ERRATA TO THE ARTICLE "SEIBERG-WITTEN-FLOER STABLE HOMOTOPY OF THREE-MANIFOLDS WITH $b_1 = 0$"

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There is a significant error at the top of p.921 in the paper. The sentence "Changing everything by a gauge, we can assume without loss of generality that $i^*(\hat{a}) \in \ker d^*$" is not correct. There is no gauge freedom available on X, because on p.916 we already fixed the gauge by imposing the Coulomb-Neumann condition on forms, $\hat{a} \in \Omega^1_g(X)$.

Furthermore, if we do not assume that $i^*(\hat{a}) \in \ker d^*$, then the estimates on the second term on the right hand side of Equation (17) do not go through. Precisely, in the paragraph on p.921 starting with "Similarly one can show that ...," instead of $db_n \to 0$ in L^2_k and $b_n \to 0$ in L^2_{k+1} we would have $db_n \to db$ in L^2_k and $b_n \to b$ in L^2_{k+1}, where $i^*(\hat{x}) = (a + db, \phi)$ and $a \in \ker d^*$. Knowing that $b_n \to b$ and $p^{\mu_n}_{\lambda_n}(a_n, e^{ib_n}\phi_n) \to (a, e^{ib}\phi)$, we would like to deduce that $p^0(a_n, \phi_n) \to p^0(a, \phi)$. (Here, all limits are in L^2_{k+1}.) By hypothesis, we also know that the L^2_{k+1} norms of (a_n, ϕ_n) are bounded, and that $(a_n, \phi_n) \in V_{\lambda_n}$. Thus, $p^0(a_n, \phi_n) = p^0_{\lambda_n}(a_n, \phi_n)$. We have

$$\begin{aligned}\|p^0(a_n, \phi_n) - p^0(a, \phi)\| &= \|p^0_{\lambda_n}(a_n, \phi_n) - p^0(a, \phi)\| \\ &\leq \|p^0_{\lambda_n}(a_n, e^{ib_n - ib}\phi_n) - p^0(a, \phi)\| + \|p^0_{\lambda_n}\big(0, (e^{ib_n - ib} - 1)\phi_n\big)\|,\end{aligned}$$

where all norms are L^2_{k+1}. Since $b_n \to b$ and $\|\phi_n\|_{L^2_{k+1}}$ is bounded, using the Sobolev multiplication $L^2_{k+1} \times L^2_{k+1} \to L^2_{k+1}$ we get that the second term in the last expression above converges to 0. If multiplication by e^{ib} commuted with the projection $p^{\mu_n}_{\lambda_n}$, from $p^{\mu_n}_{\lambda_n}(a_n, e^{ib_n}\phi_n) \to (a, e^{ib}\phi)$ we would get that $p^{\mu_n}_{\lambda_n}(a_n, e^{ib_n - ib}\phi_n) \to (a, \phi)$, and then (applying p^0) the first term would converge as well. It would then follow that $p^0(a_n, \phi_n) \to p^0(a, \phi)$, as desired.

This argument works for $b = 0$ but fails in general, because multiplication by e^{ib} does not commute with $p^{\mu_n}_{\lambda_n}$. The origin of the problem is that the nonlinear map C^μ defined on p.917 is not compact.

The simplest way to fix this issue is to replace the Coulomb-Neumann condition by a double Coulomb condition. This approach is the subject of Khandhawit's paper [1]. We sketch the argument here, and refer to [1] for more details.

On p.916, when we define $\Omega^1_g(X)$, instead of the condition $\hat{a}|_{\partial X}(\nu) = 0$ we impose a boundary Coulomb condition, $i^*(\hat{a}) \in \ker(d^*)$. We also ask that the integral of $\hat{a}|_{Y_i}(\nu)$ is zero on each connected component $Y_i \subseteq \partial X$. (This is automatic when ∂X is connected.) The new gauge condition satisfies a Fredholm property similar to Proposition 5; see [1, Proposition 2]. Moreover, the nonlinear map C^μ from p.917 is now compact, and we can delete $\mathrm{pr}_{\ker d^*}$ from the second term on the right hand side of Equation (17) on p.921. Then, it is easy to show that this term converges to zero. A new difficulty appears in the argument at the top of p.922, when we glue a half-trajectory on $[0, \infty) \times Y$ with a monopole on X that may have a non-trivial dt component on the boundary. Nevertheless, the gluing can be done after changing the half-trajectory on $[0,\infty)\times Y$ by a suitable gauge transformation; see [1, Corollary 2].

There were a few other minor errors in the article:

(1) On p.898, the metric $\tilde{g}$ on V was defined by the formula
$$\|(b,\psi)\|_{\tilde{g}} = \|(b,\psi)+(-id\xi, i\xi\phi)\|_{L^2},$$
measuring the norm of the projection of (b,ψ) to the local Coulomb slice at (a,ϕ). However, this formula does not yield a non-degenerate metric. There is still a residual S^1 gauge action on V, and the vectors tangent to the S^1-orbits, such as $(0,i\phi)$, would have length zero. We can correct this by adding a circular projection term, given by the square of the inner product with $(0,i\phi)$. Precisely, we set:
$$\|(b,\psi)\|^2_{\tilde{g}} = \|(b,\psi)+(-id\xi, i\xi\phi)\|^2_{L^2} + \big(\mathrm{Re}\langle i\phi,\psi\rangle\big)^2.$$
Since the gradient of the CSD functional is perpendicular to the S^1-orbits, it is still true that the trajectories of the $\tilde{g}$-gradient of $CSD|_V$ are the Coulomb projections of the trajectories of CSD on $i\Omega^1(Y)\oplus\Gamma(W_0)$.

(2) In the middle of p.907, when we define the desuspension of X by E in the category $\mathfrak{C}_0$, the alternative definition as $\Omega^E X$ is incorrect. The correct definition is the one given in the previous line, $\Sigma^{-E}X = (E^+\wedge X, 2\dim E, 0)$. In general, $\Sigma^{-E}X$ and $\Omega^E X$ may not even have the same homology, so they are not isomorphic in $\mathfrak{C}_0$.

(3) At the bottom of p.917, the set $\tilde{K}$ should be the preimage of $B(U_n,\epsilon_n)\times V^\mu_\lambda$ under the map $\mathrm{pr}_{U_n\times V^\mu_\lambda} SW^\mu$, not under the linear map L^μ.

(4) Lemma 4 on p.918 is incorrect as stated. There can be trajectories that start outside the ball $\overline{B(2R)}$, go inside $\overline{B(2R)}$ at some time t_0, and converge to a point in $B(R)$. For the lemma to be true, we need an additional hypothesis, that $x_n(t_0)$ is the restriction of an approximate Seiberg-Witten solution on a compact 4-manifold X with boundary Y. An argument of this type is used in the proof of Lemma 2 in [1].

(5) At the top of p.924, the proof that the class Ψ is independent of the choices made in the construction was incomplete. One needs to show independence of the index pair (N,L) chosen in Theorem 4. This is done by Khandhawit in Proposition 5 from [1, Appendix A].

Acknowledgements. I would like to thank Tirasan Khandhawit for pointing out two of the errors discussed above, as well as for fixing them in his paper [1]. I am also grateful to Mikio Furuta, Tye Lidman and Jianfeng Lin for discussions that led to finding the other issues.

References

[1] Tirasan Khandhawit, *A new gauge slice for the relative Bauer-Furuta invariants*, Geom. Topol. **19** (2015), no. 3, 1631–1655.

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