ERRATA TO THE ARTICLE "A GLUING THEOREM FOR THE RELATIVE BAUER-FURUTA INVARIANTS"

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The main results in the paper still hold, but there were several errors in the proofs. We list these errors here, and explain how they can be corrected. The page numbers refer to the arXiv version.

(1) Let us clarify the connectedness assumptions in the introduction. In Theorem 1, when we write a closed four-manifold X as $X_1 \cup_Y X_2$, we should assume that Y is connected (in addition to having $b_1(Y) = 0$). If Y is disconnected, then in general $b_1(X) \neq b_1(X_1) + b_1(X_2)$ and for example we may not have $\mathbf{T}(X) = \mathbf{T}(X_1) \wedge \mathbf{T}(X_2)$. Gluing along disconnected boundaries leads to other difficulties and will not be discussed in this paper.

In Theorem 2, we can allow the initial manifold Y_1 and the final manifold Y_3 to be disconnected, but we still need to assume that Y_2 (the manifold along which we glue the cobordisms X_1 and X_2) is connected.

(2) This paper relies on the previous work of the author [15]. There was an error in the published version of [15]; this was corrected by Khandhawit in [4]. An errata to [15] is available in the latest arxiv version [14].

Precisely, in order to define the relative Bauer-Furuta invariants in [15], it is more convenient to choose a double Coulomb gauge slice instead of the Coulomb-Neumann slice. The current paper needs to be adjusted accordingly. Thus, on page 4, the space $\Omega^1_g(X_1)$ should be defined as the space of 1-forms a_1 on X_1 in double Coulomb gauge, that is, such that $a_1 \in \ker d^*$ and also the restriction of a_1 to the boundary Y is in the kernel of the three-dimensional d^* operator. Later on (on p.17), when we have to use Ω^1_g on a four-manifold with possibly disconnected boundary, in addition to the two Coulomb conditions we ask that, on each boundary component Y_i, the normal component $a_1|_Y(\nu)$ integrates to zero. (This is automatic when the boundary is connected, by Stokes' theorem.)

The rest of the proof of Theorem 1 (and also of Theorem 2) can be adapted to this new setting. A complete proof, using the double Coulomb gauge, can be found in the work of Khandhawit, Lin, and Sasahira [5]. In fact, they prove a more general gluing theorem, where Y can be any three-manifold. In the case $b_1(Y) = 0$, Theorem 1.3 in [5] specializes to give Theorem 2 in this paper.

(3) There were several other issues with the proof of Theorem 1, which were fixed in [5] in their context:

(i) The definition of the duality map in Section 2.5 relies on the fact that the index pairs are constructed from smooth manifolds. On the other hand, in the construction of the relative Bauer-Furuta invariants in [15], we use index pairs that are not of manifold type. We need to relate these different index pairs by certain homotopy equivalences. In [5], the authors deal with this problem by introducing the concept of T-tameness;

(ii) On p.10, in the definition of U''_t, the linear part of Seiberg-Witten map, when restricted to U''_t, is not a Fredholm map. This is due to the presence of the term $(1-t)d(\gamma(x_1)+\gamma(x_2)) + t(\beta(x_1)-\beta(x_2))$, where the first part is order 1 and the second part is order 0. In [5, p.66], the fix for this problem is to replace d by a special degree-0 operator $\bar{d}$, which is not a differential operator. The definition of $\bar{d}$ involves the spectral decomposition of the Laplacian d^*d;

(iii) After the final homotopy on p.16, the boundary condition is deformed to $x_1|_{Y_2} - x_2|_{Y_2} = 0$. Moving this condition from the map to the domain, we get an extra component $(x_1, x_2) \to x_1|_Y - x_2|_Y$ in the Seiberg-Witten map. Via the finite dimensional approximation, this component gives the "approximated boundary condition" $x_1|_{Y_2} - x_2|_{Y_2} \in V^{\mu_n}_{\lambda_n}$. However, in the definition of $\Psi(X_1) \wedge \Psi(X_2)$, we actually used a different boundary condition $x_1|_{Y_2} \in V^{\mu_n}, x_2|_{Y_2} \in V_{\lambda_n}$. A homotopy connecting these two different boundary conditions appears in Step 7 of [5].

(4) The first condition in the definition of a very compact map (Definition 4 on p.21) needs to be strengthened. Suppose E is a bundle over a manifold M. Let I be a compact interval, and consider the pullback π^*E of E under the projection $\pi : I \times M \to M$. Let $\widehat{V}$ be $\Gamma(\pi^*E)$, with Sobolev completions $L^2_k(\widehat{V})$. The map $c : \Gamma(E) \to \Gamma(E)$ induces a map

$$\hat{c} : \widehat{V} \to \widehat{V},$$

given by slicewise application of c. In Definition 4, instead of only asking for c to extend to a compact map

$$c : L^2_{k+1}(V) \to L^2_{k+1}(V)$$

we should ask for $\hat{c}$ to extend to a continuous map

$$\hat{c} : L^2_k(\widehat{V}) \to L^2_k(\widehat{V}).$$

Note that this implies that c extends to a map from $L^2_k(V)$ to itself, and using Rellich's Lemma we get the original condition.

