PERIODIC FLOER PRO-SPECTRA FROM THE SEIBERG-WITTEN EQUATIONS

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ABSTRACT. Given a three-manifold with $b_1 = 1$ and a nontorsion $spin^c$ structure, we use finite dimensional approximation to construct from the Seiberg-Witten equations two invariants in the form of a periodic pro-spectra. Various functors applied to these invariants give different flavors of Seiberg-Witten Floer homology. We also construct stable homotopy versions of the relative Seiberg-Witten invariants for certain four-manifolds with boundary.

1. INTRODUCTION

In [6], Cohen, Jones, and Segal posed the question of constructing a "Floer homotopy type." They conjectured that Floer homology (in either of the two variants known at the time, symplectic or instanton) should be the homology of an object called a *pro-spectrum*. However, the passage from homology to homotopy in either of these cases seems a difficult task at the moment. Because of their remarkable compactness properties, the Seiberg-Witten equations are better suited for this task. Using the technique of finite dimensional approximation, Furuta and Bauer were able to define stable homotopy invariants for four-manifolds ([5], [11], [12]). In [19], the second author has associated to each rational homology 3-sphere a certain equivariant spectrum whose homology is the Seiberg-Witten Floer homology.

This paper is a continuation of [19]. Here we define the stable homotopy generalization of the Seiberg-Witten Floer homology for closed, oriented 3-manifolds Y with $b_1(Y) = 1$, together with $spin^c$ structures $\mathfrak{s}$. We assume that $c_1(\mathfrak{s})$, the first Chern class of the corresponding determinant line bundle, is nontorsion.

It turns out that the usual homotopy category $\mathfrak{S}$ of spectra is not good enough to support our invariants. To ensure the existence of good colimits, we divide out the sets of morphisms in $\mathfrak{S}$ by the class of phantom maps, and we call the resulting category $\mathfrak{S}'$. Furthermore, to ensure the existence of good inverse limits, we enlarge the class of objects to include *pro-spectra*. The resulting category is called Pro-$\mathfrak{S}'$. Its exact definition will be given in section 3.

The main result of this article is:

Theorem 1. *a) Given a Riemannian metric g on Y and a $spin^c$ connection A_0, finite dimensional approximation for the Seiberg-Witten map produces an invariant $SWF(Y, \mathfrak{s}, g, A_0)$ in the form of a pointed S^1-equivariant pro-spectrum well-defined up to canonical equivalence in Pro-$\mathfrak{S}'$.*

When the metric g or the connection A_0 change, the invariant $SWF(Y, \mathfrak{s}, g, A_0)$ can change only by suspending or desuspending by a finite dimensional complex representation of S^1.

Furthermore, the pro-spectrum is periodic modulo ℓ, where

$$\ell = g.c.d.\{(h \cup c_1(\mathfrak{s}))[Y] \,|\, h \in H^1(Y; \mathbb{Z})\},$$

in the sense that there is a complex representation E of S^1 of real dimension ℓ and a natural equivalence:

$$SWF \to \Sigma^E(SWF).$$

b) There is a different version $SWF_0(Y, \mathfrak{s}, g, A_0)$ which is a spectrum obtained by doing finite dimensional approximation for the Seiberg-Witten map with a nonexact perturbation in the cohomology class $-c_1(\mathfrak{s})$. It has the same properties as the ones mentioned above for $SWF(Y, \mathfrak{s}, g, A_0)$, and there is a natural morphism:

$$j : SWF_0 \to SWF$$

which induces isomorphisms on homology and on the equivariant Borel homology.

c) Under change of orientation, $SWF(Y, \mathfrak{s}, g, A_0)$ and $SWF(\bar{Y}, \bar{\mathfrak{s}}, g, A_0)$ are duals, but the analogous statement for SWF_0 is false.

Our construction of SWF runs parallel to the one for rational homology spheres in [19]. However, there is an important complication, given by the loss of compactness of the Seiberg-Witten moduli space. If we work in Coulomb gauge, the solutions of the Seiberg-Witten equations are the critical points of the Chern-Simons-Dirac functional

$$CSD : V = (\Omega^1(Y; i\mathbb{R})/\operatorname{Im} d) \oplus \Gamma(W_0) \to \mathbb{R},$$

where W_0 is a *spin^c* bundle on Y with determinant line bundle L.

There is a residual gauge action of $\mathbb{Z} = H^1(Y; i\mathbb{Z}) \subset \Omega^1(Y; i\mathbb{R})$ on V with respect to which CSD is periodic modulo $4\pi^2\ell$; more precisely, the action of $u \in H^1(Y; i\mathbb{Z})$ changes CSD by $4\pi^2([u] \cup c_1(\mathfrak{s}))[Y]$. The Seiberg-Witten moduli space is therefore periodic modulo $\mathbb{Z}$. It would be compact if we divided out by the residual gauge action, but this would destroy the linear structure of the configuration space and hinder the application of finite dimensional approximation.

In [19], SWF was defined as the Conley index of a flow on a finite dimensional space. To be able to define the Conley index in the current setting we need to restrict our attention to a bounded subset of the configuration space. We manage to do this by cutting the configuration space between two levels of the Chern-Simons-Dirac functional. (This procedure was used before by Fintushel and Stern in [9] to define $\mathbb{Z}$-graded instanton homology and by Marcolli and Wang in [21] in the context of Seiberg-Witten theory.) We approximate the gradient flow of CSD in a bounded set between the two levels by a flow on a finite-dimensional approximation. The pro-spectrum SWF is then obtained from the Conley index of this flow by taking direct and inverse limits as the levels of CSD go to $\pm\infty$.

If we perturb the Seiberg-Witten equations by a nonexact ν with $[\nu] = -c_1(\mathfrak{s})$, the Chern-Simons-Dirac functional is invariant under the residual gauge action, so we can no longer use it to cut the configuration space. Instead, we cut between the levels of $CSD + f$, for a certain function f. Depending on our choice of f, we are free to take either direct or inverse limits as the levels go to $\pm\infty$. Direct limits at $+\infty$ and inverse limits at $-\infty$ give us back SWF, while direct limits in both directions produce the new invariant SWF_0. This is different from SWF: for example, when $Y = S^1 \times S^2$ with any nontorsion *spin^c* structure, $SWF = *$ is trivial, while SWF_0 is nontrivial.

We choose to mention SWF_0 here because of the connection to Ozsváth-Szabó theory. In [23] and [24], Ozsváth and Szabó have constructed several versions of Floer homology for three-manifolds, which they denoted by $\widehat{HF}, HF^+, HF^-, HF^\infty$. Their theory is supposed to give the same output as Seiberg-Witten theory. In [24], they have made the precise conjecture relating HF^+ and HF^- to two versions of the Seiberg-Witten Floer homology for rational homology 3-spheres.

More generally, we conjecture that all variants of the Ozsváth-Szabó Floer homology are different functors applied to our invariant. Given an integer-graded generalized homology theory h_* for S^1-equivariant pro-spectra, we can apply h_* to either SWF or SWF_0 and obtain a sequence of abelian groups periodic modulo ℓ. Since a change in the Riemannian metric only changes SWF, SWF_0 by suspending or desuspending by an even dimensional representation, $h_*(\text{SWF})$ and $h_*(\text{SWF}_0)$ are invariants of Y and $\mathfrak{s}$ only, thought of either as having a relative grading modulo ℓ or an absolute grading modulo 2.

Remark 1. *All generalized homology theories h_* are meant to be reduced, but we suppress the conventional tilde from notation for simplicity.*

In section 5, we explore some of these theories: the ordinary (non-equivariant) homology H_*, the equivariant Borel homology $H_*^{S^1}$, the co-Borel homology $cH_*^{S^1}$, and the Tate homology $tH_*^{S^1}$.

Conjecture 1. *Let Y be a 3-manifold with $b_1(Y) = 1$ and a nontorsion spin^c structure $\mathfrak{s}$. Then:*

$$\widehat{HF}_n = H_n(SWF_0) = H_n(SWF); \quad HF_n^+ = H_{n+1}^{S^1}(SWF_0) = H_{n+1}^{S^1}(SWF);$$

$$HF_n^- = cH_{n+2}^{S^1}(SWF_0); \quad HF_n^\infty = tH_n^{S^1}(SWF_0).$$

The same should be true for manifolds with $b_1(Y) = 0$ if we replace SWF_0 by SWF.

