A CONCORDANCE INVARIANT FROM THE FLOER HOMOLOGY OF DOUBLE BRANCHED COVERS

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ABSTRACT. Ozsváth and Szabó defined an analog of the Frøyshov invariant in the form of a correction term for the grading in Heegaard Floer homology. Applying this to the double cover of the 3-sphere branched over a knot K, we obtain an invariant δ of knot concordance. We show that δ is determined by the signature for alternating knots and knots with up to nine crossings, and conjecture a similar relation for all H-thin knots. We also use δ to prove that for all knots K with $\tau(K) > 0$, the positive untwisted double of K is not smoothly slice.

1. INTRODUCTION

In [22], Ozsváth and Szabó associated an invariant $d(Y, \mathfrak{t}) \in \mathbb{Q}$ to every rational homology three-sphere Y endowed with a Spin^c structure. In this paper we study the knot invariant

$$\delta(K) = 2d(\Sigma(K), \mathfrak{t}_0),$$

where $\Sigma(K)$ is the double cover of S^3 branched over the knot K, and $\mathfrak{t}_0$ is the Spin^c structure induced by the unique Spin structure on $\Sigma(K)$. In [22] it is shown that d induces a group homomorphism from the three-dimensional Spin^c homology bordism group to $\mathbb{Q}$. An immediate consequence of this fact and the basic properties of d is the following:

Theorem 1.1. *The invariant $\delta(K)$ descends to give a surjective group homomorphism $\delta : \mathcal{C} \to \mathbb{Z}$, where $\mathcal{C}$ is the smooth concordance group of knots in S^3.*

It is interesting to compare δ to three other homomorphisms from $\mathcal{C}$ to the integers. The first is the classical knot signature σ, which we normalize to $\sigma' = -\sigma/2$. The second is the invariant τ defined using the knot Floer homology of Ozsváth-Szabó and Rasmussen [25], [31]. The third is Rasmussen's invariant s coming from Khovanov homology [32], which we normalize to $s' = -s/2$. For alternating knots it is known that $\tau = s' = \sigma'$. We show that a similar result holds for δ :

Theorem 1.2. *If the knot K is alternating, then $\delta(K) = \sigma'(K)$.*

The similarities between the four invariants hold for a much larger class of knots. Indeed, in [32] Rasmussen conjectured that $s' = \tau$ for all knots. On the other hand, there are several known examples where $s' = \tau \neq \sigma'$. Following [12], we call a knot H-thin (homologically thin) if its Khovanov homology is supported on two adjacent diagonals, and H-thick otherwise. Alternating knots are H-thin by the work of Lee [15], as are most knots up to ten crossings [2], [12]. For all H-thin knots for which s' was computed, it turned out to be equal to σ'.

The invariant δ can be computed algorithmically for Montesinos and torus knots, using the fact that their double branched covers are Seifert fibrations. We performed this computation for all Montesinos knots in a certain range, and found that $\delta = \sigma'$ for most of

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them. Interestingly, all the exceptions were H-thick. We also got $\delta = \sigma'$ for many H-thick examples, such as 8_{19} and 9_{42}.

In another direction, we used the method of Ozsváth and Szabó from [28] to compute δ for several H-thin knots of a special form. Together with the calculations for alternating, Montesinos and torus knots, this is enough to cover all the knots with up to nine crossings. We find:

Theorem 1.3. *If the knot K admits a diagram with nine or fewer crossings, then $\delta(K) = \sigma'(K)$.*

We also managed to calculate δ for all but eight of the ten-crossing knots. Among these we only found two examples with δ different from σ', the H-thick knots 10_{139} and 10_{145}. Based on Theorem 1.2 and all our computations, we make the following:

Conjecture 1.4. *For any H-thin knot K, $\delta(K) = \sigma'(K)$.*

On the other hand, an important difference between δ and the other three invariants σ', τ, and s' is that the absolute value of δ does not provide a lower bound for the slice genus of a knot. For example, the slice genus of the knot 10_{145} is two, but $\delta(10_{145}) = -3$. However, since δ is a concordance invariant, it can still be used as an obstruction to sliceness.

We can also say something about δ for Whitehead doubles. We denote by $Wh(K)$ the untwisted double of K with a positive clasp. We show:

Theorem 1.5. *For any knot K we have $\delta(Wh(K)) \leq 0$, and the inequality is strict if $\tau(K) > 0$. If K is alternating, then $\delta(Wh(K)) = -4\max\{\tau(K), 0\}$.*

It is well-known that the Alexander polynomial of $Wh(K)$ is always 1, and consequently $Wh(K)$ is topologically slice [7]. Our interest lies in the following conjecture, which appears as Problem 1.38 in Kirby's list [13].

