexamples to the triangulation conjecture were also found in dimension four: In the mid 1980's, Andrew Casson introduced a new invariant of homology 3-spheres. This can be used it to show that, for example, Freedman's E_8 manifold is not triangulable.

This left open the question of triangulability for manifolds in dimensions greater than 4. In the 1970's, this problem had been shown to be equivalent to a different problem, about 3-manifolds and homology cobordism. The equivalence was discovered by Ron Stern (a UCLA Ph.D.) together with his collaborator David Galewski, and independently by Takao Matumoto. In technical terms, they showed that all manifolds of dimension > 4 are triangulable if and only if the 3-dimensional homology cobordism group admits an element of order two and Rokhlin invariant one. Furthermore, Galewski and Stern gave an explicit example of a 5-dimensional manifold that is not triangulable, if the answer to the question above were negative. By taking products with tori, one would also obtain counterexamples in all higher dimensions.

Indeed, the answer to the question about homology cobordism turned out to be negative. The proof involves techniques from gauge theory, namely, a new version of Floer homology called Pin(2)-equivariant Seiberg-Witten Floer homology. Gauge theory is the study of certain elliptic partial differential equations that first appeared in physics—they govern the weak and strong interactions between particles. In the 1980's, Donaldson pioneered the use of gauge theory in low-dimensional topology. Out of gauge theory came Floer homology, an invariant associated to three-manifolds that is particularly useful in studying cobordisms. (A cobordism between two three-manifolds Y and Y' is a four-manifold with initial boundary Y and final boundary Y'.) Floer homology is what Atiyah called a topological quantum field theory (TQFT). The main property of a TQFT is that a cobordism from Y to Y' induces a map between the respective invariants (in this case, their Floer homologies). This should be contrasted with what happens in ordinary homology, where we need an actual map (not a cobordism!) between Y and Y' to get a map between their homologies. The various kinds of Floer homologies (instanton, Seiberg-Witten, Heegaard Floer) are the main tool for studying cobordisms between 3-manifolds, and the answer to the Galewski-Stern-Matumoto problem is only one of their many applications.