In the last section we discuss relative invariants of four-manifolds with boundary and prove:

Theorem 2. *Consider a Riemannian 4-manifold X together with a spin^c structure, such that the boundary $(Y, \mathfrak{s})$ has $b_1(Y) = 1, c_1(\mathfrak{s})$ nontorsion, and the image of $H^1(X; \mathbb{R})$ in $H^1(Y; \mathbb{R})$ is zero. Let $T(Ind)$ be the Thom spectrum for the Dirac index bundle on X. The Seiberg-Witten equations on X induce a morphism*

$$\Psi : \Sigma^{-b_2^+(X)} T(Ind) \to SWF(Y, \mathfrak{s}, g, A_0).$$

The reader may wonder what happens when $b_1(Y) \geq 1$ or $b_1(Y) = 0$ and $c_1(\mathfrak{s})$ is torsion. As Cohen, Jones and Segal pointed out in [6], in general there is an obstruction to defining Floer homotopy which lies in KO^1 of the configuration space, and usually factors through the map $K^1 \to KO^1$. For the Seiberg-Witten case, we present a computation of this obstruction in the appendix. It turns out that the obstruction is zero if and only if for every

$$a_1, a_2, a_3 \in c_1(\mathfrak{s})^\perp = \{a \in H^1(Y; \mathbb{R}) : a_j \cup c_1(\mathfrak{s}) = 0\},$$

we have

$$a_1 \cup a_2 \cup a_3 = 0.$$

For example, the obstruction is zero whenever $b_1(Y) \leq 3$. In such cases, we expect to be able to mod out by the residual gauge action in the cohomological directions in which CSD is unchanged. We can then use the notion of Conley index over a base from [22], where the base is a Picard torus. The result should be a "fiberwise deforming" pro-spectrum over the torus. When we vary the metric, this can change by suspending or desuspending by an arbitary complex bundle over the torus. We hope to explain this case in a subsequent paper.

However, the obstruction can be nonzero. This is the case, for example, when $Y = T^3$ and $\mathfrak{s}$ is torsion. In such a situation, it seems that to define a similar invariant one needs an even more general notion. Furuta has proposed in [12] the concept of *pro-spectrum with*

parametrized universe. It should be noted that, according to [6], one cannot expect to define stable Floer homotopy groups in this setting, but at most complex oriented generalized Floer homology theories, such as Floer complex bordism and Floer K-homology.

2. Seiberg-Witten trajectories

Let Y be a closed, oriented, Riemannian 3-manifold with $b_1(Y) = 1$. Fix a nontorsion $spin^c$ structure $\mathfrak{s}$ on Y with spinor bundle W_0 and set $L = \det(W_0)$. We identify the space of $spin^c$ connections on W_0 with $i\Omega^1(Y)$ via the correspondence $A \to A - A_0$, where A_0 is a fixed reference connection. We denote Clifford multiplication by $\rho : TY \to \operatorname{End}(W_0)$ and the Dirac operator corresponding to the connection $A_0 + a$ by $\slashed{\partial}_a = \rho(a) + \slashed{\partial}$.

The gauge group $\mathcal{G} = \operatorname{Map}(Y, S^1)$ acts on the space $i\Omega^1(Y) \oplus \Gamma(W_0)$ by

$$u(a, \phi) = (a - u^{-1}du, u\phi).$$

We will work with the completions of $i\Omega^1(Y) \oplus \Gamma(W_0)$ and $\mathcal{G}$ in the L^2_{k+1} and L^2_{k+2} norms, respectively, where $k \geq 4$ is a fixed integer. In general, we denote the L^2_k completion of a space E by $L^2_k(E)$.

Unlike in [19], here it becomes necessary to perturb the Seiberg-Witten equations by an exact 2-form ν on Y in order to obtain a genericity condition. The perturbed Chern-Simons-Dirac functional is defined on $L^2_{k+1}(i\Omega^1(Y) \oplus \Gamma(W_0))$ as

$$CSD_\nu(a, \phi) = -\frac{1}{2}\int_Y a \wedge (2F_{A_0} + da + 2\pi i\nu) + \int_Y \langle \phi, \slashed{\partial}_a\phi \rangle d\mathrm{vol}.$$

The change in CSD_ν under the action of the gauge group is

$$CSD_\nu(u \cdot (a, \phi)) - CSD_\nu(a, \phi) = 4\pi^2([u] \cup c_1(\mathfrak{s}))[Y].$$

The gradient of CSD_ν with respect to the L^2 metric is the vector field

$$\nabla CSD_\nu(a, \phi) = (*da + *F_{A_0} + i\nu + \tau(\phi, \phi), \slashed{\partial}_a\phi),$$

where τ is the bilinear form defined by $\tau(\phi, \psi) = \rho^{-1}(\phi\psi^*)_0$ and the subscript 0 denotes the trace-free part.

The *perturbed Seiberg-Witten equations* on Y are the equations for the critical points of CSD_ν :

$$*da + *F_{A_0} + i\nu + \tau(\phi, \phi) = 0, \quad \slashed{\partial}_a\phi = 0.$$

The basic compactness result for the solutions (a, ϕ) to the Seiberg-Witten equations ([16]) is that one can always find a gauge transformation u such that $u(a, \phi)$ is smooth and bounded in all C^m norms by constants which depend only on ν and the metric on Y.

As mentioned in the introduction, we intend to cut the moduli space between two levels of CSD_ν. In order for this to be possible, we need to impose a constraint on ν.

Definition 1. *A perturbation ν is called* **good** *if the set of critical points of CSD_ν is discrete modulo gauge.*

The set of good perturbations is Baire in the space of exact 2-forms. Indeed, according to [10], all critical points are nondegenerate for a Baire set of perturbations, and any nondegenerate critical point is isolated.

From now on we will always assume that ν is good. Let us study the trajectories of the downward gradient flow of CSD_ν, given by:

$$(1) \qquad x = (a, \phi) : \mathbb{R} \to L^2_{k+1}(i\Omega^1(Y) \oplus \Gamma(W_0)), \frac{\partial}{\partial t}x(t) = -\nabla CSD_\nu(x(t)).$$

Such Seiberg-Witten trajectories can be interpreted as solutions of the four-dimensional Seiberg-Witten equations on the infinite cylinder $\mathbb{R} \times Y$.

We borrow the terminology of [19] and say that a Seiberg-Witten trajectory $x(t)$ is *of finite type* if both $CSD_\nu(x(t))$ and $\|\phi(t)\|_{C^0}$ are bounded functions of t.

The following proposition was proved in [19] for the case $b_1(Y) = 0$, but the proof works in general, with only minor changes:

Proposition 1. *There exist $C_m > 0$ such that for any $(a, \phi) \in L^2_{k+1}(i\Omega^1(Y) \oplus \Gamma(W_0))$ which is equal to $x(t_0)$ for some $t_0 \in \mathbb{R}$ and some Seiberg-Witten trajectory of finite type $x : \mathbb{R} \to L^2_{k+1}(i\Omega^1(Y) \oplus \Gamma(W_0))$, there exists (a', ϕ') smooth and gauge equivalent to (a, ϕ) such that $\|(a', \phi')\|_{C^m} \leq C_m$ for all $m > 0$.*

As in [19], it is useful to project our configuration space to the *Coulomb gauge slice*. Let $\mathcal{G}_0$ be the group of gauge transformations of the form $u = e^{i\xi}$, where $\xi : Y \to \mathbb{R}$ satisfies $\int_Y \xi = 0$. The Coulomb gauge slice is the space

$$V = i \ker d^* \oplus \Gamma(W_0) \cong i\Omega^1(Y) \oplus \Gamma(W_0)/\mathcal{G}_0.$$

For every $(a, \phi) \in i\Omega^1(Y) \oplus \Gamma(W_0)$, there is a unique element $\Pi(a, \phi) \in V$ which is gauge equivalent to (a, ϕ) by a transformation in $\mathcal{G}_0$. We call it the *Coulomb projection* of (a, ϕ).

There is a residual gauge action of $H^1(Y; \mathbb{Z}) \times S^1$ on V as follows: if we choose basepoints on Y and S^1, then $h \in H^1(Y; \mathbb{Z})$ is the homotopy class of a unique pointed harmonic map $u : Y \to S^1$. Then h acts on (a, ϕ) via the gauge transformation u, while $e^{i\theta} \in S^1$ sends $(a, \phi) \in V$ to $(a, e^{i\theta}\phi)$.