Conjecture 1.6. *$Wh(K)$ is (smoothly) slice if and only if K is slice.*

A quick corollary of Theorem 1.5 is a result in the direction of this conjecture:

Corollary 1.7. *If the knot K has $\tau(K) > 0$, then $Wh(K)$ is not slice.*

Previously, Rudolph proved in [35] that $Wh(K)$ is not slice whenever the Thurston-Bennequin invariant $TB(K)$ of the knot K is nonnegative. (See also [1], [16] for different proofs.) Since $TB(K) \leq 2\tau(K) - 1$ by the work of Plamenevskaya [30], Rudolph's result is a consequence of ours. Furthermore, Corollary 1.7 can also be applied to knots such as $6_2, 7_6, 8_{11}$ and the mirrors of $8_4, 8_{10}, 8_{16}$, which according to the knot table in [5] have $\tau > 0$ but $TB < 0$.

As suggested to us by Jacob Rasmussen, Theorem 1.5 can be used to produce examples of knots for which δ is a nontrivial obstruction to sliceness, while the other three invariants are not. Indeed, we have:

Corollary 1.8. *Let $K_1 = T(2, 2m + 1)$ and $K_2 = T(2, 2n + 1)$ be two positive torus knots with $m, n \geq 1$, $m \neq n$. Then the connected sum $Wh(K_1)\#(-Wh(K_2))$ has $\sigma' = \tau = s' = 0$ but $\delta \neq 0$.*

Let us denote by $\mathcal{C}_{ts}$ the smooth concordance group of topologically slice knots. Endo [4] proved that $\mathcal{C}_{ts}$ has a $\mathbb{Z}^\infty$ subgroup. However, it is not known whether it has a $\mathbb{Z}^\infty$ summand. It was shown by Livingston [16] that $\tau : \mathcal{C}_{ts} \to \mathbb{Z}$ is a surjective homomorphism, and therefore $\mathcal{C}_{ts}$ has a $\mathbb{Z}$ summand. Using the pair (δ, τ), we show:

Corollary 1.9. *The group $\mathcal{C}_{ts}$ has a $\mathbb{Z} \oplus \mathbb{Z}$ summand.*

2. General facts

In this section we prove Theorem 1.1. First, we claim that $\delta(K)$ is an integer for any knot K. Note that the double branched cover $Y = \Sigma(K)$ is a rational homology three-sphere with $|H_1(Y)| = \det(K)$ an odd integer. Any Y with $|H_1(Y)|$ odd admits a unique Spin structure $\mathfrak{t}_0$. This can be distinguished in the set of all Spin^c structures by the requirement that $c_1(\mathfrak{t}_0) = 0 \in H^2(Y; \mathbb{Z})$. Let X be a four-manifold equipped with a Spin^c structure $\mathfrak{s}$ such that $\partial X = Y$ and $\mathfrak{s}|_Y = \mathfrak{t}_0$. Then $k = c_1(\mathfrak{s})$ is an element in the kernel of the map $H^2(X; \mathbb{Z}) \to H^2(Y; \mathbb{Z})$. According to [22], we have

$$\delta(K) = 2d(Y, \mathfrak{t}_0) \equiv \frac{k^2 - \mathrm{sgn}(X)}{2} \pmod{4}.$$

Here $\mathrm{sgn}(X)$ denotes the signature of X, and k is a characteristic element for the intersection form on X.

We choose X to be the double cover of B^4 branched along the pushoff of a Seifert surface for K. It is shown in [10] that X is a spin manifold (so that $k = 0$ is characteristic) with $\mathrm{sgn}(X) = \sigma(K)$. It follows that $\delta(K)$ is an integer and, furthermore,

$$\delta(K) \equiv \sigma'(K) \pmod{4} \tag{1}$$

Next, assume that K is a slice knot, i.e. it sits on the boundary S^3 of B^4 and bounds a smooth disk $D \subset B^4$. Let X be the double cover of B^4 branched over D. Then X is a rational homology four-ball with boundary $Y = \Sigma(K)$. Since d is an invariant of Spin^c rational homology bordism [22], it follows that $d(Y, \mathfrak{t}_0) = 0$ and therefore $\delta(K) = 0$.

The additivity property for connected sums $\delta(K_1 \# K_2) = \delta(K_1) + \delta(K_2)$ is a consequence of the additivity of d (Theorem 4.3 in [22]). Also, Proposition 4.2 in [22] implies that $\delta(-K) = -\delta(K)$, where by $-K$ we denote the mirror of K. If K_1 and K_2 are two cobordant knots, then $K_1 \#(-K_2)$ is slice, hence $\delta(K_1) = \delta(K_2)$. This shows that δ is a well-defined group homomorphism from the concordance group $\mathcal{C}$ to $\mathbb{Z}$. Surjectivity is an easy consequence of Theorem 1.2 below. For example, $\delta(T(3,2)) = 1$ for the right-handed trefoil.

