A GLUING THEOREM FOR THE RELATIVE BAUER-FURUTA INVARIANTS

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Abstract. In a previous paper we have constructed an invariant of four-dimensional manifolds with boundary in the form of an element in the stable homotopy group of the Seiberg-Witten Floer spectrum of the boundary. Here we prove that when one glues two four-manifolds along their boundaries, the Bauer-Furuta invariant of the resulting manifold is obtained by applying a natural pairing to the invariants of the pieces. As an application, we show that the connected sum of three copies of the K3 surface contains no exotic nuclei. In the process we also compute the Floer spectrum for several Seifert fibrations.

1. Introduction

In [1], Bauer and Furuta have defined an invariant Ψ of closed four-manifolds which takes values in an equivariant stable cohomotopy group of spheres. As shown by Bauer in [2], this invariant is strictly stronger than the Seiberg-Witten invariant. For example, Ψ can be used to distinguish between certain connected sums of homotopy K3 surfaces; on the other hand, it is well-known that such connected sums have trivial Seiberg-Witten invariants. Essential in computing the invariant for connected sums was Bauer's gluing theorem, which states that the invariant of a connected sum is equal to the smash product of the invariants for each of the two pieces.

As a first step in computing the Bauer-Furuta invariants of other 4-manifolds, one would be interested in determining their behavior with respect to decompositions along more general 3-manifolds. The purpose of the present article is to generalize Bauer's gluing theorem in this direction. We will only discuss gluing along rational homology 3-spheres, but the method of proof seems suitable for further generalizations. In particular, we do not require any special properties for the metric on the 3-manifold.

In [10] and [8], we have extended the definition of Ψ to compact four-manifolds with boundary. If X has boundary Y, then Ψ takes the form of a stable, S^1-equivariant morphism between a Thom spectrum $\mathbf{T}(X)$ associated to the Dirac operator on X and $\mathrm{SWF}(Y)$, which is an invariant of 3-manifolds called the Seiberg-Witten Floer spectrum. More precisely, $\mathbf{T}(X)$ is the formal desuspension of order $b_2^+(X)$ of the Thom spectrum corresponding to the virtual index bundle on $H^1(X;\mathbb{R})/H^1(X;\mathbb{Z})$ coming from the Dirac operators. Note that all of our invariants depend on the choice of *spin*^{c} structures on the respective manifolds, as well as on choices of orientations for H^2_+, but we omit them from notation for simplicity.

Let X_1 and X_2 be two compact, orientable four-manifolds with boundaries Y and $-Y$, respectively, where $b_1(Y)=0$. We can form the manifold $X = X_1 \cup_Y X_2$ and it is easy to see that $\mathbf{T}(X) = \mathbf{T}(X_1) \wedge \mathbf{T}(X_2)$. Furthermore, we know from [10] that the spectra $\mathrm{SWF}(Y)$ and $\mathrm{SWF}(-Y)$ are Spanier-Whitehead dual to each other. There is a natural duality morphism:

$$\eta : \mathrm{SWF}(Y) \wedge \mathrm{SWF}(-Y) \to \mathbf{S},$$

where $\mathbf{S}$ is the sphere spectrum.

We will prove the following gluing result:

Theorem 1. *The Bauer-Furuta invariant of X, as a morphism*

$$\Psi(X) : \boldsymbol{T}(X) \to \boldsymbol{S},$$

is given by the formula:

$$\Psi(X) = \eta \circ \big(\Psi(X_1) \wedge \Psi(X_2)\big).$$

The main idea in the proof is to do finite dimensional approximation on X_1, X_2, and X at the same time, respecting a fiber product formula for the respective Sobolev spaces (Lemma 3).

In [10] we have noted that the invariant Ψ can also be interpreted in terms of cobordisms. If X is a 4-dimensional cobordism between Y_1 and Y_2, then the restriction of $\Psi(X)$ to a fiber over the Picard torus gives a morphism between $\mathrm{SWF}(Y_1)$ and $\mathrm{SWF}(Y_2)$ with a possible change in degree, i.e. a morphism

$$\mathcal{D}_X : \mathrm{SWF}(Y_1) \to \Sigma^{b_2^+(X), -d(X)} \mathrm{SWF}(Y_2).$$

Here $\Sigma^{m,n}$ denotes suspension by m real representations and n complex ones (this could mean formal desuspension if $n < 0$), while $d(X) = (c^2 - \sigma(X))/8, c$ being the first Chern class of the determinant line bundle for the *spin*^{c} structure on X.

For example, in our case $\Psi(X_1)$ gives rise to a morphism between $\mathbf{S}$ and $\mathrm{SWF}(Y)$ and, via the duality map, $\Psi(X_2)$ gives a morphism $\mathcal{D}_{X_2}$ between $\mathrm{SWF}(Y)$ and $\mathbf{S}$ (both up to a change in degree). We can then rephrase Theorem 1 by saying that the Bauer-Furuta invariant of X is the composition of $\mathcal{D}_{X_1}$ with the relevant suspension of $\mathcal{D}_{X_2}$.

More generally, the same method of proof applies to compositions of cobordisms where the initial and the final 3-manifolds are nonempty. Thus we have the following:

Theorem 2. *Let X_1 be a cobordism between Y_1 and Y_2, and X_2 a cobordism between Y_2 and Y_3, where the Y_j's are homology 3-spheres. Denote by X the composite cobordism between Y_1 and Y_3. Then, for $b = b_2^+(X_1)$ and $d = d(X_1)$, we have:*

$$\mathcal{D}_X = (\Sigma^{b,-d} \mathcal{D}_{X_2}) \circ \mathcal{D}_{X_1}.$$

In order to be able to use these gluing results, in sections 6 and 7 we develop some techniques for computing the Seiberg-Witten Floer spectrum for rational homology 3-spheres. This computation can be easily carried out for elliptic manifolds using a metric of positive scalar curvature. It can also be done for some Brieskorn spheres, using the work of Mrowka, Ozsváth, and Yu from [12]. We describe explicitely $\mathrm{SWF}(-Y)$ when $Y = \Sigma(2, 3, 6n \pm 1)$.

In the last section we give a few topological applications of Theorem 1. When Y is a lens space, for example, we recover some versions of the adjunction inequality for Seiberg-Witten basic classes.

When Y is the Brieskorn sphere $\Sigma(2, 3, 11)$, a form of the gluing theorem for relative Seiberg-Witten invariants was proved by Stipsicz and Szabó in [18]. Their methods show, for example, that the K3 surface does not contain any embedded copy of an exotic nucleus $N(2)_{p,q}$, obtained from the standard nucleus $N(2)$ by doing logarithmic transformations of multiplicities p and q along the fibers, for $p, q \geq 1$ relatively prime, $(p, q) \neq (1, 1)$. Using Theorem 1, we can strengthen this result:

Theorem 3. *The connected sum $K3 \# K3 \# K3$ does not split off any exotic nucleus $N(2)_{p,q}$, for $p, q \geq 1$ relatively prime, $(p, q) \neq (1, 1)$.*

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2. The setup

As in the introduction, we let X_1 and X_2 be two compact, orientable four-manifolds with boundaries Y and $-Y$, respectively. We assume $b_1(Y) = 0$, and give X_1 and X_2 metrics which are cylindrical near their boundaries. We also endow each manifold with a $spin^c$ structure and a base $spin^c$ connection in a compatible way, i.e. so that the restrictions of the 4-dimensional objects to their boundaries are the 3-dimensional objects. We choose the base connection on Y to be flat.

To distinguish between objects on different manifolds, we will conventionally use the subscript j the ones on $X_j (j = 1, 2)$, while leaving the three-dimensional ones unmarked. For example, we will denote the Dirac operator associated to the respective base connection by $\partial\!\!\!/_1$ on X_1, $\partial\!\!\!/_2$ on X_2, and by $\partial\!\!\!/$ on Y. The base connections themselves will be A_1, A_2, and A, respectively.

2.1. **The Seiberg-Witten Floer spectrum.**

We recall from [10] the definition of the invariant $\mathrm{SWF}(Y)$. Let V be the space of pairs $x = (a, \phi)$ consisting of a 1-form a and a spinor ϕ on Y, with $d^*a = 0$. Coulomb projection is denoted $\Pi : i\Omega^1(Y) \oplus \Gamma(W) \to V$, and its linearization by Π'. For $\mu \gg 0 \gg \lambda$, we consider the finite dimensional approximation V_λ^μ to V, made of eigenspaces of the operator $(*d, \partial\!\!\!/)$ with eigenvalues between λ and μ. We denote by p_λ^μ the orthogonal projection to V_λ^μ. Similarly, V^μ (resp. V_λ) will be the direct sum of all eigenspaces with eigenvalues at most μ (resp. at least λ), and by p^μ and p_λ the corresponding projections.

The Seiberg-Witten map on V can be written as a sum of a linear Fredholm part l and a compact part c. Both l and c are thought of as mapping the Sobolev $L^2_{k+1/2}$ completion of V to the $L^2_{k-1/2}$ completion of V, where $k > 3$ is an integer. (In the original formulation, k was taken to be a half-integer, but everything in [10] works for any real $k > 3$. In Lemma 3 below we will need k to be an integer.)

We can consider the Seiberg-Witten flow on $L^2_{k+1/2}(V_\lambda^\mu)$:

$$\frac{\partial}{\partial t} x(t) = -(l + p_\lambda^\mu c) x(t).$$

We write $x(t) = \varphi_t(x(0))$ for $t \in \mathbb{R}$.

The following compactness result is proved in [10]:

Proposition 1. *Given any $R \gg 0$, and for any $\mu, -\lambda$ sufficiently large compared to R, if a flow trajectory $x : \mathbb{R} \to L^2_{k+1/2}(V_\lambda^\mu)$ satisfies $x(t) \in \overline{B(2R)}$ for all t, then in fact $x(t) \in B(R)$ for all t.*

This says that the set S of flow trajectories contained in $B = \overline{B(2R)}$ is an isolated invariant set, i.e.

$$S = \{x \in B : \varphi_t(x) \in B \text{ for all } t \in \mathbb{R}\} \subset \mathrm{int} B.$$

Given an isolated invariant set for a flow, one can associate to it an invariant called the Conley index. It takes the form of a pointed space N/L, where (N, L) is a pair of compact subspaces $L \subset N \subset B$ with certain properties. There are many suitable index pairs (N, L), but the quotient N/L is well-defined up to canonical homotopy equivalence. The spectrum $\mathrm{SWF}(Y)$ is then basically the suspension spectrum associated to N/L, shifted in dimension by desuspending by some finite dimensional subspace. For more details, we refer to [10].

2.2. **The relative Bauer-Furuta invariants.** We continue by summarizing the definition of the morphisms Ψ, again following [10].

Let us first introduce some notation which will be useful later on. For $j = 1, 2$, given a spinor/1-form pair x_j on X_j, we denote by $\phi(x_j) \in \Gamma(W)$ the restriction of the spinor component to the boundary. Then, we let $\omega(x_j)$, $\beta(x_j)$ and $\gamma(x_j)$ be the $i \ker d^* \in i\Omega^1(Y)$, $i \ker d \in i\Omega^1(Y)$ and $i\Omega^0(Y) \cdot dt$ components of the 1-form on the boundary, respectively. Note that if the 1-form component of x_j is in the kernel of d^*, then automatically $\gamma(x_j) \in i\,\mathrm{Im}\, d^* \subset i\Omega^0(Y)$. We denote collectively $\alpha = (\omega, \phi)$ and

$$i^* = (\alpha, \beta, \gamma).$$

In this notation, for example, Coulomb projection is

$$\Pi i^*(x_j) = (\omega(x_j), e^{d^{-1}\beta(x_j)}\phi(x_j)),$$

with linearization $\alpha = \Pi' i^*$.

Let $\Omega^1_g(X_1)$ be the space of 1-forms a_1 on X_1 in Coulomb gauge: $a_1 \in \ker d^*, a_1|_{\partial X_1}(\nu) = 0$, where ν is the normal vector to the boundary.

The morphism $\mathcal{D}_{X_1}$ arises as the finite dimensional approximation of the Seiberg-Witten map:

$$SW : i\Omega^1_g(X_1) \oplus \Gamma(W_1^+) \to i\Omega^2_+(X_1) \oplus \Gamma(W_1^-) \oplus V^\mu$$
$$(a_1, \phi_1) \to (F^+_{A_1+a_1} - \sigma(\phi_1, \phi_1), \not{\partial}_1\phi_1 + a_1 \cdot \phi_1, p^\mu \Pi i^*(a_1, \phi_1))$$

We will sometimes denote the map to the first two factors as sw, so that $SW = (sw, p^\mu \Pi i^*)$.

The finite dimensional approximation goes as follows. We first extend SW to a map, still called SW, between the L^2_{k+1} completion of the domain and the L^2_k and $L^2_{k+1/2}$ completions of $i\Omega^2_+(X_1) \oplus \Gamma(W_1^-)$ and V^μ, respectively. We decompose SW as $L + C$, where L is its linearization and C is compact. Then we pick U_1 a finite dimensional subspace of $L^2_k(i\Omega^2_+(X_1) \oplus \Gamma(W_1^-))$, let $U_1' = L^{-1}(U_1)$, and

$$SW_{U_1} = \mathrm{pr}_{U_1 \times V^\mu_\lambda} \circ SW : U_1' \to U_1 \times V^\mu_\lambda.$$

We denote by $B(U_1, \epsilon)$ and $S(U_1, \epsilon)$ the closed ball and the sphere of radius ϵ in U_1 (in the L^2_k norm), respectively. Set

$$\mathcal{M}_1 = SW^{-1}_{U_1}(B(U_1, \epsilon) \times V^\mu_\lambda)$$

and let M_1' and K_1' be the intersections of $\mathcal{M}_1$ with $B(U_1', R)$ and $S(U_1', R)$ respectively, for some $R > 0$ (in the L^2_{k+1} norm). We denote by M_1, K_1 the images of M_1' and K_1' under the composition of SW_{U_1} with projection to the factor V^μ_λ. One can find an index pair (N_1, L_1) which represents the Conley index for V^μ_λ in the form N_1/L_1 such that $M_1 \subset N_1$ and $K_1 \subset L_1$.

