TRIANGULATIONS AND FLOER HOMOLOGY

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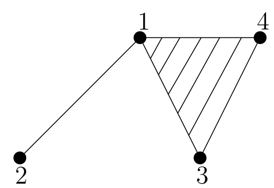
Abstract. We review a few problems about triangulations of manifolds, and their connection to homology cobordism. We explain how questions about homology cobordism can be approached using Floer homology.

1. Triangulations of manifolds

Recall that a *simplicial complex* K is specified by a set of vertices V and a collection S of finite nonempty subsets of V, such that if $\sigma \in S$ and $\tau \subset \sigma$ then $\tau \in S$. We then construct K by attaching a d-dimensional simplex for each set in S of cardinality $d+1$. For example, if

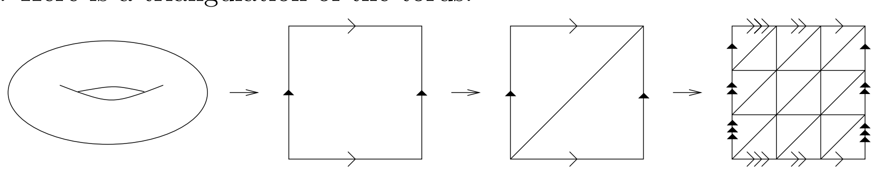
$$V = \{1, 2, 3, 4\}, \quad S = \{\{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{3, 4\}, \{1, 3, 4\}\}$$

then the simplicial complex is



A *triangulation* of a topological space X is a homeomorphism from X to a simplicial complex. This gives a way of describing the space in terms of combinatorial data.

Example 1. Here is a triangulation of the torus:

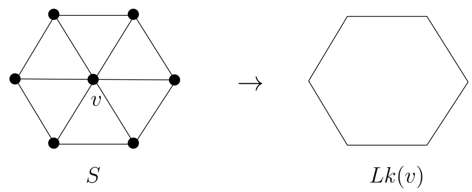


Note that in the third picture, the torus is already split into triangles, but these are not uniquely determined by their simplices. This problem is fixed in the last picture.

The most commonly studied spaces in topology are manifolds. There are different kinds of manifolds, distinguished by the condition we impose on the transition functions between charts. For example, we have

- *Topological manifolds* if there is no condition (the transition functions are C^0);
- *Smooth manifolds* if the transition functions are C^∞;
- *PL (piecewise linear) manifolds* if the transition functions are piecewise linear.

There is a particular kind of triangulations that appear naturally in the context of manifolds. These triangulations are usually called *combinatorial*, but here we will use the term *PL* as it is more accurate. To define them, we first introduce a few notions: in a simplicial complex, the *closed star* of a simplex σ is the union of all the (closed) simplices that contain σ, and the *open star* is the union of the corresponding open simplices. The *link* of σ is the set difference between the closed star and the open star:



We say that a triangulation of a space X is PL if the link of every simplex (or, equivalently, of every vertex) is piecewise-linearly homeomorphic to a sphere. If a PL triangulation exists, then the space X looks locally like a cone on a sphere, i.e., it is a manifold; in fact, it is a PL manifold. One can also prove the converse, that every PL manifold has a PL triangulation.

It is not immediately obvious that some manifolds can have non-PL triangulations, but this can indeed happen in dimensions ≥ 5. One starts with a triangulation of a homology sphere X where $\pi_1(X) \neq 1$. (For example, X could be the Poincaré sphere.) By taking two cones on each simplex in X we get a triangulation of the suspension ΣX. This suspension is not a manifold, but the double suspension $\Sigma^2 X$ is a topological manifold homeomorphic to a sphere, by the double suspension theorem of Edwards [Edw06] , [Edw80] and Cannon [Can79]. The triangulation of $\Sigma^2 X$ induced from that of X is not PL, because the link ΣX of any cone point of $\Sigma^2 X$ is not a sphere (in fact, it is not even a manifold).

It is natural to ask whether all manifolds can be triangulated. There are several versions of this question.

Question 1 (Poincaré [Poi99])**.** *Does every smooth manifold admit a triangulation?*

The answer is yes, as was found by Cairns [Cai35] and Whitehead [Whi40]. Every smooth manifold has a PL-structure, and therefore has a PL triangulation.