3. Alternating knots

This section contains the proof of Theorem 1.2. The main input comes from the paper [27], where Ozsváth and Szabó calculated the Heegaard Floer homology for the double branched covers of alternating knots.

Let K be a knot with a regular, alternating projection. The projection splits the two-sphere into several regions, which we color black and white in chessboard fashion. Our coloring convention is that at each crossing, the white regions should be to the left of the overpass (see Figure 1). We form a graph Γ as follows. We form the set of vertices $V(\Gamma)$ by assigning a vertex to each white region. We index the vertices by setting $V(\Gamma) = \{v_0, v_1, \ldots, v_m\}$, so that the number of white regions is $m + 1$. Let us denote by R_i the white region corresponding to v_i. We draw an edge between the vertices $i, j \in V(\Gamma)$ for every crossing at which the regions R_i and R_j come together. We denote by e_{ij} the number of edges joining the vertices v_i and v_j. We can assume that the diagram contains no reducible crossings, and therefore $e_{ii} = 0$ for all i.
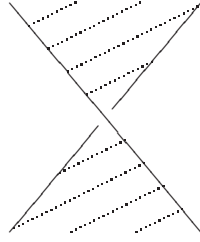
FIGURE 1. **Coloring convention for alternating knot diagrams.**

Choose an orientation on K, and consider the oriented resolution of the knot projection. Note that at every positive crossing the two white regions get joined, while at every negative crossing they are separated. Now form a new graph $\tilde{\Gamma}$ by taking its set of vertices $V(\tilde{\Gamma})$ to be the set of white regions in the resolved diagram, and by assigning exactly one edge whenever two white domains are adjacent at some negative crossing in the original knot projection. In other words, we look at the natural projection map $\pi : V(\Gamma) \to V(\tilde{\Gamma})$. Given $x \neq y \in V(\tilde{\Gamma})$, there is at most one edge between x and y. Such an edge exists if and only if there are $v_i \in \pi^{-1}(x), v_j \in \pi^{-1}(y)$ such that v_i and v_j are joined by at least one edge in G. This is shown in Figure 2 for the two-bridge knot 8_{13}.
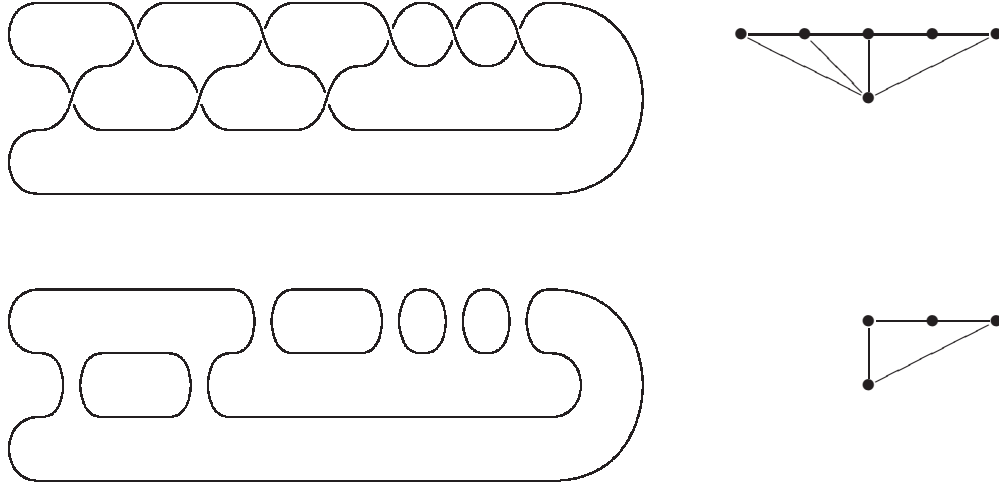

FIGURE 2. **The knot** 8_{13}**.** The top row contains the knot diagram and the white graph Γ, and the second row contains the oriented resolution and the graph $\tilde{\Gamma}$.

Lemma 3.1. *Each circuit in $\tilde{\Gamma}$ has an even number of edges.*

Proof. Label the edges of Γ by $+$ or $-$ according to the sign of the corresponding crossing in the knot diagram. Going from Γ to $\tilde{\Gamma}$ consists in collapsing all positive edges. Thus, a circuit in $\tilde{\Gamma}$ can be lifted to one in Γ, possibly at the expense of introducing some positive edges.