Now we have a map:

$$SW_{U_1} : B(U_1', R)/S(U_1', R) \to \Big(B(U_1, \epsilon)/S(U_1, \epsilon)\Big) \wedge (N_1/L_1).$$

We will often say that we are interested in properties of this map for all U_1 *sufficiently large.* What this means is that out of every nested sequence $(U_1)_n \subset L^2_k(i\Omega^2_+(X_1) \oplus \Gamma(W_1^-))$ such that $\mathrm{pr}_{(U_1)_n} \to 1$ pointwise as $n \to \infty$, these are properties of $(U_1)_n$ for all $n \gg 0$.

In a similar vein, we can choose U_2 a finite dimensional approximation on X_2 and we obtain a map:

$$SW_{U_2} : B(U_2', R)/S(U_2', R) \to \Big(B(U_2, \epsilon)/S(U_2, \epsilon)\Big) \wedge (N_2/L_2).$$

Finally, finite dimensional approximation on X using subspaces U gives a map:

$$SW_U : B(U', R)/S(U', R) \to B(U, \epsilon)/S(U, \epsilon).$$

2.3. **Equivariant maps between spheres.** The map SW_U can be thought of as an equivariant map between two spheres with semifree S^1 actions. In general, given such an equivariant map $f : E'^{+} \to E^{+}$, we can choose an identification of E with the standard $\mathbb{C}^n \oplus \mathbb{R}^m$, for some $n, m \geq 0$. Up to homotopy, there is a $\mathbb{Z}/2$ ambiguity in this choice, because we can compose with an orientation reversing isomorphism. Similarly, we can identify E' with some $\mathbb{C}^{n'} \oplus \mathbb{R}^{m'}$.

If we are given an orientation on $E' \oplus E$, we choose the isomorphisms above so that their direct sum is an orientation preserving map

$$(E' \oplus E) \to (\mathbb{C}^{n+n'} \oplus \mathbb{R}^{m+m'}).$$

In this way f defines a canonical element in the equivariant stable homotopy groups of spheres. Indeed, due to the additive structure of the equivariant stable category, if we compose with an orientation reversing map both on the image and on the target, we get back the same element. Note that this is not true unstably, as shown by the example of the Hopf map $S^3 \to S^2$, which does not depend on the orientation of the target.

In our case, the orientation on $U' \oplus U$ is fixed once we specify an orientation for $H^2_+(X)$. Similarly, if we fix an orientation for $H^2_+(X_1)$, the maps SW_{U_1} give a canonical element in the stable homotopy groups of $\mathrm{SWF}(Y)$.

2.4. **From fibers to bundles.** In [10] we have shown that as we increase U_1, the maps SW_{U_1} give rise to the same element in the stable homotopy groups of $\mathrm{SWF}(Y)$. These maps depend on the base connection A_1, which we can allow to vary by adding harmonic forms to it. The end result is a collection of maps parametrized by the Picard torus of X_1, which gives (after stabilization) the morphism $\Psi(X_1) : \mathbf{T}(X_1) \to \mathrm{SWF}(Y)$. Similarly we can obtain $\Psi(X_2)$ from a collection of maps parametrized by the Picard torus of X_2.

In order to prove Theorem 1, we will focus on showing the corresponding statement fiberwise. In other words, we fix A_1, A_2 such that they both restrict to A on the cylindrical ends, we glue them together to give a base connection A_X on X, and then we will show:

Theorem 4. *There exist $\mu, -\lambda, U, U_1, U_2$ sufficiently large and $\epsilon > 0$ sufficiently small such that the maps $\eta \circ (SW_{U_1} \wedge SW_{U_2})$ and SW_U are stably homotopic.*

By "stably homotopic" we mean that the two maps induce the same element in the equivariant stable homotopy groups of spheres.

The method of proof can be applied to a bundle of maps over the Picard torus rather than to single maps, yielding a proof of Theorem 1.

2.5. **The duality map.** In order to prove Theorem 4, we need an explicit description of the duality map η.

The origin of η is the duality theorem between the Conley indices for the forward and reverse flows ([5]). In our situation, the flow on V^μ_λ coming from the Seiberg-Witten flow for Y is the reverse of the one coming from the Seiberg-Witten flow for $-Y$.

According to [11] and [5], the duality between the Conley indices N_1/L_1 and N_2/L_2 can be represented as follows. One can choose the index pairs so that $N_1 = N_2 = N$ is a manifold with boundary whose interior is an open subset of V^μ_λ and $\partial N = L_1 \cup L_2$, where L_j are manifolds with boundary $\partial L_1 = \partial L_2 = L_1 \cap L_2$. The duality is represented by a map:

$$\eta : N/L_1 \wedge N/L_2 \to B(\epsilon)/S(\epsilon),$$

where $B(\epsilon)$ and $S(\epsilon)$ are the ball and the sphere of radius ϵ in V_λ^μ.

Here is one way of constructing η. Pick $\tilde{N}$ any compact subset contained in the interior of N. Take a small tubular neighborhood $[0,\delta] \times \partial N$ of ∂N in N, disjoint from $\tilde{N}$. Then

$$N_1 = N \setminus \big([0,\delta) \times \mathrm{int}L_2\big)$$

is homotopy equivalent to N via a map $m_1 : N \to N_1$ which can be chosen to be the identity on $\tilde{N}$ and such that $m_1(L_1)$ lies in the interior of L_1.

Similarly,

$$N_2 = N \setminus \big([0,\delta) \times \mathrm{int}L_1\big)$$

is homotopy equivalent to N via a map $m_2 : N \to N_2$ which can be chosen to be the identity on $\tilde{N}$ and such that $m_2(L_2)$ lies in the interior of L_2. We can assume that N_1 is separated from $m_2(L_2)$ by a distance at least ϵ, and that the same is true for N_2 and $m_1(L_1)$. We can also assume that $|m_j(x) - x| < 2\epsilon$, for any $x \in N$ and $j = 1, 2$.

Then we define:

$$\eta : N \times N/((N \times L_2) \cup (L_1 \times N)) \to B(\epsilon)/S(\epsilon)$$

$$\eta(x,y) = \begin{cases} m_1(x) - m_2(y) & \text{if } |m_1(x) - m_2(y)| < \epsilon; \\ * & \text{otherwise.} \end{cases}$$

Note that when $x, y \in \tilde{N}$ and $|x-y| < \epsilon$, then $\eta(x,y) = x - y$.

Now we have explicit descriptions of both

$$\eta \circ (SW_{U_1} \wedge SW_{U_2}) : \Big(B(U_1', R)/S(U_1', R)\Big) \wedge \Big(B(U_2', R)/S(U_2', R)\Big) \to$$
$$B(U_1, \epsilon)/S(U_1, \epsilon) \wedge B(U_2, \epsilon)/S(U_2, \epsilon) \wedge \Big(B(V_\lambda^\mu, \epsilon)/S(V_\lambda^\mu, \epsilon)\Big)$$

and

$$SW_U : B(U', R)/S(U', R) \to B(U, \epsilon)/S(U, \epsilon).$$

Note that we have the freedom to choose U, U_1, U_2 as we want, provided they are sufficiently large.

3. Compactness of the glued up moduli space

The definition of the map η in section 2.5 involves the homotopy equivalences m_1 and m_2, which are somewhat inconvenient to deal with. Nevertheless, we know that if x and y lie in a fixed compact subset $\tilde{K}$ in the interior of N, then we have a simple description:

$$\eta(x,y) = \begin{cases} x - y & \text{if } |x-y| < \epsilon; \\ * & \text{otherwise.} \end{cases}$$

It turns out that this description is sufficient when working with the map $\eta \circ (SW_{U_1} \wedge SW_{U_2})$. Indeed, the pairs (x,y) to which we apply the map η in this case are boundary values of connection-spinors pairs (a_j, ϕ_j) on X_j such that $sw(a_j, \phi_j)$ are small. Furthermore, $\eta(x,y) = *$ unless $|m_1(x) - m_2(y)| < \epsilon$. Since $|m_1(x) - x| < \epsilon$ and $|m_2(y) - y| < \epsilon$, this condition implies $|x - y| < 3\epsilon$.

Therefore, we have approximate solutions of the Seiberg-Witten equations on both sides with approximate boundary values x and y close to each other. The idea is that by gluing together the two solutions we obtain an approximate Seiberg-Witten solution on X, whose Sobolev norms we can control. In particular, we would also be able to control the size of x

and y. If we prove that they lie in a compact set in V_λ^μ, then by choosing the radius R in the definition of SW_{U_j} large enough, we could assume that this compact set is $\tilde{K} \subset N$.

The result that we need is Proposition 2 below. We denote by Π' the linearization of the Coulomb projection, i.e. the orthogonal projection from $i\Omega^1(Y) \oplus \Gamma(W)$ onto V.

Proposition 2. *There exists a constant $C > 0$ such that, given any $R \gg C$, for every $U_1, U_2, \mu, -\lambda$ sufficiently large and for every ϵ sufficiently small, whenever $x_j \in U_j'(j = 1, 2)$ satisfy:*

$$\|x_j\|_{L^2_{k+1}} < R; \|pr_{U_j} sw(x_j)\|_{L^2_k} < \epsilon;$$
$$\|p^\mu_\lambda \Pi i^*(x_1) - p^\mu_\lambda \Pi i^*(x_2)\|_{L^2_{k+1/2}} < 3\epsilon,$$

we have the bounds:

$$\|p^\mu_\lambda \Pi i^*(x_j)\|_{L^2_{k+1/2}} \leq C.$$

Note that the requirement $x_j \in U_j'$ automatically implies that:

$$p^\lambda \Pi' i^*(x_1) = p_\mu \Pi' i^*(x_2) = 0.$$

We start by proving the following:

Lemma 1. *There is a constant $C > 0$ such that for any smooth $x_1 \in i\Omega^1_g(X_1) \oplus \Gamma(W_1^+)$ we have:*

$$\|p_0 \Pi' i^*(x_1)\|_{L^2_{k+1/2}} \leq C \cdot \big(\|(d^+ \oplus \not{\partial}_1)x_1\|_{L^2_k} + \|x_1\|_{L^2}\big).$$

Proof. Let us first prove the statement in the particular case when $(d^+ \oplus \not{\partial}_1)x_1 = 0$. Consider the spinorial part ϕ_1 of x_1. On the cylindrical end $[-1, 0] \times Y$ near the boundary, we can write ϕ_1 as a function $\phi : [-1, 0] \to \Gamma(W)$. If ϕ_λ are the eigenvectors of $\not{\partial}$ on $\Gamma(W)$ corresponding to eigenvalue λ, with $\|\phi_\lambda\|_{L^2} = 1$, we have an orthogonal decomposition:

$$\phi(t) = \sum_\lambda c_\lambda(t)\phi_\lambda.$$

The equation $\not{\partial}_1(\phi_1) = 0$ implies $c_\lambda'(t) + \lambda c_\lambda(t) = 0$, so $c_\lambda(t) = e^{-\lambda t} c_\lambda(0)$. Note that:

$$\|p_0 \Pi' i^*(\phi_1)\|^2_{L^2_{k+1/2}} \sim \sum_{\lambda > 0} \lambda^{2k+1} c_\lambda(0)^2,$$

while

$$\|\phi_1\|^2_{L^2} \sim \sum_\lambda \frac{e^{2\lambda} - 1}{2\lambda} c_\lambda(0)^2,$$

so clearly the first expression is controlled linearly by the second.

A similar discussion applies to a_1, the differential form part of x_1. In this case on the cylindrical end we can decompose a_1 as

$$a_1 = \omega(t) + \beta(t) + \gamma(t)dt, \omega(t) \in \ker d^*, \beta(t) \in \ker d, \gamma(t) \in i\Omega^0(Y).$$

We are only interested in bounding the positive part of $\omega(0)$, and the operator d^+ acts on the $\omega(t)$ part as $(\partial/\partial t) \wedge + * d$, so indeed we can proceed just as we did in the spinorial case.

Now consider any smooth $x_1 \in i\Omega^1_g(X_1) \oplus \Gamma(W_1^+)$, and let $y_1 = (d^+ \oplus \not{\partial}_1)x_1$.

There is an $a \ll 0$ so that the Fredholm map:

$$(d^+ \oplus \not{\partial}_1, p^a \Pi' i^*) : L^2_{k+1}(i\Omega^1_g(X_1) \oplus \Gamma(W_1^+)) \to L^2_k(i\Omega^2_+(X_1) \oplus \Gamma(W_1^-)) \oplus L^2_{k+1/2}(V^a)$$

is surjective.

Thus, we can find x_1' with:

$$(d^+ \oplus \not{\partial}_1)x_1' = y_1, p^a\Pi' i^*(x_1') = 0$$

such that we have a bound:

$$\|x_1'\|_{L^2_{k+1}} < C' \cdot \|y\|_{L^2_k}.$$

Since $x_1 - x_1'$ satisfies $(d^+ \oplus \not{\partial}_1)(x_1 - x_1') = 0$, we have:

$$\|p_0\Pi' i^*(x_1 - x_1')\|_{L^2_{k+1/2}} \leq C\|x_1 - x_1'\|_{L^2}.$$

Both the $L^2_{k+1/2}$ norm of $p_0\Pi' i^* x_1'$ and the L^2 norm of x_1' are controlled linearly by the L^2_{k+1} norm of x_1', and hence by the L^2_k norm of y. We deduce that (for a new constant C):

$$\|p_0\Pi' i^*(x_1)\|_{L^2_{k+1/2}} \leq C \cdot \big(\|y\|_{L^2_k} + \|x_1\|_{L^2}\big),$$

as desired.□

Proof of Proposition 2. Fix $R \gg 0$. Assume the statement is not true, so that there exist $\mu_n, -\lambda_n \to \infty, \epsilon_n \to 0, U_{j,n}$ with $\mathrm{pr}_{U_{j,n}} \to 1$, and $x_{j,n}$ satisfying the conditions above but with

$$\|p^{\mu_n}_{\lambda_n}\Pi i^*(x_{j,n})\|_{L^2_{k+1/2}} > C.$$

By Rellich's lemma, we can pick a subsequence of the $x_{j,n}$, still denoted as such for simplicity, so that $x_{j,n}$ converges to some x_j in the L^2_k norm. Hence $sw(x_{j,n}) \to sw(x_j)$ in L^2_{k-1}.