Question 2 (Kneser [Kne26])**.** *Does every topological manifold admit a triangulation?*

We can ask this about arbitrary triangulations, or about the more natural PL triangulations. In the PL case, the answer depends on the dimension n of the manifold:

- for $n = 0, 1$: Yes, trivially.
- for $n = 2$ (Radó [Rad25]) Yes. Every two-dimensional surface has a piecewise linear structure and therefore it is triangulable.
- for $n = 3$ (Moise [Moi52]) Yes. Every three dimensional manifold is smooth, hence piecewise linear, and hence triangulable.
- for $n = 4$ (Freedman [Fre82]) No. Freedman constructed the four-manifold E_8 which has no piecewise linear structure.
- for $n \geq 5$ (Kirby-Siebenmann [KS69]) No. For a topological manifold M, the *Kirby-Siebenmann class* $\Delta(M) \in H^4(M, \mathbb{Z}/2)$ is an obstruction class to having a PL structure. In dimensions $n \geq 5$ this is the only obstruction, and there exist manifolds with $\Delta(M) \neq 0$; e.g. $M = T^{n-4} \times E_8$. (In dimension 4, smooth and PL structures are equivalent, and gauge theory provides additional obstructions to their existence.)

For arbitrary triangulations, the answers are as follows:

- for $n \leq 3$: Yes, as above.

- for $n = 4$ (Casson [AM90]) No. The Casson invariant can be used to show that Freedman's E_8 manifold is not triangulable.
- for $n \geq 5$ (Manolescu [?]) No. The ideas in the proof will be sketched below.

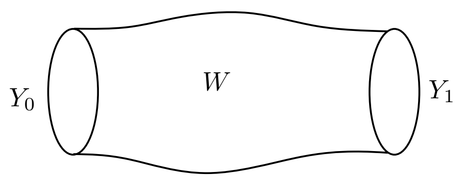
2. The homology cobordism group

To study triangulations of manifolds in dimensions ≥ 5, the key idea is to look at the possible links of simplices of codimension $n+1$. They are n-dimensional homology spheres. Triangulation questions can thus be reduced to questions about a group generated by n-dimensional homology spheres, called the *homology cobordism group* $\Theta^n_{\mathbb{Z}}$. We will describe here the PL version of this group; there is also a smooth version, which is related to the set of exotic spheres. (In dimensions ≤ 6, the PL and smooth theories coincide, and the two versions are the same)

The homology cobordism group is defined as

$$\Theta^n_{\mathbb{Z}} = \{Y^n \text{ oriented, PL}, H_*(Y) = H_*(S^n)\} \ / \ \sim,$$

where the equivalence relation is given by $Y_0 \sim Y_1 \iff$ there exist compact, oriented, PL cobordism W^{n+1} with $\partial W = (-Y_0) \cup Y_1$ and $H_*(W, Y_i; \mathbb{Z}) = 0$, $i = 0, 1$.



Observe that $[Y] = 0$ in $\Theta^n_{\mathbb{Z}}$ if and only if Y bounds a PL homology ball.

The set $\Theta^n_{\mathbb{Z}}$ is given an abelian group structure under the operation of connected sum, with zero being the class $[S^n]$ and the inverse given by orientation reversal: $-[Y] = [-Y]$.

It turns out that $\Theta^n_{\mathbb{Z}} = 0$ for $n \neq 3$ (cf. [Ker69]), but $\Theta^3_{\mathbb{Z}} \neq 0$. The latter fact is a consequence of Rokhlin's Theorem:

Theorem 2 (Rokhlin [Rok52])**.** *Let W be a closed, smooth, spin 4-manifold. Then, the intersection form*

$$Q : H_2(W; \mathbb{Z}) \otimes H_2(W; \mathbb{Z}) \to \mathbb{Z}$$

has signature $\sigma(W)$ divisible by 16.

If W is as above except with boundary ∂W a homology 3-sphere Y, we only know that $\sigma(W)$ is divisible by 8. Rokhlin's theorem implies that $\mu(Y) = \sigma(W)/8 \pmod 2$ only depends on Y (not on W).

This produces the so-called Rokhlin homomorphism:

$$\mu : \Theta^3_{\mathbb{Z}} \to \mathbb{Z}/2, \quad \mu(Y) = \sigma(W)/8 \pmod 2.$$

For example, we have $\mu(S^3) = 0$, but there are also homology 3-spheres (such as the Poincaré sphere) on which μ evaluates to 1. A consequence of that is that $\Theta^3_{\mathbb{Z}} \neq 1$.