The statement of the lemma is equivalent to saying that all circuits in Γ have an even number of negative edges. It suffices to prove this for elementary circuits in Γ, those whose vertices correspond to the white regions adjacent to a particular black region, in cyclic order. In other words, we need to show that each black region R has an even number of negative crossings among its vertices. Let $E_1, E_2, \ldots, E_n$ be the edges on the boundary of R in cyclic order. Each of them is oriented; let us define the sign of an edge E_i to be $+$ if E_i is oriented counterclockwise around R and $-$ if it is oriented clockwise. Since the knot projection is alternating, a negative crossing at a vertex of R corresponds to a change of sign between the respective edges E_i and E_{i+1}. As we go around the boundary, there must be an even number of such changes. This completes the proof. $\qquad\square$

Lemma 3.1 implies that $\tilde{\Gamma}$ can be made into a bipartite graph as follows. The distance between two vertices on a graph is defined to be the minimal number of edges in a path from one vertex to the other. We define a map $\tilde{\eta} : V(\tilde{\Gamma}) \to \{0,1\}$ by setting $\tilde{\eta}(x) =$ the parity of the distance between $\pi(0)$ and x. This produces a partition of $V(\tilde{\Gamma})$ into two classes such that $\tilde{\eta}(x) \neq \tilde{\eta}(y)$ whenever x and y are joined by an edge. We can pull back $\tilde{\eta}$ to a partition of $V(\Gamma)$ described by the map $\eta = \tilde{\eta} \circ \pi : V(\Gamma) \to \{0,1\}$.

Lemma 3.2. *An edge in Γ between v_i and v_j corresponds to a positive crossing if and only if $\eta(v_i) = \eta(v_j)$.*

Proof. If two domains R_i and R_j meet at a positive crossing, then $\pi(v_i) = \pi(v_j)$, which implies $\eta(v_i) = \eta(v_j)$.

If R_i and R_j meet at a negative crossing c, we claim that any other crossing c' where R_i and R_j meet is also negative. Indeed, let us draw a loop γ on the two-sphere by going from c to c' inside R_i and then going from c' to c in R_j. Let D one of the two disks bounded by γ. Since c is negative, the underpass and the overpass at c must be going either both into D or both out of D. It follows that the same is true for the underpass and the overpass at c', which means that c' is negative. This proves our claim.

Therefore, if R_i and R_j meet at a negative crossing, the corresponding v_i and v_j must project to different vertices $x = \pi(v_i), y = \pi(v_j) \in V(\tilde{\Gamma})$, where x and y are joined by an edge in $\tilde{\Gamma}$. We observed above that in this situation $\tilde{\eta}(x) \neq \tilde{\eta}(y)$. $\qquad\square$

The Goeritz matrix $G = (g_{ij})$ of the alternating projection is the $m \times m$ symmetric matrix with entries:

$$g_{ij} = \begin{cases} -\sum_{k=0}^{m} e_{ik} & \text{if } i = j; \\ e_{ij} & \text{if } i \neq j. \end{cases}$$

The quadratic form associated to G is negative-definite. Indeed, if we have a vector $w = (w_i), i = 1 \ldots m$, then

$$(2) \qquad w^T G w = \sum_{i=1}^{m} \sum_{j=1}^{m} g_{ij} w_i w_j = -\sum_{i=1}^{m} e_{0i} w_i^2 - \sum_{1 \leq i < j \leq m} e_{ij}(w_i - w_j)^2.$$

Definition 3.3. *Given an $m \times m$ symmetric matrix $M = (m_{ij})$ with integer entries, a vector $w = (w_i), i = 1 \ldots m$, with $w_i \in \mathbb{Z}$ is called **characteristic** for M if $(Mw)_i \equiv m_{ii}$ (mod 2) for all i.*

Lemma 3.4. *If $w = (w_i)$ is a characteristic vector for the Goeritz matrix G of an alternating projection, then $w_i \equiv \eta(v_i)$ (mod 2).*

Proof. The determinant of G is the same as the determinant of K (c.f. [3, Corollary 13.29]), which is an odd number. This means that the system of equations

$$(3) \qquad \sum_{j=1}^{m} g_{ij} w_j = g_{ii} \quad (i = 1, \dots, m)$$

has a unique solution over the field $\mathbb{Z}/2$. Therefore, it suffices to show that $w_i = \eta(v_i)$ satisfy (3). Using the fact that $\eta(v_0) = 0$, we can rewrite this condition as:

$$\sum_{j=0}^{m} e_{ij}(\eta(v_j) - \eta(v_i) + 1) \equiv 0 \pmod{2} \quad \text{for } i = 1, \dots, m.$$

The left hand side of this congruence is the number of positive crossings around the white region R_i. The boundary of R_i is a polygon with an orientation on each edge. A corner of R_i corresponds to a positive crossing if and only if the orientations of the two incident edges do not match up at that corner. The number of those corners has to be even. $\qquad \square$

Gordon and Litherland [8, Theorem 6] proved a formula for the signature of a knot in terms of the Goeritz matrix. In the case of an alternating projection (with our coloring convention), their formula reads:

$$(4) \qquad \sigma(K) = -m + \mu,$$

where μ is the number of negative crossings.