As in [19], one can find a metric on V such that the downward gradient trajectories of $CSD_\nu|_V$ with respect to this metric are exactly the Coulomb projections of the trajectories of CSD_ν on $i\Omega^1(Y) \oplus \Gamma(W_0)$. Given a tangent vector at some point in V, its length in the new metric is the L^2 length of its projection to the orthogonal complement of the tangent space to the gauge equivalence class through that point in $i\Omega^1(Y) \oplus \Gamma(W_0)$. The gradient of $CSD_\nu|_V$ with respect to this metric on V is of the form $l + c$, where $l, c : V \to V$ are given by

$$l(a, \phi) = (*da, \not{\partial}\phi)$$
$$c(a, \phi) = (\pi \circ (F_{A_0} + \tau(\phi, \phi)) + i\nu, \rho(a)\phi - i\xi(\phi)\phi).$$

Here π denotes the orthogonal projection from $\Omega^1(Y)$ to $\ker d^*$. Also, $\xi(\phi) : Y \to \mathbb{R}$ is defined by $d\xi(\phi) = i(1 - \pi) \circ \tau(\phi, \phi)$ and $\int_Y \xi(\phi) = 0$.

Let us look at finite type trajectories $x : \mathbb{R} \to L^2_{k+1}(V)$ for some fixed $k \geq 4$. From Proposition 1 we know that they are locally the Coulomb projections of smooth trajectories contained in a bounded set modulo the residual gauge action. In other words, if we denote by $Str_m(R)$ the union of balls

$$\{(a, \phi) \in V | \exists h \in H^1(Y; \mathbb{Z}) \text{ such that } \|h \cdot (a, \phi)\|_{L^2_m} \leq R\},$$

then the following statement is true: all trajectories x as above are smooth in t and there are uniform constants $C_m > 0$ such that $x(t) \in Str_m(C_m)$ for each m.

3. The construction of SWF

This section contains the proof of Theorem 1 (a). Let h be the generator of $H^1(Y; \mathbb{Z})$ which satisfies

$$\ell = (h \cup c_1(\mathfrak{s}))[Y] > 0.$$

Let $u : Y \to S^1$ be the pointed harmonic map in the homotopy class h. Then $u^{-1}du = ih$, where we think of h as a harmonic 1-form. The Seiberg-Witten moduli space is compact

modulo the residual gauge action $u \cdot (a, \phi) \to (a - ih, u\phi)$. We have $CSD_\nu(u \cdot (a, \phi)) = CSD_\nu(a, \phi) + 4\pi^2\ell$.

We follow [19] and consider the orthogonal projections from V to the finite dimensional subspaces V_λ^μ spanned by the eigenvectors of l with eigenvalues in the interval $(\lambda, \mu]$. We can smooth out these projections to obtain a family p_λ^μ which is continuous in μ and λ and still satisfies $V_\lambda^\mu = \mathrm{Im}\ (p_\lambda^\mu)$.

3.1. **The Conley index.** As mentioned in the introduction, we intend to cut the moduli space between two levels of the CSD_ν functional. Since the set of critical points is discrete modulo gauge, we can choose $v \in \mathbb{R}$ which is not the value of CSD_ν at any critical point. Because of the periodicity in the residual gauge direction, the same must be true for the values $v + 4\pi^2 n\ell, n \in \mathbb{Z}$. Choose $m, n \in \mathbb{Z}, m < n$, and consider the set

$$T(R) = \{(a, \phi) \in Str_{k+1}(R) | CSD_\nu(a, \phi) \in (m, n)\}.$$

Recall that the "strip" $Str_{k+1}(R)$ is the union of the residual gauge translates of the ball of radius R in the L_{k+1}^2 norm. On that ball, CSD_ν takes values in a compact interval $I \subset \mathbb{R}$. On a translate of the ball, it takes values in the corresponding interval $I + 4\pi^2 n\ell$, for some $n \in \mathbb{Z}$. It follows that $T(R)$ intersects only finitely many such translates, hence it is bounded in the L_{k+1}^2 norm. Furthermore, if R is sufficiently large, all the Seiberg-Witten trajectories contained in $T(2R)$ are of finite type, hence contained in $T(R)$.

Now we are in the right setting for doing finite dimensional approximation: the gradient flow of CSD_ν on the bounded set $T(2R)$ is of the form $-(\partial/\partial t)x(t) = (l+c)x(t)$, where l is linear Fredholm and self-adjoint and $c : L_{k+1}^2(V) \to L_k^2(V)$ is compact. Let us consider the trajectories of the gradient flow of CSD_ν restricted to V_λ^μ which are contained in $T(2R)$. The following compactness result is the analogue of Proposition 2 in [19], and the proof is completely similar:

Proposition 2. *For any $-\lambda$ and μ sufficiently large, if a trajectory $x : \mathbb{R} \to L_{k+1}^2(V_\lambda^\mu)$,*

$$(l + p_\lambda^\mu c)(x(t)) = -\frac{\partial}{\partial t}x(t)$$

satisfies $x(t) \in T(2R)$ for all t, then in fact $x(t) \in T(R)$ for all t.

Let S be the invariant part of $T = T(2R) \cap V_\lambda^\mu$ under the flow, i.e. the set of critical points of $CSD_\nu|_{V_\lambda^\mu}$ contained in T together with the gradient trajectories between them. Then S is contained in the interior of T by Proposition 2 and the fact that no gradient trajectory can be tangent to a level set of CSD_ν.

Because of these properties, we can associate to S a *Conley index* $I(S)$, which is the pointed space N/L, where (N, L) is an index pair for S, i.e. a pair of compact subsets $L \subset N \subset T$ satisfying the following conditions:

(1) $S \subset \mathrm{int}(N \setminus L)$;
(2) L is an exit set for N, i.e. for any $x \in N$ and $t > 0$ such that $\varphi_t(x) \notin N$, there exists $\tau \in [0, t)$ with $\varphi_\tau(x) \in L$. Here we denote by ϕ the downward gradient flow on T.
(3) L is positively invariant in N, i.e. if for $x \in L$ and $t > 0$ we have $\varphi_{[0,t]}(x) \subset N$, then in fact $\varphi_{[0,t]}(x) \subset L$.

For the basics of Conley index theory, the reader is referred to [7] or [26]. Section 5 of [19] also gives an overview of the relevant properties. The most important ones are the existence of the index pair and the fact that the Conley index is independent of N and L up to canonical homotopy equivalence.

We are interested in the Conley index $I_\lambda^\mu(m, n)$ of $S = S_\lambda^\mu(m, n)$ because its homology is the same as the Morse homology computed by counting critical points and trajectories between them in the usual way. In our case, if we are able to take the limits $n, \mu \to \infty, m, \lambda \to -\infty$, the homology of the resulting object should be some version of Seiberg-Witten Floer homology.

Let us first see what happens as $\mu \to \infty$ and $\lambda \to -\infty$. This process was studied in detail in [19]. If μ increases so that V_λ^μ increases in dimension by 1, the flow on the new T is homotopic to the product of the flow on the old T and a linear flow on the complementary subspace. Since the linear operator l on this subspace has only positive eigenvalues, the respective Conley index is trivial, which implies that I_λ^μ does not change with μ, up to homotopy equivalence.

However, if we decrease λ so that V_λ^μ increases in dimension by 1, then l has negative eigenvalues on the complementary subspaces and the new Conley index I_λ^μ is the suspension of the old one. In order to obtain an invariant object, we need to desuspend by V_λ^0. Let $\mathcal{S}(I_\lambda^\mu)$ be the S^1-equivariant spectrum of I_λ^μ, in the sense of [18]. Set:

$$J_\lambda^\mu(m, n) = \Sigma^{-V_\lambda^0} \mathcal{S}(I_\lambda^\mu).$$

Then $J(m, n) = J_\lambda^\mu(m, n)$ is independent of μ and λ, up to canonical equivalence.

3.2. Some algebraic-topological preliminaries.

Let $\mathfrak{S}$ be the homotopy category of S^1-equivariant spectra with semifree S^1 actions, modelled on the standard universe $\mathbb{R}^\infty \oplus \mathbb{C}^\infty$. Here $\mathbb{R}$ and $\mathbb{C}$ are the real and complex representations of S^1.

As it stands, the category $\mathfrak{S}$ does not have colimits. However, as explained in [20] for the nonequivariant case, given a sequential direct system of spectra in $\mathfrak{S}$:

$$X_1 \to X_2 \to \cdots$$

there is a *minimal weak colimit* $X = \text{wcolim } X_i$ with maps $X_i \to X$ inducing isomorphisms at the level of all homotopy groups. The minimal weak colimit is not usually a colimit: given a system of commuting morphisms $X_i \to Y$, they factor through a morphism $X \to Y$, but this morphism is not unique. Furthermore, while $X = \text{wcolim } X_i$ is unique up to equivalence, it is not unique up to canonical equivalence. Indeed, there can be nontrivial self-homotopy equivalences of X which are the identity on all homotopy groups: they could be of the form $\text{id} + f$, where f is a *phantom map*.