In the 1970's, Galewski-Stern [GS80] and Matumoto [Mat78] expressed the main questions about triangulations of manifolds in terms of questions about the structure of $\Theta^3_{\mathbb{Z}}$ (and the Rokhlin homomorphism). The following are consequences of their work:

- There exist non-triangulable manifolds in dim $\geq 5 \iff$ the exact sequence

$$0 \longrightarrow \ker(\mu) \longrightarrow \Theta^3_{\mathbb{Z}} \xrightarrow{\mu} \mathbb{Z}/2 \longrightarrow 0$$

does not split; i.e. there is no $[Y] \in \Theta^3_{\mathbb{Z}}$, with $2[Y] = 0$ and $\mu(Y) = 1$. (In [**?**], the author showed that the sequence does not split; see Section 3.)

- A manifold M of dimension ≥ 5 is triangulable $\iff$ a certain obstruction is zero in $H^5(M; \ker(\mu))$. This could be replaced with an equivalent obstruction in $H^5(M; \mathbb{Z})$, *if we knew* that $\Theta^3_{\mathbb{Z}}$ had no torsion elements at all with $\mu = 1$.
- Triangulations on M (if they exist) are classified by elements in $H^4(M; \ker(\mu))$.

3. Methods for studying $\Theta^3_{\mathbb{Z}}$

The above triangulation problems provide the motivation for understanding $\Theta^3_{\mathbb{Z}}$. While the structure of this group is not fully understood, much progress has been made over the years. We review here some of the main results, and refer to [Ş24] for a comprehensive survey.

The definition of $\Theta^3_{\mathbb{Z}}$ involves four-dimensional PL (or, equivalently, smooth) cobordisms. The tools for studying smooth four-dimensional topology involve gauge theory or symplectic geometry. They can be grouped into three major theories:

(1) Yang-Mills theory;
(2) Seiberg-Witten theory;
(3) Heegaard Floer homology.

3.1. **Yang-Mills theory.** The Yang-Mills equation is a nonlinear PDE, which is elliptic (modulo the action of the gauge group), and can be written on any 4-manifold. Counting its solutions tells us something about the smooth structure on the 4-manifold.

Here are some of the results about $\Theta^3_{\mathbb{Z}}$ obtained from analyzing the Yang-Mills equation:

- Fintushel-Stern, Furuta): $\Theta^3_{\mathbb{Z}}$ has a $\mathbb{Z}^\infty$ subgroup;
- Frøyshov [Frø02]: $\Theta^3_{\mathbb{Z}}$ has a $\mathbb{Z}$ summand;
- Frøyshov [Frø23]: There exist elements of $\Theta^3_{\mathbb{Z}}$ not representable by Brieskorn spheres;
- Nozaki-Sato-Taniguchi [NST24]: There exist elements of $\Theta^3_{\mathbb{Z}}$ not representable by surgeries on knots;
- Daemi [Dae20]: If W is a homology cobordism between homology spheres with $\mu = 1$, then $\pi_1(W) \neq 1$.

3.2. **Seiberg-Witten theory.** The Seiberg-Witten equations are a system of two nonlinear PDEs, again elliptic (mod gauge), and sensitive to the smooth structure on the 4-manifold. They have an S^1 gauge symmetry. In the presence of a spin structure on the manifold, they have an additional conjugation symmetry. Altogether, this amounts to a symmetry under the Lie group

$$\mathrm{Pin}(2) = S^1 \cup jS^1 \subset \mathbb{C} \oplus j\mathbb{C} = \mathbb{H}.$$

If Y is a 3-dimensional manifold, one can study the Seiberg-Witten equations on $\mathbb{R} \times Y$, and obtain an invariant called Seiberg-Witten Floer homology (or monopole Floer homology). Different constructions of this have appeared in [KM07], [Frø10], [MW01], [Man03].

Let $\mathbb{F}$ be a field. The standard (S^1-equivariant) version of Seiberg-Witten Floer homology uses only the S^1 symmetry, and is a a module over

$$H^*_{S^1}(pt; \mathbb{F}) = H^*(\mathbb{CP}^\infty; \mathbb{F}) = \mathbb{F}[U].$$

When Y is a homology 3-sphere, $SWFH^*_{S^1}(Y; \mathbb{F})$ is of the form $\mathbb{F}[U] \oplus (\mathbb{F}[U]\text{-torsion part})$. By considering the grading of the free part $\mathbb{F}[U]$, we obtain a single numerical invariant