On the other hand, Ozsváth and Szabó show in [27, 28] that the correction terms of $\Sigma(K)$ may be computed using the Goeritz matrix G. In [28, Proposition 3.2] they show that there is an affine identification

$$\phi : \mathbb{Z}^m / G \cdot \mathbb{Z}^m \longrightarrow \mathrm{Spin}^c(\Sigma(K)),$$

taking zero to the spin structure and with

$$4d(\Sigma(K), \phi(\xi)) = m + \max_{v} \{(Gw_0 + 2v)^T G^{-1}(Gw_0 + 2v)\},$$

where w_0 is any characteristic vector for G and the maximum is taken over $v \in \mathbb{Z}^m$ with $[v] = \xi \in \mathbb{Z}^m / G \cdot \mathbb{Z}^m$. Applying this with $\xi = 0$ we find

$$(5) \qquad 2\delta(K) = m + \max_{w} \{w^T G w\},$$

where the maximum is taken over characteristic elements w. From equation (2) we see that the maximum can be attained by choosing all the w_i to be either 0 or 1. Using the description of the characteristic vectors in Lemma 3.4, it follows that we should take $w_i = \eta(v_i)$ for all i. Equation (2) shows that the maximum appearing in (5) is minus the total number of edges between vertices v_i and v_j with $\eta(v_i) \neq \eta(v_j)$. By Lemma 3.2, this is exactly $-\mu$. Comparing (4) with (5), we get that $\delta(K) = -\sigma(K)/2$. This completes the proof of Theorem 1.2.

4. Montesinos and torus knots

4.1. **Montesinos knots.** For a detailed exposition of the properties of Montesinos knots and links we refer to [3]. In the following definition, assume that e is any integer and (α_i, β_i) are coprime pairs of integers with $\alpha_i > \beta_i \geq 1$.

Definition 4.1. *A Montesinos link $M(e; (\alpha_1, \beta_1), (\alpha_2, \beta_2), \dots, (\alpha_r, \beta_r))$ is a link which has a projection as shown in Figure 3(a). There are e half-twists on the left side. A box* $\boxed{\alpha, \beta}$ *represents a rational tangle of slope α/β: given a continued fraction expansion*

$$\frac{\alpha}{\beta} = [a_1, a_2, \dots, a_m] := a_1 - \cfrac{1}{a_2 - \cfrac{\ddots}{} \cfrac{}{- \cfrac{1}{a_m}}} \,,$$

the rational tangle of slope α/β consists of the four string braid $\sigma_2^{a_1} \sigma_1^{a_2} \sigma_2^{a_3} \sigma_1^{a_4} \dots \sigma_i^{a_m}$, which is then closed on the right as in Figure 3(b) if m is odd or (c) if m is even.
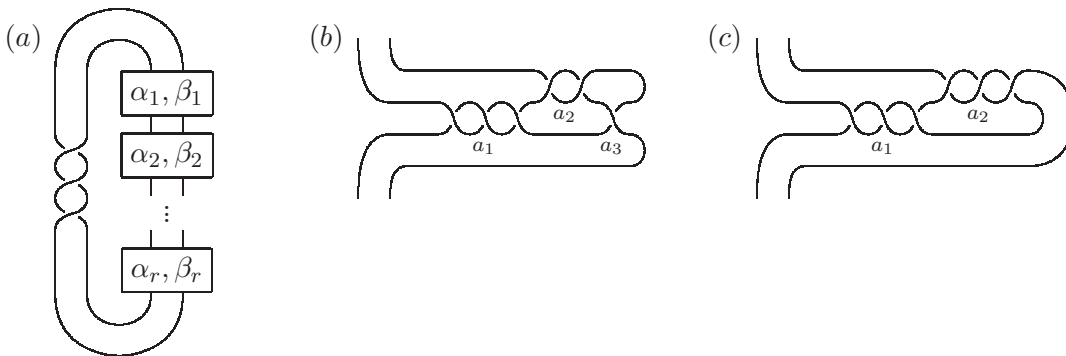

FIGURE 3. **Montesinos links and rational tangles.** Note that $e = 3$ in (a). Also (b) and (c) are both representations of the rational tangle of slope 10/3:

$$10/3 = [3, -2, 1] = [3, -3]$$

(and one can switch between (b) and (c) by simply moving the last crossing).