Also, we know that $\mathrm{pr}_{U_{j,n}} sw(x_{j,n}) \to 0$ in L^2_k, and $\mathrm{pr}_{U_{j,n}} \to 1$, so we can deduce that $sw(x_j) = 0$. Furthermore, $p^{\mu_n}_{\lambda_n} \to 1$ as well, so $\Pi i^*(x_1) = \Pi i^*(x_2)$. It follows that after adding a closed 1-form to them (i.e. changing the gauge), we can glue x_1 and x_2 together to form a solution of the Seiberg-Witten equations on the closed manifold X. The moduli space of such solutions is compact, so we get a L^2_{k+1} bound on their size which is independent of R, and this persists after changing the gauge back. Since we know that $x_{j,n} \to x_j$ in L^2_k, we obtain an L^2_k bound on $x_{j,n}$ and an $L^2_{k-1/2}$ bound on the boundary values $p^{\mu_n}_{\lambda_n}\Pi i^*(x_{j,n})$.

To obtain the stronger $L^2_{k+1/2}$ bound which we need, we proceed as follows. By the compactness of $sw - (d^+ \oplus \not{\partial}_j)$, the L^2_k bound on $x_{j,n}$ and the fact that $sw(x_{j,n}) \to 0$ in L^2_k, we get an L^2_k bound on $(d^+ \oplus \not{\partial}_j)x_{j,n}$ which is independent of R. From Lemma 1 it follows that we have such a bound on $\|p_0\Pi' i^*(x_{1,n})\|_{L^2_{k+1/2}}$. If we apply Lemma 1 to X_2 instead of X_1, we get a bound of the same type on $\|p^0\Pi' i^*(x_{2,n})\|_{L^2_{k+1/2}}$.

Now, let us look at the map

$$\Pi - \Pi' = (0, (e^{d^{-1}\beta} - 1)\phi).$$

Changing everything on X_1 and X_2 by the same gauge, we can assume $\beta(x_1) = \beta(x_2) = 0$. We know that the $\phi(x_{j,n})$ are bounded in $L^2_{k+1/2}$ norm, while $\beta(x_{j,n})$ converges to 0 weakly in $L^2_{k+1/2}$, implying $d^{-1}\beta(x_{j,n}) \to 0$ strongly in $L^2_{k+1/2}$. From this we get that $(\Pi - \Pi')i^*(x_{j,n} - x_j) \to 0$ in $L^2_{k+1/2}$, which gives a bound

$$\|(\Pi - \Pi')i^*(x_{j,n})\|_{L^2_{k+1/2}} < C',$$

with C' independent of R.

Combining this with the hypothesis, we get:

$$\|p_{\lambda_n}^{\mu_n}\Pi' i^*(x_{1,n}) - p_{\lambda_n}^{\mu_n}\Pi' i^*(x_{2,n})\|_{L^2_{k+1/2}} < 3\epsilon_n + C'$$

Given the $L^2_{k+1/2}$ bounds on $p_0\Pi' i^*(x_{1,n})$ and $p^0\Pi' i^*(x_{2,n})$ obtained above, as well as the hypothesis

$$p^{\lambda_n}\Pi' i^*(x_{1,n}) = p_{\mu_n}\Pi' i^*(x_{2,n}) = 0,$$

we obtain an $L^2_{k+1/2}$ bound on $p_{\lambda_n}^{\mu_n}\Pi' i^*(x_{i,n})$ and hence on $p_{\lambda_n}^{\mu_n}\Pi i^*(x_{i,n})$ too. This completes the proof. □

4. Composing homotopies

In this section we present the proof of Theorem 4. We need to show that $\eta\circ(SW_{U_1}\wedge SW_{U_2})$ and SW_U are stably homotopic.

We will do this by a series of homotopies and identifications of domains/targets for different maps. All of the maps that we consider will be coming from the following type of construction. We start with a continuous map between two separable Hilbert spaces $f : H' \to H$ such that the zero set $K = f^{-1}(\{0\})$ is compact and such that f decomposes into a linear Fredholm part l and a compact part c. (The prototype is the Seiberg-Witten map on a closed manifold.) We can then conisder finite dimensional approximations E for H and $E' = l^{-1}(E)$ for H' and the corresponding maps

$$f_E = l + \mathrm{pr}_E c : E' \to E.$$

These might not have compact zero sets. However, if we choose a large ball $B(H', r)$ containing $f^{-1}(\{0\})$, it is true that for E sufficiently large, the intersection $f_E^{-1}(\{0\}) \cap B(E', r)$ is compact. By abuse of language, we will say that f_E has **stably compact zero set.** Of course, this definition is meaningful only if f_E is part of a collection of finite dimensional approximations for a map f. In practice, we will only write down f_E, the way E approximates a Hilbert space H being usually self-understood from the context.

For $\epsilon > 0$ small, we can then construct the map:

$$\tilde{f}_E : B(E', r)/\partial B(E', r) \to B(E, \epsilon)/\partial B(E, \epsilon),$$

by sending $x \in f_E^{-1}(B(E, \epsilon))$ to $f_E(x)$ and everything else to the basepoint. Such a map defines an element e in the equivariant stable homotopy group of spheres once we specify an orientation for $E_1 \oplus E_2$ or, equivalently, one for $(\ker l) \oplus (\mathrm{coker} l)$. For different E's approximating the same Hilbert space, and for any $R \gg 0$ and small ϵ, the maps $\tilde{f}_E$ are stably homotopic, in the sense that they define the same element e.

All the maps that we want to compare are of the type $\tilde{f}_E$, but for simplicity we will write down the expressions for f_E, and we will say that two f and g are **stably c-homotopic** if $\tilde{f}$ and $\tilde{g}$ are stably homotopic.

In this language, given the description of $\eta \circ (SW_{U_1} \wedge SW_{U_2})$ in Proposition 2, we need to show that the following two maps are stably c-homotopic:

$$\begin{array}{rcl} U_1' \times U_2' & \to & U_1 \times U_2 \times V_\lambda^\mu \\ (x_1, x_2) & \to & \left(\mathrm{pr}_{U_1} sw(x_1), \mathrm{pr}_{U_2} sw(x_2), p_\lambda^\mu \Pi i^*(x_1 - x_2)\right) \end{array} \tag{1}$$

and

$$U' \to U; \quad x \to \mathrm{pr}_U sw(x). \tag{2}$$

4.1. **Some preliminaries.** As we mentioned above, from now on we will only work with maps between finite dimensional vector spaces approximating Hilbert spaces, and such that their zero sets are stably compact. When we say that one such map is linear, we imply that the limiting map on Hilbert spaces is also linear.

In comparing such maps we will be using repeatedly the following observation.

Observation 1. *Let $f : A \to B$ be continuous and $g : A \to C$ linear and surjective such that*

$$h = (f, g) : A \to B \oplus C$$

has stably compact zero set. An orientation on $A \oplus B \oplus C$ induces one on $\ker g \oplus B$ via the identification $g : (A/\ker g) \to C$.

Then h and

$$f : \ker g \to B$$

are stably c-homotopic.

Proof. Let D be the orthogonal complement of $\ker g$ in A. Since $g|D : D \to C$ is an isomorphism, we have that f is stably c-homotopic to:

$$h' : (\ker g) \oplus D \to B \oplus C, \quad h'(x, y) = (f(x), g(y)).$$

On the other hand, h can be written as:

$$h : (\ker g) \oplus D \to B \oplus C, \quad h(x, y) = (f(x + y), g(y)).$$

We can interpolate from h to h' via the maps $(1 - t)h + th', t \in [0, 1]$. This is possible because the zero sets of all these maps are identical, hence stably compact. Indeed, if $x \in \ker g$ and $y \in D$, then $g(y) = 0$ implies $y = 0$, and from here $(1-t)f(x+y)+tf(x) = 0$ means simply $f(x) = 0$. □

We will refer to the operation of replacing h with $f|_{\ker g}$ as "moving the condition $g = 0$ from the map to the domain."

The comparison of (1) and (2) will be done in several steps.

4.2. **Changing the boundary conditions.** The conditions $\gamma(x_j) = 0$ in the definitions of

$$U_1' = \{x_1 \in L^2_{k+1}(i\Omega^1(X_1) \oplus \Gamma(W_1^+)) : (d^+ \oplus \not{\partial}_j)x_1 \in U_1; d^*x_1 = 0; \\ p^\lambda\alpha(x_1) = 0; \gamma(x_1) = 0\}$$

and

$$U_2' = \{x_2 \in L^2_{k+1}(i\Omega^1(X_2) \oplus \Gamma(W_2^+)) : (d^+ \oplus \not{\partial}_2)x_2 \in U_2; d^*x_2 = 0; \\ p_\mu\alpha(x_2) = 0; \gamma(x_2) = 0\}$$

make these spaces unsuitable for gluing.

However, we can construct isomorphisms

$$\Phi_t : U_1' \times U_2' \to U_t'', t \in [0, 1],$$

where

$$\begin{aligned} U_t'' \quad = \quad & \{(x_1, x_2) : x_j \in L^2_{k+1}(i\Omega^1(X_j) \oplus \Gamma(W_j^+)); (d^+ \oplus \not{\partial}_j)x_j \in U_j; \\ & d^*x_j = 0; p^\lambda\alpha(x_1) = 0; p_\mu\alpha(x_2) = 0; \\ & \gamma(x_1) = \gamma(x_2); (1 - t)d(\gamma(x_1) + \gamma(x_2)) + t(\beta(x_1) - \beta(x_2)) = 0\} \end{aligned}$$

by

$$\Phi_t(x_1, x_2) = (x_1 + du_1, x_2 + du_2),$$

where u_j are harmonic functions on X_j determined by the boundary conditions on Y:

$$u_1' = u_2';$$

$$(1-t)d(\gamma(x_1)+\gamma(x_2)+u_1'+u_2') + t(\beta(x_1)-\beta(x_2)+du_1-du_2) = 0.$$

Here prime denotes the normal derivative at the boundary. Finding the harmonic functions above is a mixed Neumann-Dirichlet problem, whose solution exists and is unique up to addition of constants. Hence the maps Φ_t are well-defined. Their inverses can be easily constructed in a similar way, so they are bijections.

Denoting $U'' = U_1''$, we have that (1) is stably c-homotopic to the map:

$$\begin{array}{rcl} U'' & \to & U_1 \times U_2 \times V_\lambda^\mu \\ (x_1,x_2) & \to & \big(\mathrm{pr}_{U_1} sw(x_1), \mathrm{pr}_{U_2} sw(x_2), p_\lambda^\mu \Pi i^*(x_1-x_2)\big) \end{array} \tag{3}$$

via the homotopy

$$\begin{array}{rcl} U_1' \times U_2' & \to & U_1 \times U_2 \times V_\lambda^\mu \\ (x_1,x_2) & \to & \big(\mathrm{pr}_{U_1} sw(\Phi_t(x_1)), \mathrm{pr}_{U_2} sw(\Phi_t(x_2)), p_\lambda^\mu \Pi i^* \Phi_t(x_1-x_2)\big) \end{array}$$

We need to check that the homotopy goes only through maps whose zero sets are stably compact. The proof is similar to that of Proposition 2. The essential point is to check that in the limit $\mu, -\lambda, \sigma, U_1, U_2 \to \infty$, the Seiberg-Witten equations on X_1 and X_2 with mixed boundary conditions on Y give a compact moduli space. In our case, the boundary conditions are:

$$\begin{array}{rcl} \Pi i^*(x_1) & = & \Pi i^*(x_2); \\ (1-t)(\gamma(x_1)+\gamma(x_2)) & + & t d^*\beta(x_1-x_2) = 0; \\ \gamma(x_1) & = & \gamma(x_2). \end{array} \tag{4}$$

Let $x_n = (x_{1,n}, x_{2,n})$ be a sequence of such pairs of solutions. Gauge transform them to $y_n = (y_{1,n}, y_{2,n})$ so that $y_{1,n} = y_{2,n}$ on Y. Then y_n are actual continuous monopoles on X, and modulo gauge we can choose a convergence subsequence of them, so that they are smooth and the convergence is in C^∞. Then, on each side we can gauge transform them back in a unique way so that they satisfy (4). Since this kind of gauge projection is continuous, we find that the original subsequence of the x_n's was also convergent. This implies that the Seiberg-Witten moduli space with boundary conditions (4) is compact.