Definition 2. *A **phantom map** $f : X \to Y$ is a morphism in $\mathfrak{S}$ with the property that for every finite CW-spectrum $Z \in Ob\ \mathfrak{S}$ and morphism $g : Z \to X$, the composite $f \circ g : Z \to Y$ is zero. We denote the abelian group of phantom maps from X to Y by $ph(X, Y)$.*

Note that this maps are called **f**-phantoms in [20]. We can get a better behavior if we replace the category $\mathfrak{S}$ by $\mathfrak{S}'$, whose objects are the same as those of $\mathfrak{S}$, but whose sets of morphisms are:

$$\text{Mor}_{\mathfrak{S}'}(X, Y) = \text{Mor}_{\mathfrak{S}}(X, Y)/ph(X, Y).$$

From now on we will denote $\text{Mor}_{\mathfrak{S}'}(X, Y)$ by $[X, Y]'_{S^1}$ for simplicity. The weak colimit becomes an actual colimit in the category $\mathfrak{S}'$. In other words, for a sequential direct system X_i as before, there exists X, unique up to canonical equivalence, such that for every $Y \in \mathfrak{S}$:

$$[X, Y]'_{S^1} = \varprojlim [X_i, Y]'_{S^1}.$$

Moreover, for every generalized homology theory h_*, there is a natural isomorphism:

$$h_*(X) = \varinjlim h_*(X_i).$$

The category $\mathfrak{S}'$ still does not have good inverse limits. There is a notion of weak inverse limit, but its cohomology is not the direct limit of the cohomologies of the terms.

The solution to this problem is to introduce pro-spectra, along the lines of Cohen, Jones, and Segal ([6]). (For the general definition of a pro-category, we refer to [1].) Basically, a *pro-spectrum* X is an inverse system of spectra $\{X_p\}, p \in \mathbb{Z}$:

$$\cdots \leftarrow X_{p-1} \leftarrow X_p \leftarrow \cdots$$

We could call X the *pro-limit* of X_p. The set of morphisms between two pro-spectra $X = \{X_p\}$ and $Y = \{Y_q\}$ is defined as

$$\mathrm{Mor}_{\mathrm{Pro}\text{-}\mathfrak{S}'}(X, Y) = \lim_{\infty \leftarrow q} \lim_{p \to \infty} [X_p, Y_q].$$

In fact, it is easy to check that the pro-limit is in fact the inverse limit of X_p in the category Pro-$\mathfrak{S}'$ of pro-spectra. As before, we denote $\mathrm{Mor}_{\mathrm{Pro}\text{-}\mathfrak{S}'}(X, Y)$ by $[X, Y]'_{S^1}$ for simplicity. The category Pro-$\mathfrak{S}'$ is closed under both sequential colimits and limits.

3.3. **Taking co- and pro-limits.** Let us come back to the spectrum $J(m, n)$, which still depends on m and n, and let us see what happens as we vary n. In a finite dimensional approximation V_λ^μ, the isolated invariant set $S_\lambda^\mu(m, n)$ is an attractor subset of $S = S_\lambda^\mu(m, n+1)$. This means that it "attracts" nearby points of S under the downward gradient flow, which is obviously true because CSD_ν is decreasing along flow lines. There is a corresponding repeller subset $S_\lambda^\mu(n, n+1)$. There is a coexact sequence for the Conley indices of an attractor-repeller pair (see [7]):

$$I(S_\lambda^\mu(m, n)) \to I(S_\lambda^\mu(m, n+1)) \to I(S_\lambda^\mu(n, n+1)) \to \Sigma I(S_\lambda^\mu(m, n)) \to \dots$$

For μ and $-\lambda$ sufficiently large, the first map in the above sequence gives a morphism $J(m, n) \to J(m, n+1)$. This gives a sequential direct system:

$$J(m, n) \to J(m, n+1) \to J(m, n+2) \to \dots.$$

we can take its colimit in $\mathfrak{S}'$:

$$J(m, \infty) = \mathrm{colim}_n J(m, n).$$

We may expect a similar construction for $m \to -\infty$. Indeed, we have exact triangles:

$$J(m-1, m) \to J(m-1, n) \to J(m, n) \to \Sigma J(m-1, m) \to \dots$$

The maps obtained by the composition $J(m-1, n) \to J(m, n) \to J(m, \infty)$ induce a well-defined morphism $J(m-1, \infty) \to J(m, \infty)$ in $\mathfrak{S}$.

Using the resulting maps we can take the pro-limit of $J(m, \infty)$ as $m \to -\infty$. Let us define $SWF(Y, \mathfrak{s}, g)$ as being the pro-spectrum $J(-\infty, \infty) = \{J(m, \infty)\}$.

3.4. **Invariance.** Of course, we have to check that our invariant is independent of the choices made in its construction. The first step is:

Proposition 3. *The pro-spectrum $J(-\infty, \infty)$ does not depend on the value v of CSD_ν where we do the cutting, up to canonical equivalence in Pro-$\mathfrak{S}'$.*

Proof. From our construction it is clear that nothing changes if we replace v by $v + 4\pi^2 n\ell, n \in \mathbb{Z}$ (the direct limits and the pro-limits are the same). Thus it suffices to study the case of $v, v' \in \mathbb{R}$ which are not values of CSD_ν at critical points and which satisfy $v < v' < v + 4\pi^2\ell$. Let us switch notation and denote by $J(a, b)$ the Conley index obtained as before from the approximate trajectories between the levels $CSD_\nu = a$ and $CSD_\nu = b$,

for some $\mu, -\lambda \gg 0$. (For example, $J(v + 4\pi^2 m\ell, v + 4\pi^2 n\ell)$ is what we previously denoted $J(m, n)$.) We have a sequential direct system:

$$J(v + 4\pi^2 m\ell, v + 4\pi^2 n\ell) \to J(v + 4\pi^2 m\ell, v' + 4\pi^2 n\ell) \to$$

$$\to J(v + 4\pi^2 m\ell, v + 4\pi^2 (n+1)\ell) \to J(v + 4\pi^2 m\ell, v' + 4\pi^2 (n+1)\ell) \to \dots$$

Its colimit is the same as that of its subsequence

$$J(v + 4\pi^2 m\ell, v + 4\pi^2 n\ell) \to J(v + 4\pi^2 m\ell, v + 4\pi^2 (n+1)\ell) \to \dots$$

which gives back $J(v + 4\pi^2 m\ell, \infty)$, the one previously denoted $J(m, \infty)$.

Taking the pro-limits as $m \to -\infty$ according to the levels of both $v + 4\pi^2\mathbb{Z}$ and $v' + 4\pi^2\mathbb{Z}$ gives the same pro-spectrum as taking the pro-limit according to each of them separately (they would be subsystems of the same inverse system.) $\square$

Then, from the invariance of the Conley index under deformations it is straightforward to deduce:

Proposition 4. *$SWF(Y, \mathfrak{s}, g, A_0)$ does not change (up to canonical equivalence) as we vary the other parameters involved in the construction, such as the perturbation ν or the radius R in Proposition 2.*

3.5. **Changing the connection and the metric.** Let us explain what happens when we vary the base connection A_0. It suffices to consider nearby connections A_0, A_0'. For these we can find μ and $-\lambda$ sufficiently large so that V_λ^μ does not changes dimension as we choose the base connection to be $A_0 + t(A_0' - A_0), t \in [0, 1]$. Using the invariance under continuation of the Conley index, we get that all the corresponding Conley indices are equivalent. However, the number of negative eigenvalues $n_\lambda = \dim V_\lambda^0$ by which we desuspend in the construction of SWF varies according to the spectral flow of the Dirac operator $\partial\!\!\!/_t, t \in [0, 1]$. Let E be the spinorial part of V_λ^0 for $\partial\!\!\!/_0$ and E' that for $\partial\!\!\!/_0$. Then E, E' are complex S^1-representations, and the difference in their dimensions is the spectral flow. After taking the co- and pro-limits, it follows that SWF changes according to the formula:

$$\mathrm{SWF}(Y, \mathfrak{s}, g, A_0') = \Sigma^{E'-E}\mathrm{SWF}(Y, \mathfrak{s}, g, A_0).$$

Changes in the Riemannian metric have a similar effect.

3.6. **Periodicity.** Let us consider a particular change in connection: the homotopy $A_t = A_0 - ith, t \in [0, 1]$. The spectral flow of the Dirac operators (over $\mathbb{R}$) along this homotopy is ℓ. Hence:

$$(2) \qquad (I_\lambda^\mu(m, n) \text{ with } A_0) \cong \Sigma^E(I_\lambda^\mu(m, n) \text{ with } A_0 - ih),$$

where E is (noncanonically) isomorphic to $\mathbb{C}^{\ell/2}$.