Let K be the link $M(e; (\alpha_1, \beta_1), (\alpha_2, \beta_2), \dots, (\alpha_r, \beta_r))$. Note that K is alternating if $e \notin \{1, 2, \dots, r-1\}$: for an alternating projection in the case $e \leq 0$, take a continued fraction expansion

$$\frac{\alpha_i}{\beta_i} = [a_1^i, a_2^i, \dots, a_{m_i}^i]$$

with $a_1^i, a_3^i, \dots$ positive and $a_2^i, a_4^i, \dots$ negative, for each i (as in [3, 12.13]). Then note that the reflection $-K$ of K is obtained by replacing e with $r - e$ and β_i with $\alpha_i - \beta_i$.

Also, K is a knot if and only if either exactly one of $\alpha_1, \dots, \alpha_r$ is even, or if all of $\alpha_1, \dots, \alpha_r, e + \sum_{i=1}^{r} \beta_i$ are odd. We will restrict our attention to knots. Since $\delta(K)$ is determined for alternating knots by Theorem 1.2, and since $\delta(-K) = -\delta(K)$, we also restrict to $1 \leq e \leq \lfloor r/2 \rfloor$.

The double cover of S^3 branched along K is a Seifert fibred space which is given as the boundary of a plumbing of disk bundles over S^2 (see for example [20]). This plumbing is determined (nonuniquely) by the Montesinos invariants which specify K. After possibly reflecting K we may choose the plumbing so that its intersection pairing is represented by

Montesinos knot	*Knotscape* notation	δ	σ'
$M(1;(3,1),(3,1),(4,1))$	$10n27 = 10_{139}$	-1	3
$M(1;(3,1),(3,1),(5,2))$	$10n14 = 10_{145}$	3	-1
$M(1;(2,1),(3,1),(7,1))$	$12n242$	0	4
$M(1;(5,2),(5,2),(5,2))$	$12n276$	2	-2
$M(1;(3,1),(4,1),(5,1))$	$12n472$	0	4
$M(1;(3,1),(3,1),(6,1))$	$12n574$	0	4
$M(1;(2,1),(5,1),(5,1))$	$12n725$	0	4
$M(1;(3,2),(5,1),(5,1))$	$13n3596$	4	0
$M(1;(5,2),(5,2),(6,1))$	$14n6349$	-2	2
$M(1;(3,1),(4,1),(7,1))$	$14n12201$	1	5
$M(1;(3,1),(5,1),(6,1))$	$14n15856$	1	5
$M(1;(4,1),(5,1),(5,1))$	$14n24551$	1	5
$M(1;(5,1),(5,3),(6,1))$	$15n74378$	0	4
$M(1;(5,1),(5,1),(6,1))$	$16n931575$	2	6

TABLE 1. **Montesinos knots for which $\delta \neq \sigma'$.**

a negative-definite matrix Q. It then follows from [23, Corollary 1.5] that

$$\delta(K) = \max \left\{ \frac{w^T Q w + \operatorname{rank}(Q)}{2} \right\},$$

where the maximum is taken over all $w \in \mathbb{Z}^{\operatorname{rank}(Q)}$ which are characteristic for Q in the sense of Definition 3.3.

This permits an algorithmic computation of $\delta(K)$. We ran a *Maple* program (partly written by Sašo Strle) to find δ for all Montesinos knots $M(1;(\alpha_1,\beta_1),(\alpha_2,\beta_2),(\alpha_3,\beta_3))$ with $\alpha_i \leq 7$. According to the table in [11], these include all nonalternating Montesinos knots with up to 10 crossings.

The result is that $\delta = \sigma'$ for all but the knots listed in Table 1.

We used the program *Knotscape* [6] to identify these knots. For the convenience of the reader, the two ten crossing knots are also shown in Rolfsen's notation [34].

Shumakovitch's *KhoHo* package [36] showed that all these knots are not H-thin, in agreement with Conjecture 1.4.

4.2. **Torus knots.** The double branched cover of the torus knot $T(p,q)$ is the Brieskorn sphere $\Sigma(2,p,q)$. This is a Seifert fibration, so we can use the same formula as in the Montesinos case. Therefore, for every p and q, we have an algorithmic way of computing δ. For example, the double cover of $T(3,4)$ is $\Sigma(2,3,4)$, which gives $\delta(T(3,4)) = \sigma'(T(3,4)) = -3$.

Two infinite classes of knots for which $\delta \neq \sigma'$ are provided by the torus knots $T(3,6n+1)$ for $n \geq 0$, and $T(3,6n-1)$ for $n \geq 1$. Indeed, using the recurrence formulae for the signatures of torus knots in [9] we can easily compute $\sigma(T(3,6n \pm 1)) = 8n$. Therefore, $\sigma'(T(3,6n \pm 1)) = -4n$. On the other hand, the double branched covers are the Brieskorn spheres $\Sigma(2,3,6n+1)$, whose Heegaard Floer homologies were computed by Ozsváth and Szabó in [22]. Their calculations show that $\delta(T(3,6n+1)) = 0$ and $\delta(T(3,6n-1)) = -4$ for all n.