4.3. **Moving the Coulomb gauge condition from the domain to the maps.** The map (2) is a finite dimensional approximation for

$$sw : L^2_{k+1}(i\ker d^* \oplus \Gamma(W^+)) \to L^2_k(i\Omega^2_+(X) \oplus \Gamma(W^-)).$$

An alternate approach, which turns out to be more convenient, is to consider the map:

$$s = (sw, d^*) : L^2_{k+1}(i\Omega^1(X) \oplus \Gamma(W^+)) \to L^2_k(i\Omega^2_+(X) \oplus \Gamma(W^-) \oplus i\Omega^0(X)/\mathbb{R}).$$

Here $\Omega^0(X)/\mathbb{R}$ denotes the space of functions which integrate to zero on X. Choosing a finite dimensional approximation T for $i\Omega^0(X)/\mathbb{R}$, we can consider $\tilde{U}' = (d^+ \oplus \not{\partial} \oplus d^*)^{-1}(U \times T)$. If U is large enough, then the linear map $d^* : \tilde{U}' \to T$ is surjective and Observation 1 implies that (2) is stably c-homotopic to:

$$\tilde{U}' \to U \times T; \;\; x \to (\mathrm{pr}_U sw(x), d^* x). \tag{5}$$

Starting from here, we can in fact replace (5) by other finite dimensional approximations to $s = (sw, d^*)$. We can choose any sufficiently large subspace $\tilde{U} \subset L^2_k(i\Omega^2_+(X) \oplus$

$\Gamma(W^-) \oplus i\Omega^0(X)/\mathbb{R})$, not necessarily of the form $U \times T$, and consider its preimage under the linearization $(d^+ \oplus \not{\partial} \oplus d^*)$, which we still denote $\tilde{U}'$. Then we have a map

$$\tilde{U}' \to \tilde{U}; \ \ x \to \mathrm{pr}_{\tilde{U}} s(x), \tag{6}$$

stably c-homotopic to the original (2).

Similarly, we have that (3) is stably c-homotopic to a map:

$$\begin{aligned} \tilde{U}'' \quad &\to \quad \big((\tilde{U}_1 \times \tilde{U}_2)/\mathbb{R}\big) \times V_\lambda^\mu \\ (x_1, x_2) \quad &\to \quad \big(\mathrm{pr}_{U_1} s(x_1), \mathrm{pr}_{U_2} s(x_2), p_\lambda^\mu \Pi i^*(x_1 - x_2)\big), \end{aligned} \tag{7}$$

where $\tilde{U}_j$ is a finite dimensional approximation for $i\Omega^2_+(X_j) \oplus \Gamma(W_j^-) \oplus i\Omega^0(X_j)$, the $\mathbb{R}$ in $(\tilde{U}_1 \times \tilde{U}_2)/\mathbb{R}$ stands for

$$\int_{X_1} d^* x_1 + \int_{X_2} d^* x_2 = \int_Y (\gamma(x_1) - \gamma(x_2)) \in \mathbb{R},$$

and

$$\begin{aligned} \tilde{U}'' \quad = \quad &\{(x_1, x_2) : x_j \in L^2_{k+1}(i\Omega^1(X_j) \oplus \Gamma(W_j^+)); (d^+ \oplus \not{\partial}_j \oplus d^*)x_j \in \tilde{U}_j; \\ &p^\lambda \alpha(x_1) = 0; p_\mu \alpha(x_2) = 0; \gamma(x_1) = \gamma(x_2); \beta(x_1) = \beta(x_2)\}. \end{aligned}$$

4.4. **Linearizing the Coulomb projection.** We change the map (7) using the homotopy:

$$\begin{aligned} \tilde{U}'' \quad &\to \quad \big((\tilde{U}_1 \times \tilde{U}_2)/\mathbb{R}\big) \times V_\lambda^\mu \\ (x_1, x_2) \quad &\to \quad \big(\mathrm{pr}_{U_1} s(x_1), \mathrm{pr}_{U_2} s(x_2), p_\lambda^\mu((1-t)\Pi i^* + t\alpha)(x_1 - x_2)\big), \end{aligned} \tag{8}$$

for $t \in [0, 1]$.

To make sure that the zero set is stably compact throughout this homotopy, in the limit we need to check the compactness of the space of Seiberg-Witten equations on X_1 and X_2 with boundary conditions:

$$\begin{aligned} (1-t)\Pi i^*(x_1) + t\alpha(x_1) \quad &= \quad (1-t)\Pi i^*(x_2) + t\alpha(x_2); \\ \beta(x_1) = \beta(x_2) \quad &; \quad \gamma(x_1) = \gamma(x_2). \end{aligned}$$

Recall that $\alpha(x_j) = (\omega(x_j), \phi(x_j))$ and $\Pi i^*(x_j) = (\omega(x_j), e^{d^{-1}\beta(x_j)}\phi(x_j))$. Since $\beta(x_1) = \beta(x_2)$, the boundary conditions do not actually change throughout the homotopy.

4.5. **Moving all boundary conditions from the map to the domain.** Using Observation 1 again, the map (8) at $t = 1$ is seen to be stably c-homotopic to:

$$\begin{aligned} Q' \quad &\to \quad (\tilde{U}_1 \times \tilde{U}_2)/\mathbb{R} \\ (x_1, x_2) \quad &\to \quad \big(\mathrm{pr}_{U_1} s(x_1), \mathrm{pr}_{U_2} s(x_2)\big), \end{aligned} \tag{9}$$

with

$$\begin{aligned} Q' \quad = \quad &\{(x_1, x_2) : x_j \in L^2_{k+1}(i\Omega^1(X_j) \oplus \Gamma(W_j^+)); (d^+ \oplus \not{\partial}_j \oplus d^*)x_j \in \tilde{U}_j; \\ &p^\mu \alpha(x_1) - p_\lambda \alpha(x_2) = 0; \gamma(x_1) = \gamma(x_2); \beta(x_1) = \beta(x_2))\}. \end{aligned}$$

This is true under the hypothesis that the linear map

$$\tilde{U}'' \to V_\lambda^\mu; \ \ (x_1, x_2) \to p_\lambda^\mu(\alpha(x_1) - \alpha(x_2)) \tag{10}$$

is surjective.

Set

$$\begin{aligned} \tilde{Q} \quad = \quad &\{(x_1, x_2) : x_j \in L^2_{k+1}(i\Omega^1(X_j) \oplus \Gamma(W_j^+)); (d^+ \oplus \not{\partial}_j \oplus d^*)x_j \in \tilde{U}_j; \\ &\gamma(x_1) = \gamma(x_2); \beta(x_1) = \beta(x_2))\}. \end{aligned}$$

and consider the map

$$\delta : \tilde{Q} \to V; \quad \delta(x_1, x_2) = \alpha(x_1) - \alpha(x_2). \tag{11}$$

The map δ is Fredholm. Let us choose $\tilde{U}_j$ so that $\tilde{Q}$ is sufficiently large for δ to be surjective.

The following lemma is well-known:

Lemma 2. *Let $f : H_1 \to H_2$ be a surjective Fredholm map between two Hilbert spaces. Let $c_n : H_1 \to H_2$ be a sequence of compact linear maps which converge to zero in the operator norm. Then, for $n \gg 0$, $f_n = f + c_n$ is Fredholm and surjective. Furthermore, the orthogonal projections $pr_{\ker f_n}$ converge to $pr_{\ker f}$ as $n \to \infty$.*

The maps

$$\tilde{Q} \to V; \quad (x_1, x_2) \to p^{\mu}\alpha(x_1) - p_{\lambda}\alpha(x_2) \tag{12}$$

are also Fredholm and differ from δ by compact maps which converge to 0. Hence for μ and $-\lambda$ sufficiently large compared to $\tilde{Q}$, (12) is surjective. Note that this automatically implies that (10) is surjective. Furthermore, the kernel Q' of (12) is close to $Q = \ker \delta$, in the sense that the orthogonal projection $\mathrm{pr}_{Q'} : Q \to V$ is an isomorphism which converges to the identity on Q as $\mu \to \infty, \lambda \to -\infty$.

It follows that (9) is stably c-homotopic to:

$$\begin{aligned} Q &\to (\tilde{U}_1 \times \tilde{U}_2)/\mathbb{R} \\ (x_1, x_2) &\to \big(\mathrm{pr}_{U_1} s(x_1), \mathrm{pr}_{U_2} s(x_2)\big), \end{aligned} \tag{13}$$

with

$$\begin{aligned} Q = \; &\{(x_1, x_2) : x_j \in L^2_{k+1}(i\Omega^1(X_j) \oplus \Gamma(W_j^+)); (d^+ \oplus \not{\partial}_j \oplus d^*)x_j \in \tilde{U}_j; \\ &\alpha(x_1) - \alpha(x_2) = 0; \gamma(x_1) = \gamma(x_2); \beta(x_1) = \beta(x_2)\}. \end{aligned}$$

4.6. **Gluing Sobolev spaces.** At this point we are left to show that (13) above is stably c-homotopic to the map (6):

$$\tilde{U}' \to \tilde{U}; \quad x \to \mathrm{pr}_{\tilde{U}} s(x).$$

In order to do this, we need to choose $\tilde{U}_1, \tilde{U}_2, \tilde{U}$ carefully so that we can build a suitable identification of some suspensions of the domains and the targets of the two maps.

We begin by understanding the relationship between the Sobolev spaces of forms or spinors on X_1, X_2, and X. The relevant result from analysis is the following:

Lemma 3. *Let X_1, X_2 be n-dimensional compact manifolds with common boundary Y, and let $X = X_1 \cup_Y X_2$. We assume that X_1 and X_2 have cylindrical ends near the boundary, and denote by t the variable in the direction normal to Y. If E is a vector bundle over X, we denote by $L^2_m(U; E)$ the space of L^2_m sections of E over a subset $U \subset X$. Then, for every integer $k \geq 1$, the space $L^2_k(X; E)$ can be naturally identified as the fiber product:*

$$L^2_k(X; E) = L^2_k(X_1; E) \times_{\left(\prod_{m=0}^{k-1} L^2_{m+1/2}(Y;E)\right)} L^2_k(X_2; E)$$

with respect to the maps:

$$\begin{aligned} r_j : L^2_k(X_j; E) &\to \prod_{m=0}^{k-1} L^2_{m+1/2}(Y; E) \\ r_j(u) &= \Big(u|_Y, \frac{\partial u}{\partial t}\big|_Y, \frac{\partial^2 u}{\partial t^2}\big|_Y, \dots, \frac{\partial^{k-1} u}{\partial t^{k-1}}\big|_Y\Big), j = 1, 2. \end{aligned}$$

Proof. It suffices to prove the statement for a neck surrounding the boundary, so we can assume that instead of X_1 and X_2 we have $[-1,0]\times Y$ and $[0,1]\times Y$, respectively, and we identify Y with $\{0\}\times Y$.

We need to show that if $u_j \in L^2_k(X_j;E)(j=1,2)$ are such that the restrictions of their mth derivatives to Y coincide for $0 \leq m < k$, we can combine them to form $u=(u_1,u_2)$ on $X = X_1 \cup X_2$ which is also in L^2_k.

Let us first study the case $k=1$. Observe that:

$$L^2_1([a,b]\times Y;E) = L^2_1([a,b],L^2(Y;E)) \cap L^2([a,b],L^2_1(Y;E)).$$

Since clearly $L^2([-1,1],F) = L^2([-1,0],F)\times L^2([0,1],F)$ for any F (in particular for $F = L^2_1(Y;E)$), we only have to show that:

$$L^2_1([-1,1],F) = L^2_1([-1,0],F)\times_F L^2_1([0,1],F), \tag{14}$$

where $F = L^2(Y;E)$ and the fiber product is taken with respect to the restrictions to $\{0\}$. This is well-defined, because for one-dimensional spaces $L^2_1 \subset C^0$.

In fact, (14) is true for any F. Indeed, if u is a continuous section of F over $[-1,1]$ such that its restrictions to $[-1,0]$ and $[0,1]$ are in L^2_1, then the distributional derivative u' is in L^2 when restricted to each of the two halves. Hence u is differentiable a.e. on each half. Let

$$v(t) = u(-1) + \int_{-1}^{t} u'(s)ds.$$

The functions v and u are continuous and their derivatives exist and are equal a.e. Furthermore, the distributional derivative $w=(u-v)'$ is supported only at 0. Note that $w \in L^2_{-1}$, being the derivative of a continuous function. Hence w is a multiple of the δ function. However, w integrates to 0 on a small interval around 0, so in fact $w=0$ and $u=v$. Thus u' exists as a function. Since its restriction to each half is in L^2, u' itself is in L^2 and hence $u \in L^2_1$. This takes care of the case $k=1$.

For general k, we can apply the $k=1$ statement to $u^{(m)} = (u_1^{(m)}, u_2^{(m)})$ for $m<k$ (where the superscript (m) denotes the mth derivative with respect to t). We get that $u^{(m)} \in L^2_1$ and, arguing as above, the $u^{(m)}$'s must be the actual distributional derivatives of u. This implies that $u \in L^2_k$. □

4.7. **Finite dimensional approximation on Y.** On the cylindrical end, the self-dual forms on X_j are of the form $a\wedge dt + *a$, where a are 1-forms on Y. Thus, the restriction of the bundle of self-dual 2-forms to the boundary Y can be identified with the bundle of 1-forms on Y. Under the Hodge decomposition, this is $i\Omega^1(Y) = i\ker d^* \oplus i \ker d$. Similarly, the restriction of the spinor bundle W^- to Y can be identified with the 3-dimensional spinor bundle W.

Let us denote

$$Z = i\Omega^1(Y)\oplus i\Omega^0(Y)\oplus \Gamma(W).$$

The elements of Z can be written accordingly as $z=(\omega+\beta,\gamma,\phi)$, in agreement with the notation introduced in Subsection 4.1. Here $d^*\omega = 0$ and $d\beta=0$.

On a slice of the cylindrical end, we can identify both the source and the target of the map s with Z, while the map s itself is of the form $\frac{\partial}{\partial t}+f$, with $f: Z\to Z$ the direct sum of the Seiberg-Witten and d^* maps on Y. The linearization of f is:

$$D: Z\to Z,\ \ D(\omega+\beta,\gamma,\phi) = (*d\omega + d\gamma, d^*\beta, \partial\!\!\!/\phi).$$

The fact that this is a self-adjoint linear Fredholm map allows us to do finite dimensional approximation on Z using the eigenvalues of D. We denote by Z_ν^σ the direct sum of all eigenspaces of D with eigenvalues between ν and σ, where typically $\sigma \gg 0$ and $\nu \ll 0$.