On the other hand, recall the periodicity on $V : CSD_\nu(a - ih, u\phi) = CSD_\nu(a, \phi) + 4\pi^2\ell$. This does not translate into a periodicity on V_λ^μ because $(a - ih, u\phi)$ may not be in V_λ^μ for $(a, \phi) \in V^{\mu,\lambda}$. However, if $(a, \phi) \in V_\lambda^\mu$ with base connection A_0, then it is true that $(a - ih, u\phi) \in V_\lambda^\mu$ with base connection $A_0 - ih$. Hence there is a canonical equivalence:

$$(3) \qquad (I_\lambda^\mu(m, n) \text{ with } A_0) \cong (I_\lambda^\mu(m + 1, n + 1) \text{ with } A_0 - ih).$$

Putting (2) and (3) together and taking the limits as $n, -m \to \infty$ we get the equivalence:

$$\Sigma^E(\mathrm{SWF}(Y, \mathfrak{s})) \cong \mathrm{SWF}(Y, \mathfrak{s})$$

mentioned in Theorem 1 (a).

3.7. **The S^1 action.** All the constructions above can be done in an S^1-equivariant manner, preserving the residual gauge action of S^1 on V, given by $e^{i\theta} : (a, \phi) \to (a, e^{i\theta}\phi)$. The resulting pro-spectrum SWF is S^1-equivariant.

In fact, in this case we can "divide out" by the S^1 action. Indeed, there is no reducible critical point $(a, 0)$ of CSD_ν, because there are no flat connections on W_0. This is also true for critical points in the finite dimensional approximations. It follows by S^1-equivariance that the flow lines between such points also do not intersect the plane $\phi = 0$. Thus we can replace the strip Str by the set Str' obtained from Str by deleting a neighborhood of the plane $\phi = 0$. The S^1 action is free on Str', so we can take the quotient $Str'' = Str'/S^1$. By doing all the constructions as before, but with the quotient flow on Str'', we obtain a pro-spectrum $\mathtt{swf}$. The periodicity is reflected in an equivalence $\Sigma^\ell \mathtt{swf} \cong \mathtt{swf}$.

3.8. **Duality.** Every S^1-equivariant spectrum has a Spanier-Whitehead dual defined as the function spectrum:
$$DX = F(X, S),$$
where $S = S^0$ is the sphere spectrum. It is characterized by the property that
$$[W, DX]_{S^1} = [W \wedge X, S]_{S^1}$$
for every spectrum W. Furthermore, the same is true in $\mathfrak{S}'$:
$$[W, DX]'_{S^1} = [W \wedge X, S]'_{S^1}.$$

The functor D is contravariant. We can extend its definition to pro-spectra $X = \{X_1 \leftarrow X_2 \leftarrow \dots\}$ by letting:
$$DX = \text{colim } DX_i.$$

If we have a sequential direct system of pro-spectra $X_1 \to X_2 \to \cdots$, it is not hard to check that
$$D(\text{colim} X_i) = \text{pro-lim } DX_i.$$

Proof of Theorem 1 (c). If we change the orientation on Y, the function CSD_ν switches sign and the Seiberg-Witten flow changes its direction. As a consequence, by a duality theorem for Conley indices (see [8]), the spaces $J(m, n)$ for X and $J(-n, -m)$ for $\bar{X}$ are duals. Since pro-limits and colimits are dual operations, it follows that SWF $= J(-\infty, \infty)$ for X and SWF for $\bar{X}$ are duals.

The fact that the analogue is not true for SWF_0 can be seen from the example of $S^1 \times S^2$ in Section 5.   □

4. Nonexact perturbations

4.1. **Reconstructiong SWF.** In Section 2 we have considered exact perturbations ν of the CSD functional. More generally, let us consider
$$CSD_\nu(a, \phi) = -\frac{1}{2} \int_Y a \wedge (2F_{A_0} + da + 2\pi i \nu) + \int_Y \langle \phi, \bar{\partial}_a \phi \rangle d\text{vol}$$
for any 1-form ν. The change under the gauge group is:
$$CSD_\nu(u \cdot (a, \phi)) - CSD_\nu(a, \phi) = 4\pi^2([u] \cup (c_1(\mathfrak{s}) + [\nu]))[Y].$$

An important qualitative difference appears when $[\nu] = -c_1(\mathfrak{s})$: no perturbation ν is good (in the sense of Definition 1) in this cohomology class! Indeed, there is always a line of reducible monopoles. Thus we need to introduce an additional perturbation. Recall that

h is the generator of $H^1(Y; \mathbb{Z})$ which satisfies $\ell = (h \cup c_1(\mathfrak{s}))[Y] > 0$. We can think of h as a harmonic function.

We replace CSD_ν by the functional

$$CSD_{\nu,\epsilon}(a, \phi) = CSD_\nu(a, \phi) + \epsilon \sin(2\pi \langle a, h \rangle).$$

The (twice perturbed) Seiberg-Witten equations $\nabla CSD_{\nu,\epsilon}(a, \phi) = 0$ have a discrete set of solutions for generic ν and small ϵ. We call the pair (ν, ϵ) *good* if this condition is satisfied.

However, even introducing ϵ does not make the situation completely analogous to that in the previous section. The problem is that CSD_ν is periodic:

$$CSD_\nu(a - ih, u\phi) = CSD_\nu(a, \phi).$$

This implies that cutting the configuration space between its levels does not work as well: the set $T(R) = Str_{k+1}(R) \cap \{(a, \phi) \in V | CSD_\nu(a, \phi) \in (m, n)\}$ may not be bounded. Therefore, we need to find another way of cutting. For this we introduce a new notion, that of transverse functions.

The Hodge decomposition of 1-forms gives $V = ih\mathbb{R} \oplus i \operatorname{Im} d^* \oplus \Gamma(W_0)$. Let $p : V \to ih\mathbb{R} \cong \mathbb{R}$ be the orthogonal projection to the first factor, and $T : V \to V, T(a, \phi) = (a + ih, u^{-1}\phi)$ be the generator of the residual gauge group. The strip of balls $Str = Str_{k+1}(2R)$ is as in the previous section.

Definition 3. *A **positively (resp. negatively) transverse function** is a smooth functions $f : Str \to \mathbb{R}$ satisfying the following properties:*

(1) *There exists a constant $M > 0$ such that $f(x) < 0$ whenever $p(x) < -M$ and $f(x) > 0$ whenever $p(x) > M$.*
(2) *If $f(x) \geq 0$, then $f(Tx) \geq 0$.*
(3) *We have $\langle \nabla CSD_{\nu,\epsilon}(x), \nabla f(x) \rangle > 0$ (resp. < 0 whenever $f(x) = 0$.*

The inner product in condition 3 is the one used for getting the gradient of $CSD_{\nu,\epsilon}$ in the gauge slice. Condition 3 basically says that the level sets of f at 0 are transverse to the gradient flow, and specifies the direction of the flow at these level sets. Note that because $\nabla CSD_{\nu,\epsilon}$ is invariant under T, all the translates $T^n\{x | f(x) = 0\}$ are also transverse to the flow.

Given a positively transverse function f, we obtain a nice partition of the strip Str in the following way. Let us denote

$$A_n = T^n\{x \in Str | f(x) \leq 0\}.$$

Because of Condition 2 in the definition of a transverse function, we have a nested sequence

$$\cdots \subset A_n \subset A_{n+1} \subset \ldots$$

Let $U(m, n) = A_n \setminus A_m$.

We claim that $U(m, n)$ is bounded in the L^2_{k+1} norm. Indeed, Str is the union of the residual gauge translates of a ball, so it suffices to check that $p(U(m, n))$ is bounded in $\mathbb{R}$. But this is true because of Condition 1.