The strengthened hypothesis in Definition 4 is necessary for the proof of Proposition 5, which is modelled on Proposition 3 in [15]. Precisely, in [15], Step 3 in the proof of Proposition 3 requires continuity of $\hat{c}$ in order to do elliptic bootstrapping on $M \times I$.

(5) One needs to slightly modify Proposition 5 so that it applies to the Seiberg-Witten map $l + c$. Indeed, in our setting $V = (\ker d^*) \oplus \Gamma(W)$ is not the space of *all* sections of a vector bundle E, because of the condition $d^*a = 0$. Thus, in the statement of Proposition 5, one should allow V to be the kernel of a map $\tilde{l} : \Gamma(E) \to \Gamma(F)$, where F is another vector bundle, and $l \oplus \tilde{l}$ forms a linear, self-adjoint, elliptic differential operator of order one.

(6) In the second to last paragraph on p.21, one cannot define stable and unstable Hilbert manifolds in infinite dimensional space, because in our setting the "flow" given by $l+c$ is not exactly a flow on a single Hilbert space; rather, the map $l + c$ decreases Sobolev regularity by one. Instead, the standard analogue of the Morse-Smale condition in Floer theory is to ask for the moduli spaces of flow lines between two critical points to be regular, in terms of surjectivity of a certain linear operator. See for example [6, Definition 14.5.6] for this condition in the setting of monopole Floer homology.

More importantly, there is an error in the proof of Proposition 6. At the top of page 23, we need to find a small tubular neighborhood of the set S. However, S is a stratified space inside a Hilbert manifold, and more work would be needed to describe (and ensure) the exact smoothness properties of such a neighborhood.

Proposition 6 is used in the calculations of Seiberg-Witten-Floer spectra for some Brieskorn spheres in Section 7.2. Rather than trying to fix Proposition 6, let us explain how one can do the same calculations by a different route, using some recent work of Lidman and the author [13].

The main result in [13] is an equivalence between the equivariant homology of SWF and the monopole Floer homology constructed by Kronheimer and Mrowka in [6]. This can be combined with the equivalence between monopole Floer homology and Heegaard Floer homology, which was established in [2, 3, 1, 8, 9, 10, 11, 12]. Thus, we obtain the following result, which appears as Corollary 1.4 in [13]: If Y is a rational homology sphere and $\mathfrak{s}$ is a *Spin*^{c} structure on Y, then there an isomorphism of relatively graded $\mathbb{Z}[U]$-modules:

$$\tilde{H}^{S^1}_*(\mathrm{SWF}(Y,\mathfrak{s})) \cong \mathit{HF}^+_*(Y,\mathfrak{s}). \tag{1}$$

Here, HF^+ denotes the plus version of Heegaard Floer homology, as defined by Ozsváth-Szabó in [18].

Heegaard Floer homology was computed in [17] for a large class of plumbed three-manifolds. In particular, this class includes all Brieskorn spheres $-\Sigma(p,q,r)$ with $p,q,r>1$ relatively prime. In view of (1), this tells us the equivariant homology of $\mathrm{SWF}(-\Sigma(p,q,r))$. (Alternatively, one can use the strategy in [17] to do the same calculation in monopole Floer homology, using the exact triangles from [7]. We can then obtain $\tilde{H}^{S^1}_*(\mathrm{SWF}(-\Sigma(p,q,r)))$ by applying the results from [13], but without relying on the monopole / Heegaard Floer correspondence.)

Our goal is to recover the results of Section 7.2, that is, to compute the S^1-equivariant stable homotopy types $\mathrm{SWF}(-\Sigma(2,3,6n\pm 1))$. Apart from knowledge of the equivariant homology, we need some additional input. This is the description of monopoles on Seifert fibered spaces, given by Mrowka, Ozsváth and Yu in [16]. They show that for a particular metric and connection, the Seiberg-Witten equations on $-\Sigma(p,q,r)$ have one reducible solution and several irreducibles, all non-degenerate. The techniques in [13] imply the following:

Proposition A. *Let $Y=-\Sigma(p,q,r)$ with $p,q,r>0$ relatively prime. Then* $\mathrm{SWF}(Y)$ *is the suspension spectrum associated to an S^1-space with a cell decomposition, such that:*

- *The equivariant cells are in one-to-one correspondence with the monopoles on Y as described in* [16]*;*
- *The reducible monopole produces a cell with a trivial S^1-action, whereas each irreducible produces a cell with a free S^1-action;*
- *The relative gradings of the monopoles give the dimensions of the cells;*
- *If a monopole x has lower energy (CSD functional) than another monopole y, then the cell corresponding to x is attached before the cell corresponding to y.*