4.8. **The choice of finite dimensional approximations.** We choose the subspaces

$$\tilde{U}_j \subset L_k^2(i\Omega_+^2(X_j) \oplus \Gamma(W_j^-) \oplus i\Omega^0(X_j)) \ (j = 1, 2)$$

and

$$\tilde{U} \subset L_k^2(i\Omega_+^2(X) \oplus \Gamma(W^-) \oplus i\Omega^0(X)/\mathbb{R})$$

so that they are related to each other by a fiber product construction, which models the one for the Sobolev spaces themselves.

For $j = 1, 2$ we have restriction maps as in Lemma 3:

$$r_j : L_k^2(i\Omega_+^2(X_j) \oplus i\Omega^0(X_j) \oplus \Gamma(W_j^-)) \to \mathbb{R} \times \prod_{m=0}^{k-1} L_{m+1/2}^2(Z)$$

$$x_j \to \Big((-1)^{j+1} \int_{X_j} x_j;\ x_j|_Y, \frac{\partial x_j}{\partial t}|_Y, \frac{\partial^2 x_j}{\partial t^2}|_Y, \dots, \frac{\partial^{k-1} x_j}{\partial t^{k-1}}|_Y\Big).$$

The integral $\int_{X_j} x_j$ refers to integrating the function part of x_j.

We choose the $\tilde{U}_j$ so that

$$r_j(\tilde{U}_j) = \mathbb{R} \times \prod_{m=0}^{k-1} Z_\nu^\sigma. \tag{15}$$

Once this is true, we let

$$\tilde{U} = \tilde{U}_1 \times_{(\mathbb{R} \times \prod_{m=0}^{k-1} Z_\nu^\sigma)} \tilde{U}_2.$$

Here the fiber product being taken with respect to the maps r_1 and r_2.

Given an increasing sequence of $\mu, -\lambda \to \infty$, we can choose the $\tilde{U}_j$'s as above to form a nested sequence which becomes sufficiently large. Note that this automatically implies that the sequence made out of their fiber products $\tilde{U}$ becomes sufficiently large on X.

4.9. **Rewriting the map (13).** Set

$$Q_j = \{x_j \in L_{k+1}^2(i\Omega^1(X_j) \oplus \Gamma(W_j^+)); (d^+ \oplus \not{\partial}_j \oplus d^*)x_j \in \tilde{U}_j\}.$$

Then the domain Q of the map (13) :

$$\begin{array}{rcl} Q & \to & (\tilde{U}_1 \times \tilde{U}_2)/\mathbb{R} \\ (x_1, x_2) & \to & \big(\mathrm{pr}_{U_1} s(x_1), \mathrm{pr}_{U_2} s(x_2)\big) \end{array}$$

can be expressed as

$$Q = \{(x_1, x_2) \in Q_1 \times Q_2 : i^*(x_1) = i^*(x_2)\}.$$

There is an orthogonal projection

$$\mathrm{pr}_{\tilde{U}} : (\tilde{U}_1 \times \tilde{U}_2)/\mathbb{R} \to \tilde{U} = \tilde{U}_1 \times_{\mathbb{R} \times \prod_{m=0}^{k-1} Z_\nu^\sigma} \tilde{U}_2.$$

Hence we can replace (13) by

$$\begin{array}{rcl} Q & \to & \tilde{U} \times \prod_{m=0}^{k-1} Z_\nu^\sigma \\ (x_1, x_2) & \to & \big(\mathrm{pr}_{\tilde{U}}(s(x_1), s(x_2)), i^*(s(x_1)^{(m)} - s(x_2)^{(m)})\big). \end{array} \tag{16}$$

4.10. **Changing the map (6).** In view of Lemma 3, the domain $\tilde{U}'$ of the map (6):

$$\tilde{U}' \to \tilde{U}; \;\; x \to \mathrm{pr}_{\tilde{U}} s(x)$$

can be expressed as

$$\tilde{U}' = \{(x_1, x_2) \in Q_1 \times Q_2 : i^*(x_1^{(m)}) = i^*(x_2^{(m)}), m = 0, \dots, k\}.$$

Lemma 4. *If $(x_1, x_2) \in Q$, then automatically $i^*(x_1^{(m)}) - i^*(x_2^{(m)}) \in Z_\nu^\sigma$ for $m = 0, \dots, k$.*

Proof. From the definition of Q, we have $(d^+ \oplus \not{\partial}_j \oplus d^*)x_j \in \tilde{U}_j$. In light of (15), this implies that $i^*\big(x_j^{(m+1)} + Dx_j^{(m)}\big) \in Z_\nu^\sigma$ for $m = 0, \dots, k$. Starting from the fact that $i^*(x_1) - i^*(x_2) = 0 \in Z_\nu^\sigma$, the statement follows easily by induction on m. We are using the implication $D(y) \in Z_\nu^\sigma \Rightarrow y \in Z_\nu^\sigma$. $\square$

Now we can write $\tilde{U}'$ as the kernel of the linear map

$$\begin{array}{rcl} Q & \to & \displaystyle\prod_{m=0}^{k-1} Z_\nu^\sigma \\ (x_1, x_2) & \to & i^*(x_1^{(m+1)}) - i^*(x_2^{(m+1)}). \end{array}$$

This map is surjective provided that we chose the $\tilde{U}_j$'s sufficiently large. Using Observation 1 again, (6) is stably c-homotopic to the map:

$$\begin{array}{rcl} Q & \to & \tilde{U} \times \displaystyle\prod_{m=0}^{k-1} Z_\nu^\sigma \\ (x_1, x_2) & \to & \big(\mathrm{pr}_{\tilde{U}}(s(x_1), s(x_2)), i^*(x_1^{(m+1)}) - i^*(x_2^{(m+1)})\big). \end{array} \tag{17}$$

4.11. **The final homotopy.** We complete the proof of Theorem 4 by exhibiting a homotopy between the maps (16) and (17). We simply choose the linear homotopy:

$$\begin{array}{rcl} Q & \to & \tilde{U} \times \displaystyle\prod_{m=0}^{k-1} Z_\nu^\sigma \\ (x_1, x_2) & \to & \big(\mathrm{pr}_{\tilde{U}}(s(x_1), s(x_2)), (1-t)(i^*(s(x_1)^{(m)} - s(x_2)^{(m)})) \\ & & +t(i^*(x_1^{(m+1)}) - i^*(x_2^{(m+1)}))\big). \end{array}$$

Again, we have to show that we have stably compact zero sets throughout the homotopy. In the limit this boils down to proving the compactness of the space of pairs (x_1, x_2) satisfying:

$$\mathrm{pr}_{L^2_k(X)}(s(x_1), s(x_2)) = 0;$$

$$\Big(tx_1^{(m+1)} + (1-t)s(x_1)^{(m)}\Big)|_Y = \Big(tx_2^{(m+1)} + (1-t)s(x_2)^{(m)}\Big)|_Y;$$

$$x_1|_Y = x_2|_Y.$$

Since on the boundary $\partial X_j = Y$ we have $s(x_j) = x_j' + f(x_j)$ for some function f, and $x_1 = x_2$ on Y, by induction on m we have that $x_1^{(m)} = x_2^{(m)}$ on Y for all $m \leq k$, which means that (x_1, x_2) give an element of $L^2_{k+1}(X)$. This implies that $(s(x_1), s(x_2))$ is in $L^2_k(X)$, so (x_1, x_2) satisfies the Seiberg-Witten equations on X. Thus, in fact the zero set does not change throughout the homotopy, and it is the Seiberg-Witten moduli space on X.

5. Proof of Theorem 2

Here we discuss the generalization of Theorem 4 to the gluing of cobordisms. Let X_1 be a cobordism between Y_1 and Y_2, and X_2 a cobordism between Y_2 and Y_3, where the $b_1(Y_j) = 0$. We call X the composite cobordism between Y_1 and Y_2. We want to show that the morphism $\mathcal{D}_X$ is the composition of $\mathcal{D}_{X_2}$ with the relevant suspension of $\mathcal{D}_{X_1}$.

Let us make some simplifications in notation. Let $\mathcal{B}'_j$ and $\mathcal{B}_j$ be the configuration spaces $i\Omega^1_g \oplus \Gamma(W^+)$ and $i\Omega^2_+ \oplus \Gamma(W^-)$, respectively, for the 4-manifolds X_j. When we talk about the analogous spaces for X, we just erase the subscript. Also, V_j will stand for the configuration space $i \ker d^* \oplus \Gamma(W)$ on the 3-manifold Y_j, while $(V_j)^\mu_\lambda$ will be the corresponding finite dimensional approximations. Finally, i^*_j denotes the inclusion of Y_j into one of the 4-manifolds which it bounds.

Then $\mathcal{D}_{X_1}$ is given by a map:

$$SW_{U_1} : B(U'_1, R)/S(U'_1, R) \to \Big(B(U_1, \epsilon)/S(U_1, \epsilon)\Big) \wedge (N_1/L_1) \wedge (N_2/L_2)$$

coming from the finite dimensional approximation of:

$$SW_1 = (sw, p_\lambda i^*_1, p^\mu i^*_2) : \mathcal{B}'_1 \to \mathcal{B}_1 \oplus (V_1)_\lambda \oplus (V_2)^\mu$$

by choosing some linear susbspace $U_1 \subset \mathcal{B}_1$, a preimage U'_1 under the linearized operator, and suitable index pairs (N_j, L_j) for the Conley indices of the flows on $(V_j)^\mu_\lambda$, $j = 1, 2$. Note that this gives an element in a stable homotopy group of $\mathrm{SWF}(-Y_1) \wedge \mathrm{SWF}(Y_2)$. Via the duality map on the first factor, in the stable homotopy category this is equivalent to giving a morphism $\mathcal{D}_{X_1}$ between $\mathrm{SWF}(Y_1)$ and the relevant suspension of $\mathrm{SWF}(Y_2)$.

Similarly, $\mathcal{D}_{X_1}$ is a map

$$SW_{U_2} : B(U'_2, R)/S(U'_2, R) \to \Big(B(U_2, \epsilon)/S(U_2, \epsilon)\Big) \wedge (N_2/\bar{L}_2) \wedge (N_3/L_3)$$

coming from the finite dimensional approximation of:

$$SW_2 = (sw, p_\lambda i^*_2, p^\mu i^*_3) : \mathcal{B}'_2 \to \mathcal{B}_2 \oplus (V_2)_\lambda \oplus (V_3)^\mu.$$

Here the index pair $(N_2, \bar{L}_2)$ is chosen so that $\partial N_2 = L_2 \cup \bar{L}_2$ and $\partial L_2 = \partial \bar{L}_2 = L_2 \cap \bar{L}_2$.

To get the composition $\mathcal{D}_{X_2} \circ \Sigma^* \mathcal{D}_{X_1}$, we smash SW_{U_1} and SW_{U_2} and then apply the duality map η_2 on the $(N_2/L_2) \wedge (N_2/\bar{L}_2)$ factor. As in Section 3, we would like to replace this duality map by its linear version, i.e. taking the difference in $(V_2)^\mu_\lambda$ when the two elements are within ϵ distance of each other. To do this, we need an analogue of Proposition 2.

In the case of closed X, Proposition 2 is based on the compactness of the Seiberg-Witten moduli space on X. When X has boundary, the best we can hope is that when we add a cylindrical end to each component of the boundary, the monopoles with finite energy on the resulting manifold are bounded in a suitable norm. This corresponds to the fact that the restrictions of the monopoles on X to the boundary $Y = Y_1 \cup Y_3$ can be used to construct a map to the Conley index of the Seiberg-Witten flow φ on

$$V^\mu_\lambda = (V_1)^\mu_\lambda \oplus (V_3)^\mu_\lambda.$$

Recall that the Conley index is the quotient $N/L = (N_1/L_1) \wedge (N_3/L_3)$ for an index pair (N, L) considered with respect to the isolated invariant set $S \subset V^\mu_\lambda$.

Let us make the following

Definition 1. *A pair of compact sets (K_1, K_2), $K_2 \subset K_1 \subset V^\mu_\lambda$, is called a* **pre-index pair** *if for $R \gg 0$ the following are satisfied:*

1. K_1 is contained in the closed ball $B = \overline{B(2R)}$ in V^μ_λ taken in the $L^2_{k+1/2}$ norm, and R is large enough to work in Proposition 1;

2. If $x \in K_1$ satisfies $\varphi_t(x) \in B$ for all $t > 0$, then $\varphi_t(x) \notin \partial B$ for any $t > 0$;
3. For every $x \in K_2$ there exists $t \geq 0$ such that $\varphi_t(x) \notin B$.

According to Theorem 4 in [10], given a pre-index pair (K_1, K_2), one can find an actual index pair (N, L) for S with $K_1 \subset N$ and $K_2 \subset L$.

For $R_0, C > 0$, we let K_1 be the set of pairs $(p^\mu_\lambda \Pi i_1^*(x_1), p^\mu_\lambda \Pi i_3^*(x_2)) \in V^\mu_\lambda$ coming from $x_j \in U'_j (j = 1, 2)$ which satisfy:

$$\|x_j\|_{L^2_{k+1}} < R_0; \|\mathrm{pr}_{U_j} sw(x_j)\|_{L^2_k} < \epsilon;$$

$$\|p^\mu_\lambda \Pi i_2^*(x_1) - p^\mu_\lambda \Pi i_2^*(x_2)\|_{L^2_{k+1/2}} < 5\epsilon,$$

and let K_2 be the subset of K_2 of pairs coming from $x_j \in U'_j$ which also satisfy:

$$\|p^\mu_\lambda \Pi i_2^*(x_j)\|_{L^2_{k+1/2}} \geq C$$

for $j = 1$ or 2.

The analogue of Proposition 2 is then:

Proposition 3. *There exists a constant $C > 0$ such that, given any $R_0 \gg C$, for every $U_1, U_2, \mu, -\lambda$ sufficiently large and for every ϵ sufficiently small, (K_1, K_2) is a pre-index pair.*

The proof is similar to that of Proposition 2, so we omit the details.