Note that when $\epsilon = 0$, the function CSD_ν itself is positively transverse. In general we have:

Lemma 1. *For every good pair (ν, ϵ) with $[\nu] = tc_1(\mathfrak{s}), t \geq -1$, positively transverse functions exist. When $t = -1$, negatively tranverse functions also exist.*

Proof. Let $[\alpha, \beta] \subset (0,1)$ be a small interval such that $\nabla CSD_{\nu,\epsilon}(x) \neq 0$ for all $x \in Str$ such that $p(x) \in (\alpha, \beta)$. We then define a smooth increasing function $g : [0,1] \to \mathbb{R}$ such that $g \equiv 0$ on $[0, \alpha]$, and f increases slowly on $[\alpha, \beta]$ so that

$$\|\nabla(g \circ p)(x)\| < \|\nabla CSD_{\nu,\epsilon}(x)\| \text{ if } x \in Str, p(x) \in [\alpha, \beta].$$

Then we set $g \equiv \delta$ on $[\beta, 1]$ for some $\delta = g(\beta) > 0$. Next, we extend f to the whole real line by requiring $g(u + 1) = g(u) + \epsilon$ for all $u \in \mathbb{R}$. Set :

$$f = g \circ p + CSD_{\nu,\epsilon}.$$

We claim that f is positively transverse. Condition 1 is satisfied because $\lim_{u \to \pm\infty} g(u) = \pm\infty$ and $CSD_{\nu,\epsilon}$ is either invariant under T and therefore a bounded function on Str (when $t = -1$) or satisfies $\lim_{u \to \pm\infty} CSD_{\nu,\epsilon}(u) = \pm\infty$ itself (when $t \geq -1$.)

Condition 2 is also satisfied because

$$f(Tx) \geq f(x) + \delta$$

for $x \in Str$ (when $t = -1$ we have equality.)

Finally, condition 3 is satisfied because $\|\nabla(g \circ p)\| \leq \|CSD_{\nu,\epsilon}\|$, with equality only at the critical points of $CSD_{\nu,\epsilon}$, and we can easily arrange so that that the level sets of f at 0 do not go through such points.

Similarly, for $t = -1$ one can show that the function $g \circ p - CSD_{\nu,\epsilon}$ is negatively transverse. However, this function does not satisfy conditions 1 and 2 in the case $t \geq -1$, when $CSD_{\nu,\epsilon}$ is not periodic. $\square$

Now that we have constructed a positively transverse function, note that each of the sets $U(m,n)$ can be used instead of $T(2R)$ to do finite dimensional approximation as in the previous section. Indeed, they are bounded and the flow lines are transverse to the level sets which are separating them, because of condition 3.

We can again consider the set of critical points of $CSD_{\nu,\epsilon}$ inside of U_n together with the trajectories between them (in a suitable finite dimensional approximation), and take its Conley index. After taking the relevant desuspension, we get spectra $J_n^i = J(U_n)$, the analogues of $J(m,n)$ from the previous section. We can take colimits as $n \to \infty$ and pro-limits as $m \to -\infty$. The result is a pro-spectrum $J(-\infty, \infty)$.

Proposition 5. *The pro-spectrum $J(-\infty, \infty) = SWF(Y, \mathfrak{s}, g, A_0)$ is independent of ϵ, ν and of the positively transverse function f used in its definition, as long as $[\nu] = tc_1(\mathfrak{s})$ for $t \geq -1$, and $\epsilon \neq 0$ when $t = -1$.*

Proof. The proof of invariance under changing the transverse function f is similar to the proof of Proposition 3. Independence of the perturbation follows from the invariance under continuation of the Conley index. $\square$

4.2. **The construction of SWF$_0$.** For $[\nu] = -c_1(\mathfrak{s})$, there is an alternate construction, which makes use of negatively transverse functions as well. Let f_1 be a positively transverse function and f_2 a negatively transverse one. We denote:

$$A_n = T^n\{x \in Str | f_1(x) \leq 0\}; \quad B_n = T^n\{x \in Str | f_2(x) \leq 0\}.$$

For $n \gg 0 \gg m$, we have $B_m \subset A_n$. Let $V(m,n) = A_n \setminus B_m$. There is a nested sequence:

$$V(m,n) \subset V(m-1, n+1) \subset V(m-2, n+2) \subset \ldots$$

If we take the Conley indices of the flow inside $V(m,n)$ and desuspend by V_λ^0 we get finite spectra $J'(m,n)$. Note that $V(m,n)$ is an attractor subset of $V(m-1,n+1)$. Thus we can take the colimit as $n \to \infty, m \to -\infty$. Set

$$\mathrm{SWF}_0(Y,\mathfrak{s},g,A_0) = \mathrm{colim}_{m,n} J'(m,n).$$

Unlike SWF, since there is no need of taking pro-limits, this is an actual spectrum. Apart from this, it is easy to see that the other invariance properties of SWF still hold for SWF_0.

Let us now construct the morphism $j : \mathrm{SWF}_0 \to \mathrm{SWF}$ announced in the statement of Theorem 1. Let $U(m,n)$ be the sets between the different levels of f_1 as before. Then, for $m \ll m' \ll 0$, $V(m',n)$ is an attractor subset of $U(m,n)$. This induces a map between the Conley indices:

$$J'(m',n) \to J(m,n).$$

Sending in turn $n \to \infty, m \to -\infty$, and $m' \to -\infty$, we obtain the desired morphism in Pro-$\mathfrak{S}'$:

$$j : J'(-\infty,\infty) \to J(-\infty,\infty).$$

It is not hard to check that this does not depend on the choices made in its construction.

5. FLOER HOMOLOGIES

Let X be an S^1-equivariant pointed pro-spectrum and DX its dual. In this section we discuss some of the generalized homology and cohomology theories of X, and what happens when we apply them to SWF and SWF_0. All our theories are reduced, but we do not write down the conventional tilde.

5.1. Nonequivariant homology theories.

First, we can think of X as a nonequivariant pro-spectrum. We can apply any of the usual nonequivariant generalized homology functors to X, such as stable homotopy, K-theory, or bordism. Of particular interest will be the ordinary homology.

For any generalized homology h_*, there is an associated dual cohomology theory:

$$h^n(X) = h_{-n}(DX).$$

5.2. Some equivariant homology theories.

The material here is taken from [13]. First, a bit of notation: M_+ is the disjoint union of M and a basepoint, while $\tilde{M}$ is the unreduced suspension of M with one of the cone points as basepoint.

The simplest homology theory which takes into account the S^1 equivariance is *Borel homology*:

$$H_n^{S^1}(X) = H_{n-1}(ES_+^1 \wedge_{S^1} X).$$

There is also *Borel cohomology*:

$$H_{S^1}^n(X) = H^n(ES_+^1 \wedge_{S^1} X).$$

However, these two theories are not dual to each other, as one can easily see from the example of $X = S = DS$, when both $H_*^{S^1}$ and $H_{S^1}^*$ are nonzero in infinitely many positive degrees but zero in negative degrees.

The dual homology theory to Borel cohomology is called *coBorel homology* (or *c-homology*), and is defined by:

$$cH_n^{S^1}(X) = \mathrm{colim}_m[\Sigma^{n+m}ES_+^1, K_m \wedge X]_{S^1},$$

where $K_m = K(\mathbb{Z},m)$ is the Elienberg-MacLane spectrum. Notice the analogy with usual homology:

$$H_n(X) = \mathrm{colim}_m[S^{n+m}, K_m \wedge X].$$

For example, when $X = S$, one can compute $cH_n^{S^1}(S) = \mathbb{Z}$ if $n \leq 0$ is even and $cH_n^{S^1}(S) = 0$ otherwise.

Similarly, there is a dual cohomology theory to Borel homology, which is called *coBorel cohomology* (or f-cohomology):

$$fH_{S^1}^n(X) = H_{-n}^{S^1}(DX).$$

We need to introduce one more pair of dual theories: *Tate homology* and *Tate cohomology*. These were originally defined for spaces with finite group actions by Swan. The analogue for S^1-spaces which we use below is due to Jones [14].

$$tH_n^{S^1}(X) = cH_n^{S^1}(\tilde{E}S^1 \wedge X); \;\; tH_n^{S^1}(X) = fH_{S^1}^{n+1}(\tilde{E}S^1 \wedge X).$$

The group $H_{S^1}^*(S) = \mathbb{Z}[U]$, with $\deg U = 2$, acts on Borel, coBorel, and Tate cohomologies by cup product and on the respective Borel homologies by cap product. Correspondingly, the action of U increases degree by 2 in cohomology and decreases degree by 2 in homology. We can also think of the ordinary homology and cohomology as $\mathbb{Z}[U]$ modules with the trivial U action.

There are long exact sequences of $\mathbb{Z}[U]$-modules:

$$\cdots \to H_n(X) \to H_{n+1}^{S^1}(X) \xrightarrow{\;U\;} H_{n-1}^{S^1}(X) \to H_{n-1}(X) \to \cdots$$

(4) $$\cdots \to H_n^{S^1}(X) \to cH_n^{S^1}(X) \to tH_n^{S^1}(X) \to H_{n-1}^{S^1}(X) \to \cdots$$

When applied to the invariant SWF_0, in light of Conjecture 1, these long exact sequences mimic the ones in Ozsváth-Szabó theory from [24]. There are also analogous sequences in cohomology.