5. THE OZSVÁTH-SZABÓ TECHNIQUE

In [28], Ozsváth and Szabó managed to compute the correction terms d for $\Sigma(K)$ for four knots of ten crossings that are neither alternating nor Montesinos. Their method was to find explicit sharp four-manifolds with boundary $\Sigma(K)$. If we focus on the spin structure, we see that the results

$$\delta(10_{148}) = 1, \ \delta(10_{151}) = -1, \ \delta(10_{158}) = 0, \ \delta(10_{162}) = 1$$

coincide with σ' in all four examples.

A rational homology sphere Y is called an L-space if its reduced Heegaard Floer homology $HF_{\mathrm{red}}(Y)$ vanishes, so that its Heegaard Floer invariants resemble those of a lens space. A negative-definite four-manifold X is called sharp if its boundary $\partial X = Y$ is an L-space, and if

$$d(Y, \mathfrak{t}) = \max \left\{ \frac{c_1(\mathfrak{s})^2 + b_2(X)}{4} \ \middle| \ \mathfrak{s} \in \mathrm{Spin}^c(X), \mathfrak{s}|_Y = \mathfrak{t} \right\}$$

for all $\mathfrak{t} \in \mathrm{Spin}^c(Y)$. This means that the correction terms of Y can be computed from the intersection pairing of X. We say a negative-definite form Q is sharp for Y if Q is the intersection pairing of a sharp four-manifold bounded by Y.

The following proposition summarizes the above-mentioned technique. The proof of this proposition is due to Ozsváth and Szabó and may be found in [28, §7.2].

Proposition 5.1. *Let $k \geq 2$. Suppose that a knot K has a nonalternating projection which contains a subdiagram which is a $-k/2$ twist (a 2-strand braid with k crossings, as shown below)*

and that the complement of this subdiagram is alternating. Let K', K'' respectively be the links obtained by replacing the $-k/2$ twist by
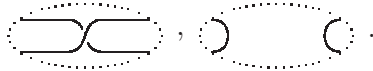
Note that the resulting projection of K'' is alternating. Let G be the negative-definite $m \times m$ Goeritz matrix associated to K'' as in Section 3, where we take v_0 and v_m to be the vertices in the white graph associated to the regions at each end of the subdiagram shown above. Let G_r be the matrix

$$\left(\begin{array}{c|c} G & \\ \hline & \begin{array}{c|c} & 1 \\ \hline 1 & r \end{array} \end{array} \right).$$

Then G_{-1} is the intersection pairing of a four-manifold bounded by $\Sigma(K')$, and G_{-k} is the intersection pairing of a manifold bounded by $\Sigma(K)$. If G_{-1} is negative-definite and sharp for $\Sigma(K')$ then G_{-k} is sharp for $\Sigma(K)$.

For each knot K in Figure 4 we find that $\Sigma(K')$ is a lens space whose correction terms may be computed using the formula from [22], so that the condition on G_{-1} may be checked. Then the matrix G_{-k} may be used to compute the correction terms of $\Sigma(K)$. In particular we find that $\delta = \sigma'$ for all of these knots.

Putting this together with the calculations for alternating, Montesinos, and torus knots, this covers all but eight of the knots up to ten crossings, and establishes the result in Theorem 1.3.