Next, instead of comparing maps with stably compact zero sets via stable c-homotopies as we did in Section 4, we need to introduce the following notion:

Definition 2. *Given two finite dimensional vector spaces E, E' with a chosen orientation for $E \oplus E'$, a map*

$$f = (g, h) : E' \to E \times V^\mu_\lambda$$

is called **of Conley type** *if*

$$K_1 = h\big(g^{-1}(B(E, \epsilon)) \cap B(E', r)\big)$$

and

$$K_2 = h\big(g^{-1}(B(E, \epsilon)) \cap \partial B(E', r)\big)$$

form a pre-index pair, for $\epsilon > 0$ small and any $r \gg 0$.

This is the analogue of maps with compact zero sets. In practice, our maps come as finite dimensional approximations to some map between Hilbert spaces, and they are only **stably of Conley type**. This means that, rather than (K_1, K_2) being a pre-index pair for every $r \gg 0$, what happens is that for every fixed $r \gg 0$, when E, E' are large enough approximations, (K_1, K_2) is a pre-index pair for that r.

Given a map f that is stably of Conley type, we can find an index pair (N, L) with $K_1 \subset N, K_2 \subset L$ and construct an element in the stable homotopy groups of the Conley index N/L :

$$\tilde{f} : B(E', r)/\partial B(E', r) \to B(E, \epsilon) \times N/(S(E, \epsilon) \times N \cup B(E, \epsilon) \times L).$$

We say that two maps f_1, f_2 are **stably Conley c-homotopic** if the corresponding maps $\tilde{f}_1, \tilde{f}_2$ are stably homotopic.

In this language, we can rephrase Theorem 2 by taking into account the result of Proposition 3:

Proposition 4. *The maps*

$$\begin{array}{rcl} U_1' \times U_2' & \to & U_1 \times U_2 \times (V_2)^\mu_\lambda \times V^\mu_\lambda \\ (x_1, x_2) & \to & \big(pr_{U_1} sw(x_1), pr_{U_2} sw(x_2), p^\mu_\lambda \Pi i_2^*(x_1 - x_2), \\ & & \qquad p^\mu_\lambda \Pi(i_1^*(x_1), i_3^*(x_2))\big) \end{array}$$

and

$$U' \to U \times V^\mu_\lambda; \quad x \to \big(pr_U sw(x), p^\mu_\lambda \Pi(i_1^*(x_1), i_3^*(x_2))\big)$$

are stably Conley c-homotopic.

For the proof, all the arguments in Section 4 carry over, with stable c-homotopies being replaced by stable Conley c-homotopies.

6. Computations in Morse homotopy

In order to be able to make use of Theorem 1, we should first develop a technique for computing the Floer spectra SWF(Y) for various homology spheres Y. In this section we outline the general method for calculating Conley indices for gradient flows, and in the next one we use it to calculate SWF(Y) for some Seifert fibrations.

6.1. **Morse homotopy in finite dimensions.** Let us briefly review the decomposition of the Conley index for a Morse-Smale downward gradient flow φ on a finite dimensional manifold M. Our exposition is inspired from [3], [4], and [6].

We assume that M is stably parallelizable. (In fact, in our applications M will always be a vector space.) Let S be a compact isolated invariant subset of M consisting of finitely many critical points $x_1, \dots, x_n$ and flow trajectories between them. Let $E = \{x_1, \dots, x_n\}$. For each $x \in E$, we can define its stable and unstable manifolds:

$$W^u(x) = \{p \in S : \lim_{t \to -\infty} \varphi_t(p) = x\},$$

$$W^s(x) = \{p \in S : \lim_{t \to \infty} \varphi_t(p) = x\}.$$

The **Morse-Smale condition** says that each critical point $x \in E$ is nondegenerate of some Morse index $\mu(x) \in \mathbb{Z}$, and that $W^u(x)$ intersects $W^s(y)$ transversely in a $(\mu(x) - \mu(y))$-dimensional manifold M_{xy}, for $x, y \in E$. The manifold M_{xy} has a natural compactification $\overline{M}_{xy}$, given by adding broken trajectories going through critical points with index between $\mu(y)$ and $\mu(x)$. This $\overline{M}_{xy}$ is a manifold with corners.

Denote

$$E_n = \{x \in E : \mu(x) = n\}.$$

Then E_n is an isolated invariant set with an associated Conley index $I(E_n)$. This is a wedge of n-dimensional spheres, one for each $x \in E_n$.

Let S_n be the union of the unstable manifolds $W^u(x)$ for all $x \in E$ with $\mu(x) \leq n$. Each S_n is also an isolated invariant set, and has a certain Conley index $I(S_n)$. If m is the maximal value of all $\mu(x)$, then $S_m = S$.

Our goal is to reconstruct the stable homotopy type of $I(S)$ from the topology and the framing of the manifolds of flow lines between critical points. We do this for each $I(S_n)$, inductively on n.

We begin with $S_0 = E_0$, which is just a finite union of critical points. Its Conley index is a corresponding wedge of 0-spheres, i.e. a finite CW complex of dimension 0.

Next, S_0 is an attractor subset of S_1, in the sense that all points in a small neighborhood of S_0 in S_1 are taken to S_0 by the flow, asymptotically as $t \to \infty$. In fact, the only points in S_1 not taken to S_0 are those in E_1. E_1 is called the repeller set and there is a Puppe sequence in homotopy relating the Conley indices:

$$I(S_0) \to I(S_1) \to I(E_1) \to \Sigma I(S_0) \to \Sigma I(S_1) \to \cdots$$

This gives $I(S_1)$ the structure of a 1-dimensional CW complex, coming from attaching to $I(S_0)$ one 1-cell for each element of E_1. The homotopy type of the suspension $\Sigma I(S_1)$ is then that of the cone of the connecting map:

$$f : I(E_1) \to \Sigma I(S_0).$$

This is the suspension of the attaching map. It consists of a collection of pointed maps between 1-spheres $I(\{x\}) \to \Sigma I(\{y\})$ for $x \in E_1, y \in E_0$. The degree of such a map (with appropriate orientations) is the signed count of the gradient flow lines between x and y.

In general, to go from $I(S_{n-1})$ to $I(S_n)$ we use the attractor-repeller sequence:

$$I(S_{n-1}) \to I(S_n) \to I(E_n) \to \Sigma I(S_{n-1}) \to \Sigma I(S_n) \to \cdots$$

It follows that $I(S_n)$ has the structure of an n-dimensional CW complex. The connecting map is the wedge of suspensions of attaching maps:

$$f_x : S^n \to \Sigma I(S_{n-1}),$$

for each $x \in E_n$.

The simplest case is when $S_{n-1} = E_j = \{y\}$ for some $j < n-1$. Then f_x is an element in the $(n-j-1)$th stable homotopy group of spheres. Franks proved in [6] that, under the Pontrjagin-Thom construction, this element corresponds to the framed $(n-j-1)$ dimensional manifold of flow lines $M_{xy}/\mathbb{R}$. ([6]) The framing comes from trivializing the normal bundle of the contractible manifold $W^s(y) \subset M$, and then restricting this framing to M_{xy}, viewed as a submanifold of $W^u(x)$.

More generally, Cohen, Jones, and Segal have shown how the stable homotopy class of f_x can be recovered from the topology of the flow using a framed topological category. For more details of their construction, we refer the reader to [3]. In our examples, however, we will only need to consider attaching maps between cells of consecutive dimensions. In fact, we make the following definition:

Definition 3. *An isolated invariant set S as above is called* **simple** *if there is some integer p such that $E_n = \emptyset$ unless $p = n$ or $p = n+1$.*

Given a simple isolated invariant set, its Conley index is the cone of a map between the corresponding wedges of p- and $(p+1)$-spheres, and this map is determined by the count of gradient flow lines.

6.2. **Morse homotopy in infinite dimensions.** In practice, to compute the Seiberg-Witten Floer spectrum one starts with a description of the critical points and flow lines in infinite dimensions. Assuming the Morse-Smale condition, in good cases one can compute a stable homotopy type using the procedure outlined above. The Floer spectrum, however, is defined from the finite dimensional approximations, so we need to check that it gives back the same information.

To do this, we consider certain perturbations of the Seiberg-Witten flow which affect neither the possibility of finite dimensional approximations nor its result, the spectrum $\mathrm{SWF}(Y)$. The following definition is useful:

Definition 4. *Let E be a bundle over a finite dimensional manifold. We denote by $V = \Gamma(E)$ the space of smooth sections of E and by $L^2_k(V)$ its L^2_k Sobolev completion. A map $c : V \to V$ is called* **very compact** *if, for every $k > 3$, c extends to a compact map:*

$$c : L^2_{k+1}(V) \to L^2_k(V)$$

and its differentials extend to continuous maps:

$$\begin{aligned} L^2_{k+1}(V) \times L^2_{k-i_1}(V) \times \dots L^2_{k-i_j}(V) &\to L^2_{k-(i_1+\dots+i_j)}(V), \\ (x; v_1, \dots v_j) &\to (d^j c)_x(v_1, \dots, v_j). \end{aligned}$$

The prototype of a very compact map is a quadratic one, or the nonlinear part of the Seiberg-Witten map. The motivation for introducing this notion is that the properties listed above were the ones used in [10] to prove Proposition 1 for approximate Seiberg-Witten flow trajectories. More generally, the same method of proof gives the following:

Proposition 5. *Let E be a bundle over a finite dimensional manifold with $V = \Gamma(E)$, $l : V \to V$ a self-adjoint, linear, elliptic differential operator of order one, and $c : V \to V$ a very compact map. We can approximate V by finite dimensional subspaces V^μ_λ using the eigenspaces of l with eigenvalues between λ and μ. For $k > 3$, let N be a bounded, closed subset of $L^2_{k+1}(V)$ such that all flow trajectories $x : \mathbb{R} \to L^2_{k+1}(V)$,*

$$\frac{\partial}{\partial t} x(t) = -(l + c)x(t)$$

which lie inside N are in fact contained in $V \cap U$, for a fixed open subset $U \subset N$.

Then, for every $\mu, -\lambda \gg 0$, if an approximate trajectory $x : \mathbb{R} \to L^2_{k+1}(V^\mu_\lambda)$,

$$\frac{\partial}{\partial t} x(t) = -(l + p^\mu_\lambda c)x(t),$$

is contained in N, it must be contained in U.

In the situation described in the proposition, we can define a Conley index $I^\mu_\lambda = I(S^\mu_\lambda)$ for the set S^μ_λ of approximate flow trajectories contained in $U \cap V^\mu_\lambda$. The suspension spectrum

$$I = \Sigma^{-V^0_\lambda} I^\mu_\lambda$$

is then independent of λ and μ, up to canonical equivalence. (See [10] for the model proof in the Seiberg-Witten case.)

On the other hand, we can also look at the original flow on $N \subset L^2_{k+1}(V)$ and at the set S of flow trajectories inside N. Assuming that this is a gradient flow, we say that its fixed points are nondegenerate if the Hessian of the respective functional is nondegenerate at those points. We can define the stable and unstable Hilbert manifolds of critical points as before, and say that the flow is Morse-Smale if they all intersect transversely in finite dimensional manifolds. In this situation, we can define a relative Morse index of critical points as in Floer theory. In principle, we should be able to calculate a stable homotopy type from the topology of spaces of flow lines in S as in Subsection 6.1. However, defining suitable framings for these spaces is a nontrivial problem. For this reason, we limit our discussion to simple flows, when we have only critical points of two successive indices. We can then define a stable homotopy tyep as before, by taking two wedges of spheres and counting gradient flow lines. We denote this stable homotopy type by $I(S)$, and call it the **Morse stable homotopy type of** S. As it stands, it is only defined up to suspension.

The following proposition shows how to get hold of I when we only have information about the flow on V :

Proposition 6. *If the infinite dimensional flow on N is Morse-Smale and S is simple, then I and $I(S)$ represent the same stable homotopy type, up to suspension.*

Proof. The idea is to deform the flow on N to an equivalent one on a finite dimensional approximation, where we know that the Conley index computes Morse stable homotopy.

Let us warm up by explaining how to do this in a simple case, when S consists of just one critical point y, so that $I(S)$ is a sphere. For $\mu, -\lambda \gg 0$, let y^μ_λ be the orthogonal projection of y onto V^μ_λ. Set

$$v^\mu_\lambda = y - y^\mu_\lambda.$$

Note that $v^\mu_\lambda \to 0$ as $\mu, -\lambda \to \infty$. For some sufficiently large μ and λ, let us consider an open convex subset U' of V with $y \in U' \subset U$ and $y^\mu_\lambda \in U' \subset (U - v^\mu_\lambda)$, the latter being the translation of U by v^μ_λ.

If the original flow φ was given by $l + c$, consider now the flow $\tilde{\varphi}$:

$$\frac{\partial}{\partial t} x = -(l+c)(x(t) + v^\mu_\lambda) = -(l + \tilde{c})x(t),$$

where the map

$$\tilde{c}(x) = c(x + v^\mu_\lambda) + l(v^\mu_\lambda)$$

is easily seen to be very compact.

We can apply Proposition 5 to the flow $\tilde{\varphi}$ on $\tilde{N} = (N - v^\mu_\lambda)$. In the infinite dimensional space, this is just a translation by v^μ_λ of the flow φ. However, their finite dimensional approximations are different. What happens is that now $\tilde{y} = y^\mu_\lambda$ is a nondegenerate critical point of $\tilde{\varphi}$ which is contained in V^μ_λ. It follows that $\tilde{y}$ remains a nondegenerate critical point for the flow of $l + p^{\mu'}_{\lambda'}\tilde{c}$, for every $\mu' \geq \mu, \lambda' \leq \lambda$. Furthermore, for $\mu', -\lambda' \gg 0$, it is the only critical point of the approximate flow in U'. One can see this, for example, by applying Proposition 5 for $\tilde{N}$ and taking as open subset an arbitrarily small ball around $\tilde{y}$. Therefore, for such μ' and λ', the Conley index of $\tilde{S}^{\mu'}_{\lambda'} = \{\tilde{x}\} \subset U'$ is a sphere. Moreover, on U' we can connect φ and $\tilde{\varphi}$ by a homotopy of flows $t\varphi + (1-t)\tilde{\varphi}, t \in [0,1]$. The usual deformation argument in Conley index theory gives that $I(S^{\mu'}_{\lambda'}) = I(\tilde{S}^{\mu'}_{\lambda'})$ is also a sphere, which exactly matches $I(S)$.