Let us conclude with a few remarks on Tate homology and cohomology: when X is a free S^1-spectrum, according to [13]:

$$tH_*^{S^1}(X) = tH_{S^1}^*(X) = 0.$$

More generally, Tate cohomology can be computed by localizing Borel cohomology:

$$tH_{S^1}^*(X) = U^{-1}H_{S^1}^*(X).$$

By the localization theorem, when X is a finite S^1-CW complex with semifree S^1 action and X^{S^1} is its fixed point set, we have:

$$tH_{S^1}^*(X) = tH_{S^1}^*(X^{S^1}) = U^{-1}H_{S^1}^*(X^{S^1}).$$

In particular, when Y is a homology 3-sphere and $\mathrm{SWF}(Y, \mathfrak{s})$ is its Seiberg-Witten Floer spectrum as defined in [19], for generic perturbations there is a unique nondegenerate reducible monopole. Thus $\mathrm{SWF}^{S^1} = S$ and

$$tH_{S^1}^*(X) = \mathbb{Z}[U, U^{-1}].$$

When $X = \mathrm{SWF}(Y, \mathfrak{s}, g, A_0)$ for $b_1(Y) = 1$ and $\mathfrak{s}$ nontorsion as in Section 3, we have $X^{S^1} = *$ and

$$tH_{S^1}^*(X) = 0.$$

5.3. **The effect of j on homology.** In Theorem 1 (b) it is claimed that $j : \mathrm{SWF}_0 \to \mathrm{SWF}$ induces isomorphisms on homology and Borel homology. Here we prove this claim. We let h_* stand for either ordinary or Borel homology.

Recall from Subsection 4.2 the construction of the map j. It comes from an attractor-repeller sequence:

$$V(m', n) \to U(m, n) \to W(m, m'),$$

for $m \ll m' \ll 0 \ll n$. Here $W(m, m') = B'_m \setminus A_m$ and let $J''(m, m')$ be the corresponding Conley index. There is a coexact sequence of Conley indices:

$$J'(m', n) \to J(m, n) \to J''(m, m') \to \Sigma J'(m', n) \to \cdots$$

Applying the functor h_* we get a long exact sequence:

(5) $$h_s(J'(m', n)) \to h_s(J(m, n)) \to h_s(J''(m, m')) \to h_{s-1}(J'(m', n)) \to \cdots$$

Lemma 2. *Fix $k \in \mathbb{Z}$. Then for every $m \ll m' \ll 0$, we have $h_k(J''(m, m')) = 0$.*

Proof. Because of periodicity,

$$h_k(J''(m, m')) = h_{k+\ell}(J''(m + 1, m' + 1)) = \cdots = h_{k-|m|\ell}(J''(0, m' - m)).$$

Now it suffices to show that there exists s_0 such that

(6) $$h_s(J''(0, m' - m)) = 0 \text{ for all } s \geq s_0.$$

For fixed $p = m' - m$, this is true because $J''(0, p)$ is a finite desuspension of a finite S^1-CW-complex, and such complexes have their homology and Borel homology bounded below.

Fix some $p_0 \gg 0$ and choose s_0 so that

$$h_s(J''(0, p_0)) = h_s(J'(0, 1)) = 0$$

for all $s \geq s_0$. By periodicity

$$h_s(J'(p, p + 1)) = h_{s-p}(J'(0, 1)) = 0$$

for all $p \geq 0$ as well. Using the long exact sequences coming form attractor-repeller pairs:

$$h_s(J'(p, p + 1)) \to h_s(J'(0, p + 1)) \to h_s(J'(0, p)) \to \ldots$$

we obtain (6) for any $p = m' - m \geq p_0$ by induction on p. $\square$

Lemma 2 says that the map $h_s(J'(m', n)) \to h_s(J(m, n))$ in (5) is an isomorphism for $m \ll m' \ll 0$. Taking direct limits as $n \to \infty$, inverse limits as $m \to -\infty$, and finally direct limits as $m' \to -\infty$, we obtain that

$$(h_s)_* j : h_s(J'(-\infty, \infty)) \to h_s(J(-\infty, \infty))$$

is an isomorphism as well.

5.4. **The case of $S^1 \times S^2$.** Let us make concrete the difference between SWF and SWF_0 by means of an example. Let $Y = S^1 \times S^2$ and $\mathfrak{s}$ a nontorsion *spin^c* structure with $\ell > 0$ an even integer. Choose g to be the product of the flat metric on S^1 and the round metric on S^2.

It is easy to see that in this case all Seiberg-Witten solutions are reducible. When $[\nu] = \epsilon = 0$, there are no reducibles either, so in fact

$$\mathrm{SWF}(Y, \mathfrak{s}, g, A_0) = *$$

Since j is an isomorphism at the level of homology and Borel homology, for $\mathrm{SWF}_0 = \mathrm{SWF}_0(Y, \mathfrak{s}, g, A_0)$ we must have:

$$H_*(\mathrm{SWF}_0) = H_*^{S^1}(\mathrm{SWF}_0) = 0.$$

From the sequence (4) it follows that

$$cH_*^{S^1}(\mathrm{SWF}_0) = tH_*^{S^1}(\mathrm{SWF}_0).$$

If Conjecture 1 is true, we expect:

$$cH_*^{S^1}(\mathrm{SWF}_0) = tH_*^{S^1}(\mathrm{SWF}_0) = \mathbb{Z}[U, U^{-1}],$$

and the periodicity map $\mathrm{SWF}_0 \to \Sigma^{-\ell}\mathrm{SWF}_0$ should be given by the action of $U^{\ell/2}$. This computation can also be carried out in the context of Seiberg-Witten theory [17].

6. Relative invariants of four-manifolds with boundary

In this section we prove Theorem 2. Let X be a compact 4-manifold with boundary Y such that the image of $H^1(X; \mathbb{R})$ in $H^1(Y; \mathbb{R})$ is 0. Assume that X has a $spin^c$ structure $\hat{\mathfrak{s}}$ which extends $\mathfrak{s}$, and that we are given orientations on $H^1(X; \mathbb{R})$ and $H_+^2(X; \mathbb{R})$.

Our goal is to construct a relative Seiberg-Witten invariant of X in the form of an element in a stable homotopy group of $\mathrm{SWF}_0(Y, \mathfrak{s})$. This construction was done in Section 9 of [19] for the case $b_1(Y) = 0$, and then corrected by Khandhawit in [15]. The current case is more or less similar, so here we will only sketch the construction. The reader is referred to [19, 15] for the analytical details.

Let us also choose a $spin^c$ connection $\hat{A}_0$ on X which restricts to A_0 on Y. Then we can define the Seiberg-Witten map

$$\begin{aligned} SW^\mu : i\Omega_g^1(X) \oplus \Gamma(W^+) &\to i\Omega_+^2(X) \oplus \Gamma(W^-) \oplus V^\mu \\ (\hat{a}, \hat{\phi}) &\to (F_{\hat{A}_0 + \hat{a}}^+ - \rho^{-1}(\phi\phi^*)_0, D_{\hat{A}_0 + \hat{a}}, p^\mu i^*(\hat{a}, \hat{\phi})) \end{aligned}$$

We need to explain the notation. The space $\Omega_g^1(X)$ consists of 1-forms on X in double Coulomb gauge, as in [15, Definition 1]. Furthermore, W^+ and W^- are the positive and negative spinor bundles on X, respectively, ρ is Clifford multiplication, and D is the four-dimensional Dirac operator. Finally, i^* is the restriction to Y, Π is the Coulomb projection for Y, and p^μ is the orthogonal projection to $V^\mu = V_{-\infty}^\mu$.

The map SW can be decomposed into a linear and a compact map between suitable Sobolev completions of the domain and the target. We can apply Furuta's technique of finite dimensional approximation and obtain a map:

$$SW_{\lambda, U}^\mu = \mathrm{pr}_{U \times V_\lambda^\mu} SW^\mu : U' \to U \times V_\lambda^\mu.$$

Here U, U' are finite dimensional spaces. Take a small ball centered at the origin $B(\epsilon) \subset U$ and consider the preimage of $B(\epsilon) \times V_\lambda^\mu$ in U'. Let $\bar{K}_1, \bar{K}_2$ be the intersections of this preimage with a large ball B' in U' and its boundary, respectively. Finally, map $\bar{K}_1, \bar{K}_2$ back to V_λ^μ by composing $SW_{\lambda, U}^\mu$ with the obvious projection. Denote the respective images by K_1 and K_2.