Proof. Monopoles on Y are stationary points of the (perturbed) Seiberg-Witten flow on Y, i.e. zeros of a map of the form $l+\tilde{c}: V\to V$, with $l=(*d,\not{\partial})$ and $\tilde{c}$ a perturbation of c. The arguments from [13, Section 7] show that, if those stationary points are non-degenerate, then there exists $R>0$ such that for all $\mu=-\lambda\gg 0$, the zeros of $l+\tilde{c}$ are in a one-to-one, grading-preserving correspondence with the zeros of $l+p^\mu_\lambda\tilde{c}$ that live inside a ball $B(R)\subset L^2_{k+1/2}(V^\mu_\lambda)$; and moreover, the zeros of $l+p^\mu_\lambda\tilde{c}$ in $B(R)$ are non-degenerate as well. In turn, this implies that the Conley index for the flow of $l+p^\mu_\lambda\tilde{c}$ inside $B(R)$ has a cell decomposition with cells corresponding to the monopoles on Y. By the usual continuation arguments, it follows that the Conley index for $l + p^\mu_\lambda \tilde{c}$ is homotopy equivalent to that for $l + p^\mu_\lambda c$, which produces the spectrum $\mathrm{SWF}(Y)$.

Let us mention a few differences between what is done in [13] and what is needed in the proof above. In [13, Section 7], one uses continuity and compactness arguments as $\mu = -\lambda \to \infty$, as well as an application of the implicit function theorem, to establish a correspondence between solutions of the Seiberg-Witten equations and approximate solutions in finite dimensions. The perturbations of the Seiberg-Witten flow used in [13] are those introduced by Kronheimer and Mrowka in [6]. In our setting, we need to use instead the perturbed connection from [16], but the same arguments apply. Also, note that in [13] one works with the Seiberg-Witten equations on the blow-up of the configuration space, again following [6]. The blow-up differs from the configuration space only on the reducible locus, so the one-to-one correspondence for irreducible stationary points follows directly from [13]. The correspondence for the reducible (i.e., the proof that $l + p^\mu_\lambda \tilde{c}$ has a single reducible, and that the reducible is non-degenerate) can be established using similar arguments to those in [13].

The statement about the order of attaching cells follows from the fact that if a zero of $l + \tilde{c}$ has lower energy than another zero, then the corresponding zeros of $l + p^\mu_\lambda \tilde{c}$ are ordered the same way by energy. (This is because energy is continuous as $\mu = -\lambda \to \infty$.) When constructing the Conley index from a Morse-Bott function using attractor-repeller pairs, the critical sets with lower energy contribute cells first. □

Observe that Proposition A does not provide a full description of the attaching maps in $\mathrm{SWF}(Y)$. Nevertheless, when combined with (1), Proposition A gives enough information to determine the stable homotopy type $\mathrm{SWF}(Y)$ when Y is of the form $-\Sigma(2, 3, 6n \pm 1)$. For example, let $Z = \mathrm{SWF}(-\Sigma(2, 3, 12j - 1))$. The description in [16] gives one reducible of index 0 and $2j$ irreducibles of index -2. From Proposition A it follows that Z is (stably homotopy equivalent) to the cone of an attaching map

$$S^{-1} \to \bigvee_{i=1}^{2j} \Sigma^{-2}(\mathbb{T}_+),$$

where $\mathbb{T}$ denotes S^1 with the free S^1-action; compare p.25-26 in the paper. Each equivariant stable homotopy group $\{S^{-1}, \Sigma^{-2}(\mathbb{T}_+)\}^{S^1}$ is isomorphic to $\mathbb{Z}$, so the attaching map is given by an element $\delta \in \mathbb{Z}^{2j}$. One can do elementary operations on the free cells without changing the stable equivalence class of Z. Thus, Z is determined (up to stable equivalence) by the divisibility of δ. In other words, we can assume that $\delta = (d, 0, \dots, 0)$ with $d \geq 0$. Given d, the reduced equivariant homology of Z is

$$\tilde{H}^{S^1}_k(Z) = \begin{cases} \mathbb{Z} & \text{if } k = 2i \geq 0, \\ \mathbb{Z}/d \oplus \mathbb{Z}^{2j-1} & \text{if } k = -2, \\ 0 & \text{otherwise.} \end{cases}$$

We now appeal to (1) and the calculations of Heegaard Floer homology in [17], which show that $d = 1$. This gives the desired description of Z. The Seiberg-Witten-Floer spectra for $-\Sigma(2, 3, 12j - 5), -\Sigma(2, 3, 12j + 1)$ and $-\Sigma(2, 3, 12j + 5)$ can be computed similarly, yielding the results in Section 7.2.

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