In the general case, S is some finite dimensional stratified space. We seek to replace the flow φ with a flow $\tilde{\varphi}$ of the form:

$$\frac{\partial}{\partial t} x = -(l+c)(x(t) + h(x(t))) = -(l + \tilde{c})x(t),$$

where $\tilde{c}$ is a very compact map of the form

$$\tilde{c}(x) = c(x + h(x)) + l(h(x)).$$

We would like the flow φ in a neighborhood of S to look the same as the flow $\tilde{\varphi}$ in a neighborhood of an isolated invariant set $\tilde{S}$ which is contained in some finite dimensional approximation V^μ_λ. It could happen that for any $\mu, -\lambda \gg 0$, the orthogonal projection of S onto V^μ_λ is not diffeomorphic to S, so we cannot simply set $\tilde{S} = p^\mu_\lambda(S)$. However, S is finite dimensional, so a simple transversality argument shows that, for μ and $-\lambda$ sufficiently large, we can find a linear subspace W with the following properties: V^μ_λ is the image of V under some orthogonal transformation A close to the identity, and the orthogonal projection $\mathrm{pr}_W : S \to \hat{S} \subset W$ is a diffeomorphism. Let $\tilde{S} \subset V^\mu_\lambda$ be the image of $\hat{S}$ under A. We can arrange so that the segments joining each $x \in S$ to $(A \circ \mathrm{pr}_W)(x) \in \tilde{S}$ form an interval bundle B over S, smoothly embedded in $L^2_{k+1}(V)$.

Take a small tubular neighborhood T of S in $L^2_{k+1}(V)$, in the form of a disc bundle $\pi : T \to S$ such that B is its subbundle. A fiber of π is a ball in a finite codimension linear subspace of $L^2_{k+1}(V)$. Note that $\tilde{S} \subset T$ and $(\pi \circ A \circ \mathrm{pr}_W)(x) = x$ for all $x \in S$. We can define a smooth "bump" function $\beta : L^2_{k+1}(V) \to [0,1]$ such that $\beta \equiv 1$ in a smaller neighborhood T' of B, $T' \subset T$, and $\beta \equiv 0$ outside T. Then we define the functions

$$\gamma : L^2_{k+1}(V) \to R = (S \times [0,1])/(S \times \{0\})$$

by

$$\gamma(x) = \begin{cases} (\pi(x), \beta(x)) & \text{if } x \in T; \\ * & \text{otherwise} \end{cases}$$

and

$$\tau : R \to L^2_{k+1}(V), \quad \tau(x,r) = r \cdot \big(x - (A \circ \mathrm{pr}_W)(x)\big).$$

Define h to be the composition

$$h = \tau \circ \gamma : L^2_{k+1}(V) \to L^2_{k+1}(V).$$

With this definition of h, the very compactness of the map $\tilde{c}$ is a consequence of the very compactness of c, together with the fact that h (and its derivatives) factor through maps to the finite dimensional space R.

Furthermore, up to suspension, $I(S)$ for the flow φ is the same as the Conley index $I(\tilde{S})$ for the flow

$$(\partial/\partial t)x(t) = -(l + p^{\mu'}_{\lambda'}\tilde{c})x(t)$$

approximating $\tilde{\varphi}$ on $V^{\mu'}_{\lambda'}$ for $\mu' \geq \mu, \lambda' \leq \lambda$. (The flows $\varphi, \tilde{v}p$ look the same in small neighborhoods of S and $\tilde{S}$, respectively.) Note that $\tilde{S}$ is the invariant part of $T' \cap V^{\mu'}_{\lambda'}$ in the flow generated by $l + p^{\mu'}_{\lambda'}\tilde{c}$. By the continuation argument, its Conley index is the same as that of the invariant part $S^{\mu'}_{\lambda'}$ of $T' \cap V^{\mu'}_{\lambda'}$ in the flow

$$(\partial/\partial t)x(t) = -(l + p^{\mu'}_{\lambda'}c)x(t).$$

The latter represents the spectrum I, and this completes the proof. □

Three remarks are in place. First, the result above should be valid without the hypothesis that S is simple, provided that one could define a good framed topological category in the infinite dimensional situation, in the spirit of [3]. Second, the result can be easily extended to G-equivariant Morse-Bott-Smale flows when G is a compact Lie group, under the hypothesis that the stable homotopy type $I(S)$ can be computed from counting gradient flow lines. (We will see such examples in the next section.) Third, sometimes we might not have enough information about the flow on V to compute the whole Morse stable homotopy type. However, we might have some partial information, e.g. the knowledge of the number of critical points and the number of flow lines between points of consecutive indices, which gives Morse-Floer homology. It may also happen that the original flow itself is not Morse-Smale. Nevertheless, the principle of Proposition 6 remains true: whatever information we get out of the infinite-dimensional flow is the same as the information coming from the Conley indices on the finite dimensional approximations.

7. Examples

Here we apply the results in the previous section to compute explicitely some Floer spectra. Let Y be a rational homology 3-sphere and $\mathfrak{c}$ a *spin*^{c} structure on it. (In the case when $H_1(Y;\mathbb{Z}) = 0$, we drop $\mathfrak{c}$ from the notation, since it is uniquely determined.) Provided we have a description of the Seiberg-Witten flow lines on Y, we can apply Proposition 6 to calculate the Floer stable homotopy type $\mathrm{SWF}(Y, \mathfrak{c})$.

The Seiberg-Witten flow is equivariant with respect to the action of the group $\mathbb{T} = S^1$, acting trivially on forms and by rotations on spinors. This action is semifree, meaning that there are only free and trivial orbits. When suspending, we only consider the representations $\mathbb{R}$ and $\mathbb{C}$ of $\mathbb{T}$. Correspondingly, there are two type of "critical points": reducible ones (on which the action is trivial) and irreducible ones (on which the action is free).

In fact, since $b_1(Y) = 0$ there is only one reducible θ. In [10] it is explained how the reducible can be given an absolute Morse index equal to $-2n(Y, \mathfrak{c}, g)$, where $n(Y, \mathfrak{c}, g)$ is a combination of the Dirac and signature eta invariants for the Riemannian metric which we use on Y. If X bounds a 4-manifold with boundary X, and X has a *spin*^{c} structure with determinant line bundle $\hat{L}$, we can express $n(Y, \mathfrak{c}, g)$ as:

$$n(Y, \mathfrak{c}, g) = \mathrm{ind}_{\mathbb{C}}(D_A^+) - \frac{c_1(\hat{L})^2 - \sigma(X)}{8} \in \frac{1}{8N}\mathbb{Z},$$

where N is the cardinality of $H_1(Y;\mathbb{Z})$.

Thus, the Conley index of the reducible is a sphere

$$S^{-n(Y,\mathfrak{c},g)\mathbb{C}} = \big(\mathbb{C}^{-n(Y,\mathfrak{c},g)}\big)^+.$$

Note that we allow for negative and even rational indices. The choice of an absolute grading for the reducible induces one for the irreducibles too.

Their Conley indices are free cells of the form $\Sigma^m(\mathbb{T}_+)$. Note that we use the notation $\mathbb{T}$ for a circle with free $\mathbb{T}$ action and S^1 for one with trivial $\mathbb{T}$ action; $\mathbb{T}_+$ stands for $\mathbb{T}$ together with a disjoint basepoint. We should also point out that for free cells suspension by $\mathbb{R}^2$ is the same as suspension by $\mathbb{C}$, so the notation Σ^m is unambiguous.

Finally, note that when we change the orientation on Y, $\mathrm{SWF}(-Y, \mathfrak{c})$ is the spectrum dual to $\mathrm{SWF}(Y, \mathfrak{c})$. Morse theoretically, all the flow lines go in the reverse direction, and the index of the reducible switches sign. For the free cells we can use the Wirthmüller isomorphism $D(\mathbb{T}_+) = \Sigma^{-1}(\mathbb{T}_+)$, which implies $D(\Sigma^m(\mathbb{T}_+)) = \Sigma^{-m-1}(\mathbb{T}_+)$.

7.1. **Elliptic 3-manifolds.** The simplest case is that when Y is a quotient of S^3 by some finite group. Then Y admits a metric of positive scalar curvature g, and therefore the only critical point is the reducible. The only thing left to compute is $n(Y, \mathfrak{c}, g)$ for the metric g.

In [10], we have done this for S^3 and for the Poincaré sphere $P = \Sigma(2, 3, 5)$:

$$\mathrm{SWF}(S^3) = S^0; \;\; \mathrm{SWF}(\Sigma(2, 3, 5)) = \mathbb{C}^+.$$

When Y is a lens space $L(n, 1)$, the invariants $n(Y, \mathfrak{c}, g)$ have been computed by Nicolaescu in [14]. We think of Y as the S^1 bundle over S^2 of degree $-n$ for $n \geq 1$. (If we change the orientation we obtain the bundle of degree n.) Observe that Y bounds a disk bundle $D(-n) \to S^2$. The *spin*^{c} structures on $D(-n)$ are denoted $\hat{\mathfrak{c}}_j, j \in \mathbb{Z}$, so that $c_1(\det(\hat{\mathfrak{c}}_j)) = -n + 2j \in \mathbb{Z} \cong H^2(D(-n);\mathbb{Z})$.

Since $H^2(Y;\mathbb{Z}) = \mathbb{Z}/n$, there are n different *spin*^{c} structures on Y, denoted $\mathfrak{c}_0, \dots, \mathfrak{c}_{n-1}$, such that $\mathfrak{c}_k$ is the restriction of $\hat{\mathfrak{c}}_j$ to the boundary for every $j \equiv k \bmod n$. We have $c_1(\det \mathfrak{c}_k) \equiv 2k \bmod n$. Then

$$\mathrm{SWF}(Y, \mathfrak{c}_k) = S^{-n_k\mathbb{C}},$$

where

$$n_k = n(Y, \mathfrak{c}_k, g) = \frac{(n-2k)^2 - n}{8n}.$$

7.2. **Some Brieskorn spheres.** Given $n \geq 1$, we denote by $Y = \Sigma(2,3,6n \pm 1)$ the Brieskorn sphere oriented as the boundary of the complex singularity $z_1^2 + z_2^3 + z_3^{6n\pm 1} = 0$. In this way, for example, $-\Sigma(2,3,6n-1)$ can be described as the oriented boundary of the nucleus of the elliptic surface $E(n)$, as well as the result of $1/n$ surgery on the right-handed trefoil. On the other hand, $-1/n$ surgery on the right-handed trefoil gives $\Sigma(2,3,6n+1)$.

We use the description of the Seiberg-Witten flow lines on Y given by Mrowka, Ozsváth, and Yu in [12]. This description uses a specific metric and a certain reducible connection on Y instead of the usual Levi-Civita connection. However, by the continuation properties of the Conley index, the invariant SWF defined this way is the same as the usual one. With this choice of metric and connections, all the critical points are nondegenerate, so the Seiberg-Witten flow is Morse-Bott. It is usually not Morse-Bott-Smale, and this makes the application of Proposition 6 more difficult. Nevertheless, we can still compute the Floer spectra.

The computation of the absolute index of the reducible in the examples below was done by Nicolaescu in [13]. We present the case of $-Y$ rather than Y so that the reader can easily compare the Borel homology of SWF to the similar computations of HF^+ in Ozsváth-Szabó theory ([16], [17]).

r=12j-1: Let us do the case $j=1$ first, which will be useful to us later. Then the Seiberg-Witten flow on $-Y$ is Morse-Bott-Smale, with the reducible of index 0 and two irreducibles of index -2. Modulo the action of $\mathbb{T}$, there is exactly one flow line from the reducible to each irreducible. It follows that there is an exact triangle in stable homotopy:

$$\cdots \to S^{-1} \to \Sigma^{-2}(\mathbb{T}_+) \vee \Sigma^{-2}(\mathbb{T}_+) \to \mathrm{SWF}(-Y) \to S^0 \to \cdots \tag{18}$$

The connecting morphism, corresponding to the attaching map from the trivial 2-cell to the free cells, can be thought of as an element in a stable cohomotopy group:

$$\{S^{-1}, \Sigma^{-2}(\mathbb{T}_+ \vee \mathbb{T}_+)\}^{\mathbb{T}} = \{\Sigma^2 D(\mathbb{T}_+ \vee \mathbb{T}_+), S^1\}^{\mathbb{T}} =$$
$$= \pi^0_{\mathbb{T}}(\mathbb{T}_+ \vee \mathbb{T}_+) = \pi^0(S^0) \oplus \pi^0(S^0) = \mathbb{Z} \oplus \mathbb{Z}.$$

Given the count of flow lines, this morphism must be $(\pm 1, \pm 1)$. The signs depend on the (noncanonical) identification of the Conley indices of the critical points with the standard cells $S^0, \Sigma^{-2}\mathbb{T}_+$. It is not important to fix a convention at this point.