Let us assume for the moment that we are in the simplest case, when $b_1(Y) = 1$ and $c_1(\mathfrak{s})$ is not torsion. Recall the notations from Section 3. Since K_1 is compact, we can choose $n, -m$, and R sufficiently large so that $K_1 \subset T(R)$. Furthermore, the analysis done in [19] (based on the compactness properties of the four-dimensional Seiberg-Witten equations) shows that there exists an index pair (N, L) for the invariant part of $T(2R)$ in the gradient flow such that $K_1 \subset L$ and $K_2 \subset N$.

Thus we can define a map

$$(U')^+ \cong B'/\partial B' \to (B(\epsilon) \times N)/(B(\epsilon) \times L \cup \partial B(\epsilon) \times N) \cong U^+ \wedge I_\lambda^\mu$$

by applying $SW_{\lambda,U}^\mu$ to the elements of $\bar{K}_1$ and sending everything else to the basepoint.

For $-\lambda, \mu \gg 0$, after taking the relevant desuspensions, this gives an element in a stable homotopy group of $J = J(m,n)$. There are such elements for any $-m, n \gg 0$, and they commute with the maps between the different $J(m,n)$ coming from the attractor-repeller exact triangles. Therefore, we can take direct limits and pro-limits and obtain a map to $\mathrm{SWF}(Y, \mathfrak{s})$. If we insist on doing the constructions equivariantly, we get an element

$$\Psi(X, \hat{\mathfrak{s}}, \hat{A}_0) \in \pi_*^{S^1}(\mathrm{SWF}(Y, \mathfrak{s})).$$

In the end we find that for any X and Y there is an element:

$$\Psi(X, \hat{\mathfrak{s}}, \hat{A}_0) \in \pi_*^{S^1}(\mathrm{SWF}(Y, \mathfrak{s})).$$

One can show by a continuity argument that Ψ is independent of the choices made in its construction, up to canonical equivalence.

Starting from here we can compose with the canonical map from stable homotopy to any other generalized homology theory h. Thus we obtain relative Seiberg-Witten invariants of X with values in any $h_*(\mathrm{SWF}(Y, \mathfrak{s}))$.

Now let us vary the base connection $\hat{A}_0$ on X by adding to it a harmonic 1-form $\alpha \in H^1(X; \mathbb{R})$ which annihilates the normal vector to the boundary.

We can collect together the maps $\Psi(X, \hat{\mathfrak{s}}, \hat{A}_0 + \alpha)$ as α varies over the Picard torus $P = H^1(X; \mathbb{R})/H^1(X; \mathbb{Z})$ and obtain a bundle of morphisms from spheres to SWF. In other words, we get a morphism from the Thom space of a vector bundle over P (the Dirac index bundle) to $\mathrm{SWF}(Y, \mathfrak{s})$:

$$\Psi(X, \hat{\mathfrak{s}}) : T(Ind) \to \mathrm{SWF}(Y, \mathfrak{s}).$$

In the case when X is closed, this is the invariant constructed by Bauer in [5].

Remark 2. *By adding a nonexact 2-form perturbation together with a small $\epsilon \sin$ perturbation to the Seiberg-Witten map, an analogous argument should give a morphism from the Picard torus to SWF_0. Thus, we expect Ψ to factor through the map $j : SWF_0 \to SWF$.*

APPENDIX A. THE K-THEORETIC OBSTRUCTION.

Let Y be a closed, oriented, Riemannian 3-manifold, endowed with a $spin^c$ structure $\mathfrak{s}$. We keep the notations from Section 2, but do not impose any condition on b_1 or on $\mathfrak{s}$.

In particular, if we fix a base connection A_0, we identify all other connections A with forms $a \in i\Omega^1(Y; \mathbb{R})$ via $a = A - A_0$. For each a there is a Dirac operator $\bar{\partial}_a$. We restrict our attention to harmonic 1-forms $a \in H^1(Y; \mathbb{R})$. In particular, if $a \in H^1(Y; \mathbb{Z})$, there is a harmonic map $u : Y \to S^1$ with $a = u^{-1}du$. In this case, for every spinor $\phi \in \Gamma(W)$,

$$\bar{\partial}_a \phi = \bar{\partial}_0(u\phi).$$

We can then form a Hilbert bundle over the Picard torus

$$P = H^1(Y; \mathbb{R})/H^1(Y; \mathbb{Z}),$$

with fiber $\Gamma(W)$ by gluing via isomorphisms of the form $\phi \to u\phi$. In this setting we get a continuous family of Dirac operators $\bar{\partial}_a$ acting on each fiber, which is parametrized by P.

As explained in [25] and [6], such a family induces a family of polarizations over P, and therefore a *structural map* $P \to U$, where U denotes the infinite unitary group.

Denote:

$$H^*(Y, \mathfrak{s}) = c_1(\mathfrak{s})^\perp = \{a \in H^*(Y) : a \cup c_1(\mathfrak{s}) = 0\};$$

$$\tilde{P} = H^1(Y, \mathfrak{s}; \mathbb{R})/H^1(Y, \mathfrak{s}; \mathbb{Z}).$$

Let

$$b_1(Y, \mathfrak{s}) = \dim H^1(Y, \mathfrak{s}; \mathbb{R}).$$

Note that $b_1(Y, \mathfrak{s})$ equals $b_1(Y)$ when $c_1(\mathfrak{s})$ is torsion and $b_1(Y) - 1$ otherwise.

The obvious covering map $\tilde{P} \to P$ corresponds to the group homomorphism $\pi_1(P) \to \pi_1(U) = \mathbb{Z}$ given by the spectral flow.

We lift the family of Dirac operators from P to $\tilde{P}$. The composition $\tilde{P} \to P \to U$ induces an element

$$q(Y, \mathfrak{s}) \in K^1(\tilde{P}).$$

According to [6], this is the obstruction for the Seiberg-Witten flow category to be framed, or, in other words, for Floer stable homtopy to be well-defined.

Note that since $\tilde{P}$ is a torus, by the Künneth formula

$$K^1(\tilde{P}) = H^1(\tilde{P}) \oplus H^3(\tilde{P}) \oplus H^5(\tilde{P}) \oplus \dots.$$

Proposition 6. *The obstruction $q(Y, \mathfrak{s}) = q \in K^1(\tilde{P})$ is given by the intersection form:*

$$\Lambda^3 H^1(Y, \mathfrak{s}; \mathbb{R}) \to \mathbb{R}, \quad (a_1, a_2, a_3) \to (a_1 \cup a_2 \cup a_3)[Y],$$

considered as an element in $(\Lambda^3 H^1(Y, \mathfrak{s}; \mathbb{R}))^* = H^3(\tilde{P}) \subset K^1(\tilde{P}).$

Proof. According to [3], the obstruction $q \in K^1(\tilde{P}) = K^0(\tilde{P} \times S^1)$ is given by the K-theoretic index of the family $\{D_{t,a}\}$ of Dirac operators on $Y \times S^1$:

$$D_{t,a} = \begin{cases} I \cos t + i(\slashed{\partial}_a + \frac{\partial}{\partial t}) \sin t, & 0 \le t \le \pi; \\ (\cos t + i \sin t)I, & \pi \le t \le 2\pi. \end{cases}$$

parametrized by $(a, t) \in \tilde{P} \times S^1$.

Since $K^0(\tilde{P} \times S^1)$ is nontorsion, it suffices to compute the Chern character of $q \in H^*(\tilde{P} \times S^1)$. This can be done using the index theorem for families ([2]). Set $m = b_1(Y, \mathfrak{s})$. Let $a_1, \dots, a_m$ be a basis for $H^1(Y, \mathfrak{s}; \mathbb{Z})$, a_{m+1} a generator of $H^1(S^1; \mathbb{Z})$, and $\alpha_1, \dots, \alpha_{m+1}$ the dual basis for $H^1(\tilde{P} \times S^1)$. The Dirac line bundle $\mathcal{L}$ over $Y \times S^1 \times \tilde{P} \times S^1$ has first Chern class:

$$c_1(\mathcal{L}) = \sum_{i=1}^{m+1} \alpha_i a_i.$$

The index theorem gives:

$$ch(q) = e^{\sum \alpha_i a_i} \hat{A}(Y \times S^1)[Y \times S^1].$$

The 3-manifold Y is parallelizable, hence so is $Y \times S^1$. Thus the $\hat{A}$ genus is 1.

In the Taylor decomposition the only term that survives is:

$$ch(q) = \frac{1}{4!}(\sum_{i=1}^{m+1} \alpha_i a_i)^4 [Y \times S^1],$$

which corresponds to the intersection form in $H^4(Y \times S^1) \cong H^3(Y)$.

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