$\delta(Y)$. This produces the Frøyshov homomorphism

$$\delta : \Theta^3_{\mathbb{Z}} \to \mathbb{Z}$$

For the Poincaré sphere one can compute $\delta(P) = 1$. It follows that δ is surjective, and therefore $\Theta^3_{\mathbb{Z}}$ has a $\mathbb{Z}$ summand:

$$\Theta^3_{\mathbb{Z}} = [P] \oplus \ker(\delta).$$

One can also use the Pin(2) symmetry and obtain a Pin(2)-equivariant Seiberg-Witten Floer homology. This is usually considered over the field $\mathbb{F}_2$, and denoted $SWFH^*_{\mathrm{Pin}(2)}(Y;\mathbb{F}_2)$. Similarly to the Frøyshov invariant, one can extract from $SWFH^*_{\mathrm{Pin}(2)}(Y;\mathbb{F}_2)$ some maps

$$\alpha, \beta, \gamma : \Theta^3_{\mathbb{Z}} \to \mathbb{Z}.$$

These happen to not be homomorphisms, but satisfy the following properties:

$$\alpha, \beta, \gamma (\mathrm{mod}\ 2) = \mu, \quad \alpha(-Y) = -\gamma(Y), \ \beta(-Y) = -\beta(Y).$$

Theorem 3 ([?])**.** *There is no 2-torsion in $\Theta^3_{\mathbb{Z}}$ with $\mu = 1$. (Hence, non-triangulable manifolds exist in dim ≥ 5.)*

Proof. A homology sphere Y with $2[Y] = 0$ would have Y homology cobordant to $-Y$, and therefore $\beta(Y) = \beta(-Y) = -\beta(Y)$. This would imply $\beta(Y) = 0$ and therefore $\mu(Y) = 0$. □

An alternate construction of the invariants α, β, γ was given by Lin in [Lin18].

3.3. **Heegaard Floer homology.** In [OS04], Ozsváth and Szabó associated to a 3-manifold Y a collection of invariants $\widehat{HF}(Y), HF^+(Y), HF^-(Y)$, which are called *Heegaard Floer homologies*. Their construction uses symplectic geometry; more precisely, Lagrangian Floer homology on the symmetric product of a Heegaard surface for Y.

Heegaard Floer homology was intended as an alternative approach to Seiberg-Witten theory and, indeed, the theories were later shown to be equivalent. For example, $HF^+(Y)$ is isomorphic to $SWFH^*_{S^1}(Y)$; see [KLT20], [CGH24], [LM18].

Its definition makes Heegaard Floer homology easier to compute than gauge-theoretic invariants. In fact, Heegaard Floer homology was shown to be algorithmically computable; see [SW10], [LOT14], [MOT09].

Furthermore, there is an analogue of the Frøyshov invariant δ in Heegaard Floer homology, called the d-correction term [OS03]. Thus, one can re-prove some of the results about $\Theta^3_{\mathbb{Z}}$ using this theory.

4. Involutive Heegaard Floer homology

While much about $\Theta^3_{\mathbb{Z}}$ remains unknown, more progress came out of an offshoot of Pin(2)-equivariant Seiberg-Witten Floer homology, called *involutive Heegaard Floer homology.* This theory is not sufficient to re-prove the triangulation result (Theorem 3), but is more computable than $SWFH^*_{\mathrm{Pin}(2)}(Y;\mathbb{F}_2)$ and has other applications.

Involutive Heegaard Floer homology was constructed by Hendricks and the author in [HM17], by making use of the conjugation symmetry on Heegaard Floer complexes. The resulting homologies are denoted $\widehat{HFI}(Y), HFI^+(Y), HFI^-(Y)$.

Conjecturally, HFI^+ is isomorphic to a $\mathbb{Z}/4$-equivariant version of Seiberg-Witten Floer homology, where $\mathbb{Z}/4 \subset \mathrm{Pin}(2)$ is the subgroup generated by j. We do not yet know how to recover the whole Pin(2) symmetry of the Seiberg-Witten equations in Heegaard Floer theory.

Nevertheless, HFI^+ suffices to give invariants $\bar{\delta}, \underline{\delta} : \Theta^3_{\mathbb{Z}} \to \mathbb{Z}$, similar to those from $\mathbb{Z}/4$-equivariant Seiberg-Witten Floer homology.

Involutive Heegaard Floer homology is computable for Seifert fibrations and surgeries on many knots (torus knots, alternating knots, connected sums of those); see for example [HMZ18], [Zem19], [DM19], [HHSZ20]. This gives more constraints on which 3-manifolds are homology cobordant to each other, and new information about $\Theta^3_{\mathbb{Z}}$.