Nonequivariantly, we can identify $\mathbb{T}_+$ with $S^1 \vee S^0$ by choosing a basepoint on $\mathbb{T}$. The triangle (18) becomes:

$$\cdots \to S^{-1} \to S^{-1} \vee S^{-2} \vee S^{-1} \vee S^{-2} \to \mathrm{SWF}(-Y) \to S^0 \to \cdots.$$

This shows that, nonequivariantly, there is an isomorphism:

$$\mathrm{SWF}(-\Sigma(2,3,11)) = S^{-2} \vee S^{-2} \vee S^{-1}.$$

More generally, for $j > 1$, there is one reducible of index 0 and $2j$ irreducibles of index -2. Again, there is one flow line from the reducible to each irreducible. However, there are also flow lines between different irreducibles, so the flow is not Morse-Bott-Smale. It turns out that we do not need to understand the topology of the spaces of flow lines in this case. Indeed, we can form attractor-repeller pairs by splitting along the levels of the CSD functional. Each irreducible contributes $\Sigma^{-2}(\mathbb{T}_+)$, and whenever we attach two of them together we do so by a morphism:

$$\Sigma^{-2}(\mathbb{T}_+) \to \Sigma^{-1}(\mathbb{T}_+).$$

All these morphisms must be trivial, which proves that all the irreducibles contribute with a wedge of the respective cells. It follows that for $Y = \Sigma(2,3,12j-1)$, the Floer spectrum admits a presentation:

$$\cdots \to S^{-1} \to \bigvee_{i=1}^{2j} \Sigma^{-2}(\mathbb{T}_+) \to \mathrm{SWF}(-Y) \to S^0 \to \cdots$$

with the connecting morphism being $(\pm 1, \dots, \pm 1) \in \mathbb{Z}^{2j}$.

r=12j-5: This case is similar to the previous one, except for a shift in index: there is one reducible of complex index 2 and $2j$ irreducibles of index 0. We get:

$$\mathrm{SWF}(-\Sigma(2,3,12j-5)) = \Sigma^{\mathbb{C}}\mathrm{SWF}(-\Sigma(2,3,12j-1)).$$

r=12j+1: There is one reducible and $2j$ irreducibles, all of index 0. The flow is not Morse-Smale-Bott even for $j=1$, because there are flow lines from the reducible to the irreducibles. The Chern-Simons-Dirac functional CSD takes a bigger value on the reducible than on any irreducible, so we can find an attractor-repeller exact triangle by splitting along a level set of CSD :

$$\cdots \to S^{-1} \to \bigvee_{i=1}^{2j} \mathbb{T}_+ \to \mathrm{SWF}(-Y) \to S^0 \to \cdots$$

The connecting morphism must be zero, simply because:

$$\pi^{\mathbb{T}}_{-1}(\mathbb{T}_+) = \pi^2_{\mathbb{T}}(\mathbb{T}_+) = \pi^2(S^0) = 0.$$

Therefore, we have even equivariantly:

$$\mathrm{SWF}(-\Sigma(2,3,12j+1)) = S^0 \vee \Big(\bigvee_{i=1}^{2j} \mathbb{T}_+\Big).$$

r=12j+5: This case is similar to the previous one, except all the critical points have index -2. We get:

$$\mathrm{SWF}(-\Sigma(2,3,12j+5)) = \Sigma^{-\mathbb{C}}\mathrm{SWF}(-\Sigma(2,3,12j+1)).$$

8. Applications

8.1. **Adjunction inequalities.** We start with some applications of Theorem 1 to gluings along lens spaces. The two propositions proved in this section are not new (they appear in [7]), but it is interesting to see how they can be obtained with our techniques.

Consider now a closed, orientable 4-manifold X with $\pi_1(X) = 1$. We can identify the set of *spin*^{c} structures $\hat{\mathfrak{c}}$ on X with the set of characteristic elements $c \in H^2(X;\mathbb{Z})$ via the correspondence $\hat{\mathfrak{c}} \to c = c_1(\hat{L})$.

Recall that an element $c \in H^2(X;\mathbb{Z})$ is called a **basic class** if the Seiberg-Witten invariant $\mathrm{SW}(X, \hat{\mathfrak{c}}) \neq 0$. The manifold X is called **of simple type** if all basic calsses satisfy $c^2 = 3\sigma(X) + 2\chi(X)$, or equivalently, if $\mathrm{SW} = 0$ whenever the formal dimension of the monopole moduli space is nonzero.

It turns out that the presence of embedded surfaces in specific homology classes of our 4-manifold X imposes some restrictions on the set of basic classes. These restrictions come from the so-called "adjunction formulae." The most general version of these formulae appears in [15]. Our tools are insufficient for proving these results in the general case, but they allow us to find the constraints imposed by the existence of embedded *spheres.*

Let $\Sigma \subset X$ be an embedded sphere. We study the case $[\Sigma]^2 = N > 0$ first. In this situation a neighborhood of Σ is the disc bundle $D(N)$ over S^2 with boundary the lens space

$L(N, N-1)$, which is $L(N,1)$ with the opposite orientation. This gives a decomposition of X into two pieces X' and $D(N)$, glued along their common boundary $L(N, N-1)$. We have $b_2^+(D(N)) = 1$ and $\mathrm{SWF}(L(N,1), \mathfrak{c}) \cong S^{-n\mathbb{C}}$, where n depends on $\mathfrak{c}$. Hence the relative invariant $\Psi(D(N))$ lives in $\pi^1_{\mathbb{T}}(S^{d\mathbb{C}})$ for some $d \in \mathbb{Z}$. These groups are torsion for every d. Since $\Psi(D(N))$ is torsion, so is $\Psi(X)$ by virtue of Theorem 1, so the Seiberg-Witten invariant is zero. We conclude:

Proposition 7. *Let X be a smooth, closed, oriented, simply connected 4-manifold with $b_2^+(X) > 1$. If there exists an embedded sphere $\Sigma \subset X$ with $[\Sigma]^2 > 0$, then X has no basic classes.*

Let us now consider the case $[\Sigma]^2 = -N < 0$, under the additional assumption that X is of simple type. Let c be a basic class of X, with corresponding *spin*^{c} structure $\hat{\mathfrak{c}}$. A neighborhood of Σ is the disc bundle $D(-N)$ over S^2 with boundary the lens space $L(N,1)$. This decomposes X into two pieces X' and $D(-N)$ and breaks $\hat{\mathfrak{c}}$ into *spin*^{c} structures $\hat{\mathfrak{c}}'$ and $\hat{\mathfrak{c}}_j$ on X' and $D(-N)$, respectively. We must have $-N + 2j = c([\Sigma])$. With the notations from Subsection 7.1, the induced *spin*^{c} structure on $L(N,1)$ is $\mathfrak{c}_k$, with $0 \le k < N-1, k \equiv j \bmod N$. Note that $b_2^+(D(-N)) = 0$, so the invariant $\Psi(D(-N), \hat{\mathfrak{c}}_j)$ lives in $\pi^0_{\mathbb{T}}(S^{i\mathbb{C}})$, where

$$i = \mathrm{ind}_{\mathbb{C}}(D^+_{\hat{A}}) = n_k + \frac{-(2j-N)^2/N + 1}{8}.$$

Recall that $n_k = ((N-2k)^2 - N)/8N$. It follows that:

$$i = \frac{(N-2k)^2 - (N-2j)^2}{8N}.$$

The following lemma appears in [2]:

Lemma 5. *Let $f : (\mathbb{R}^m \oplus \mathbb{C}^{n+d})^+ \to (\mathbb{R}^m \oplus \mathbb{C}^n)^+$ be a $\mathbb{T}$-equivariant map such that the induced map on the fixed point sets has degree 1. Then $d \le 0$ and f is $\mathbb{T}$-homotopic to the inclusion.*

Therefore, $\Psi(D(-N), \hat{\mathfrak{c}}_j)$ is the class of the inclusion. We claim that $i = 0$. Indeed, if $i \neq 0$ and $\mathrm{SW}(X, \hat{\mathfrak{c}}) \neq 0$, we could consider the *spin*^{c} structure $\hat{\mathfrak{c}}_{new}$ on X obtained from $\hat{\mathfrak{c}}'$ on X' and $\hat{\mathfrak{c}}_k$ (instead of $\hat{\mathfrak{c}}_j$) on $D(-N)$. By two applications of the gluing theorem we would get $\mathrm{SW}(X, \hat{\mathfrak{c}}_{new}) = \mathrm{SW}(X, \hat{\mathfrak{c}}) \neq 0$. But $c(\det(\hat{\mathfrak{c}}_{new}))^2 = c(\det(\hat{\mathfrak{c}}))^2 - i$, so X would not be of simple type.

Therefore, we must have $i = 0$, or $(N-2k) = \pm(N-2j)$. This happens if and only if $|N - 2j| \le N$. We deduce the following:

Proposition 8. *Let X be a smooth, closed, oriented, simply connected 4-manifold with $b_2^+(X) > 1$. If X is of simple type and there exists an embedded sphere $\Sigma \subset X$ with $[\Sigma]^2 = -N < 0$, then every basic class c of X satisfies $|c([\Sigma])| \le N$.*

8.2. **Exotic nuclei.** Here we present the proof of Theorem 3. The Bauer-Furuta invariants of $X = K3\#K3\#K3$ have been computed in [2]. They are zero except for the trivial *spin*^{c} structure $\mathfrak{c}_0$, when, nonequivariantly:

$$\Psi(X, \mathfrak{c}_0) = 12 \in \pi_3(S^0) = \mathbb{Z}/24.$$

Set $Y = -\Sigma(2,3,11)$, oriented now as the boundary of the nucleus $N(2) \subset E(2) = K3$. Assume that X decomposes as $X_1 \cup_Y X_2$, where X_1 is an exotic nucleus $N(2)_{p,q}$ with $(p,q) \neq (1,1)$, and X_2 has intersection form $6(-E_8) \oplus 8H$.

Recall from Section 7 that, nonequivariantly:

$$\mathrm{SWF}(-Y) = S^2 \vee S^2 \vee S^1.$$

Let $\mathfrak{c}'$ be the restriction of $\mathfrak{c}_0$ to X_2. Then the nonequivariant Bauer-Furuta invariant of $(X_2, \mathfrak{c}')$ lives in:

$$\pi_4(\mathrm{SWF}(-Y)) = \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/24.$$

Denote by z its projection to the $\mathbb{Z}/24$ factor.

Dually, we have that $\mathrm{SWF}(Y) = S^{-2} \vee S^{-2} \vee S^{-1}$ nonequivariantly. The Bauer-Furuta invariant of X_1 lives in:

$$\pi_{-1}(\mathrm{SWF}(Y)) = \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}.$$

We can get additional information by observing that it has to lie in the image of the equivariant group $G = \pi^{\mathbb{T}}_{-1}(\mathrm{SWF}(Y))$ under the natural forgetting map. To compute G, we look at the triangle (18):

$$\cdots \to S^{-1} \to \Sigma^{-2}(\mathbb{T}_+) \vee \Sigma^{-2}(\mathbb{T}_+) \to \mathrm{SWF}(Y) \to S^0 \to \cdots$$

Note that:

$$\pi^{\mathbb{T}}_{-1}(\Sigma^{-2}(\mathbb{T}_+)) = \pi^1_{\mathbb{T}}(D(\Sigma^{-2}(\mathbb{T}_+))) = \pi^0_{\mathbb{T}}(\mathbb{T}_+) = \mathbb{Z},$$

and the forgetting map to $\pi_{-1}(\Sigma^{-2}(\mathbb{T}_+)) = \mathbb{Z}/2 \oplus \mathbb{Z}$ is given by $1 \to (0,1)$.

Applying the functor $\pi^{\mathbb{T}}_{-1}$ to (18) we get a long exact sequence:

$$\cdots \to \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \to \pi^{\mathbb{T}}_{-1}(\mathrm{SWF}(Y)) \to 0$$

with the first map being the diagonal $1 \to (1,1)$.

Consequently, we find that $G = \pi^{\mathbb{T}}_{-1}(\mathrm{SWF}(Y)) = \mathbb{Z}$ and the forgetting map to $\mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}$ is given by $1 \to (0,0,1)$.

Let P be the primitive class in $H_2(N(2)_{p,q}; \mathbb{Z})$ such that $pqP = F$ is the class Poincaré dual to the elliptic fiber. For a *spin*^{c} structure $\mathfrak{c} = mP, m \in \mathbb{Z}$ on $X_1 = N(2)_{p,q}$, the Bauer-Furuta invariant

$$\Psi(X_1, \mathfrak{c}) \in G = \mathbb{Z}$$

is equal to the relative Seiberg-Witten invariant $SW(X_1, \mathfrak{c})$ as defined in [18]. This counts the difference in the number of monopoles on X_1 which restrict on the boundary to each of the two reducibles. The formal relative Seiberg-Witten series was computed by Stipsicz and Szabó to be:

$$SW(X_1) = \frac{\sinh^2(pqP)}{\sinh(pP) \cdot \sinh(qP)}.$$

Let

$$x_0 = SW(X_1, 0); \;\; x_1 = SW(X_1, (2pq - p - q)P) = 1.$$

Build the *spin*^{c} structure $\mathfrak{c}_1$ on X by gluing $\mathfrak{c}'$ to $(2pq - p - q)P$. Nonequivariantly, Theorem 1 says the Bauer-Furuta invariant of X with either $\mathfrak{c}_0$ and $\mathfrak{c}_1$ is obtained from those of X_1 and X_2 via the duality map:

$$\begin{aligned} \pi_{-1}(\mathrm{SWF}(Y)) \times \pi_4(\mathrm{SWF}(-Y)) &\to \pi_3(S^0) \\ (\mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}) \times (\mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/24) &\to \mathbb{Z}/24 \end{aligned}$$

given by:

$$(a_1, b_1, c_1) \times (a_2, b_2, c_2) \to \big(12(a_1 a_1' + b_1 b_1') + c_1 c_2\big) \bmod 24.$$

Therefore:

$$\Psi(X, \mathfrak{c}_0) = x_0 z \in \mathbb{Z}/24,$$

while

$$\Psi(X, \mathfrak{c}_1) = x_1 z = z \in \mathbb{Z}/24.$$

On the other hand, we knew from the beginning that $\Psi(X, \mathfrak{c}_0) = 12$, while $\Psi(X, \mathfrak{c}_1) = 0$. This is a contradiction. Hence X does not contain an exotic nucleus.

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