Here are some of the most striking applications of involutive Heegaard Floer homology:

Theorem 4 (Dai-Hom-Stoffregen-Truong [DHST23])**.** *$\Theta^3_{\mathbb{Z}}$ has a $\mathbb{Z}^\infty$ summand.*

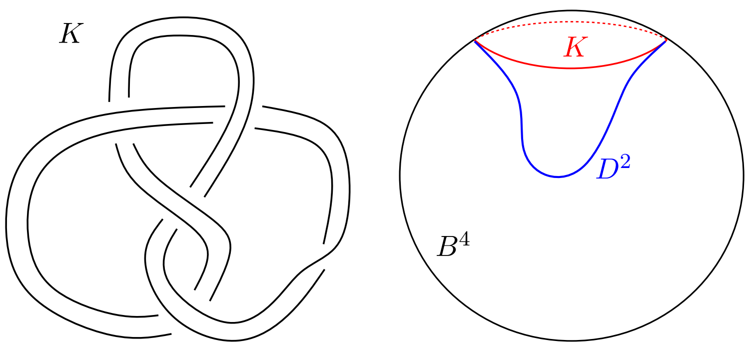
This strengthens the previous results of Fintushel-Stern and Furuta, that $\Theta^3_{\mathbb{Z}}$ has a $\mathbb{Z}^\infty$ subgroup; and that of Frøyshov, that it has a $\mathbb{Z}$ summand.

Theorem 5 (Hendricks-Hom-Stoffregen-Zemke [HHSZ20])**.** *$\Theta^3_{\mathbb{Z}}$ is not generated by Seifert fibrations.*

This strengthens a previous result of Stoffregen [Sto20], that there exist elements of $\Theta^3_{\mathbb{Z}}$ not representable by Seifert fibrations. (By contrast, Myers [Mye83] proved that every element can be represented by a hyperbolic 3-manifold.)

There are also applications to knot theory, such as:

Theorem 6 (Dai-Kang-Mallick-Park-Stoffregen [DKM+22])**.** *The $(2,1)$ cable of the figure eight knot is not slice (i.e., it does not bound a disk in B^4).*



Finally, we discuss an application to stabilization numbers of 4-manifolds.

A pair of 4-manifolds (X, X') is called *exotic* if X and X' are homeomorphic but not diffeomorphic.

Theorem 7 (Wall [Wal64])**.** *Two simply connected, smooth, homeomorphic 4-manifolds become diffeomorphic after taking connected sums with $k(S^2 \times S^2)$, for some $k \geq 0$.*

The same is true for 4-manifolds with boundary, by the work of Gompf [Gom84].

Given a pair of homeomorphic 4-manifolds X and X', the minimum k such that $X\#k(S^2\times S^2)$ and $X'\#k(S^2 \times S^2)$ are diffeomorphic is called the *stabilization number*.

In practice, for many exotic pairs, $k = 1$ suffices to make the manifolds diffeomorphic. A recent result shows that this is not always the case:

Theorem 8 (Kang [Kan22])**.** *There exist contractible, homeomorphic, smooth 4-manifolds X, X' with boundary $\partial X = \partial X' = Y$, such that $X\#(S^2 \times S^2)$ and $X'\#(S^2 \times S^2)$ are not diffeomorphic. (In other words, their stabilization number is ≥ 2.)*

The usual (Seiberg-Witten, Heegaard Floer) invariants of 4-manifolds with boundary vanish after taking connect sum with $S^2 \times S^2$, but the involutive Heegaard Floer invariants can be used to distinguish stabilizations; these are what Kang used in his proof.

5. Open problems

The following questions remain open:

(1) Does $\Theta^3_{\mathbb{Z}}$ have torsion? Is it isomorphic to $\mathbb{Z}^\infty$?
(2) Can we lift μ to a homomorphism $\tilde{\mu} \to \mathbb{Z}$, $\tilde{\mu} \equiv \mu \pmod 2$? If so, this would imply that there is no torsion in $\Theta^3_{\mathbb{Z}}$ with $\mu = 1$. In turn, it would show that a manifold of dimension ≥ 5 is triangulable $\iff$ an obstruction in $H^5(M;\mathbb{Z})$ vanishes.
(3) Can the stabilization number of an exotic pair of 4-manifolds (possibly with boundary) ever be ≥ 3?
(4) Can the stabilization number be ≥ 2 for closed 4-manifolds?

6. Acknowledgements

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