9_{47}

9_{49}

10_{149}

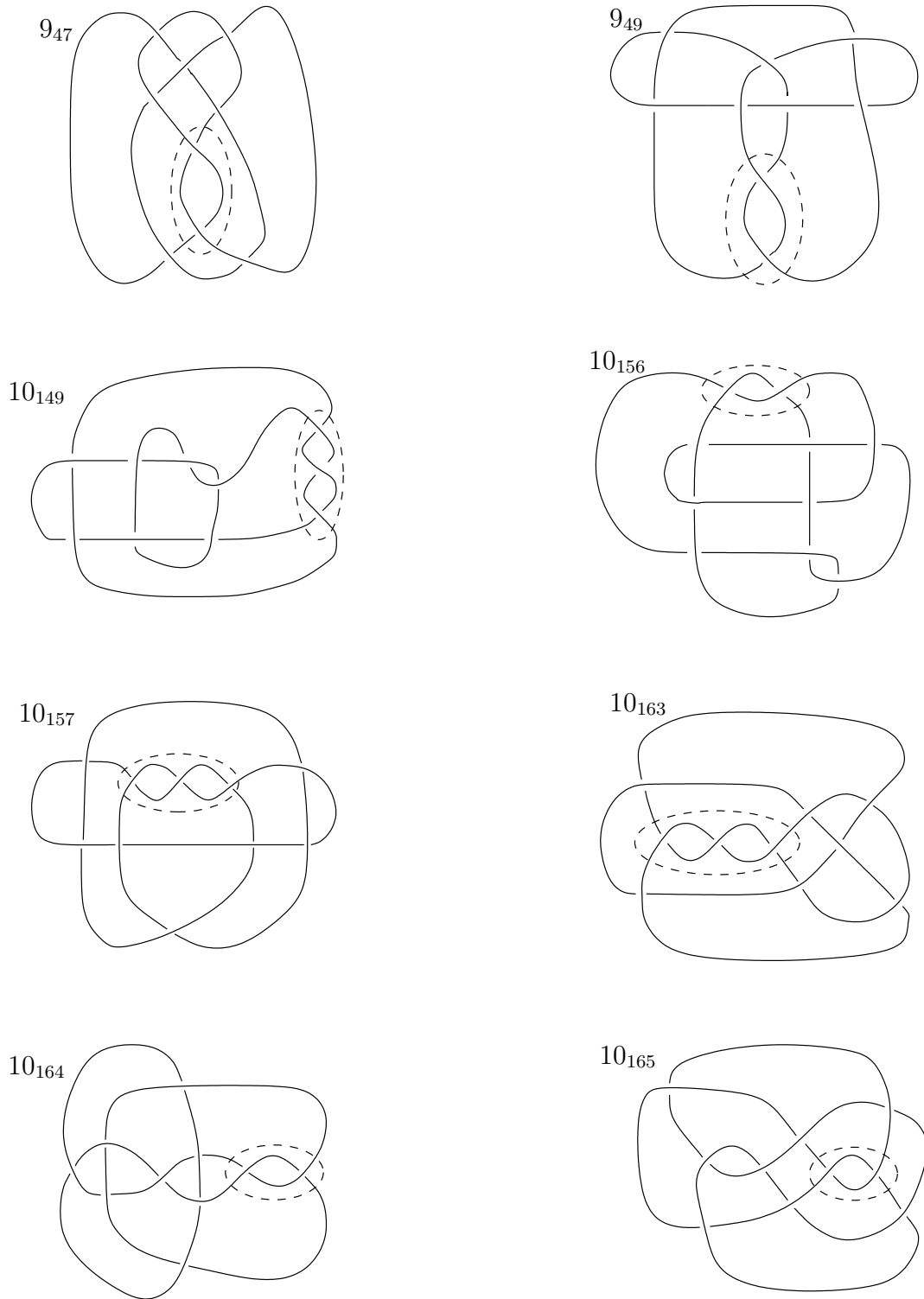10_{156}

10_{157}

10_{163}

10_{164}

10_{165}

FIGURE 4. The relevant -1 or $-3/2$ twist is indicated on each knot.

In Table 2 we exhibit all the knots of up to ten crossings for which either some of the four invariants δ, σ', τ and s' are different, or some of their values are unknown. In other words, for all the knots up to ten crossings that are not shown, we know that $\delta = \sigma' = \tau = s'$.

The notation for knots is the one in [5], the same as Rolfsen's but with the last five knots translated to account for the Perko pair. The values for σ', τ and s' are also taken from [5].

Knot	Type	δ	σ'	τ	s'
9_{42}	H-thick	-1	-1	0	0
10_{132}	H-thick	0	0	-1	-1
10_{136}	H-thick	-1	-1	0	0
10_{139}	H-thick	-1	3	4	4
10_{141}	H-thick	0	0	$?$	0
10_{145}	H-thick	3	-1	-2	-2
10_{150}	H-thin	$?$	2	2	2
10_{152}	H-thick	$?$	3	4	4
10_{153}	H-thick	$?$	0	0	0
10_{154}	H-thick	$?$	2	3	3
10_{155}	H-thin	$?$	0	0	0
10_{159}	H-thin	$?$	1	1	1
10_{160}	H-thin	$?$	2	2	2
10_{161}	H-thick	$?$	2	3	3

TABLE 2. **Knots with 10 crossings or less for which either δ, σ', τ, s' are not all equal, or some of their values are not known.**

6. WHITEHEAD DOUBLES

The double branched coverings of doubles of knots were studied extensively in [18]. In particular, it is interesting to note that the double branched cover of a twisted double of a prime knot K determines K among all prime knots.

In this section we focus on the untwisted Whitehead double $Wh(K) = Wh^+(K)$ of a knot K with a positive clasp. The untwisted double of the right handed trefoil is shown in Figure 5. (The three negative twists shown in the diagram are a consequence of the fact that the chosen diagram of the trefoil has writhe three.) Note that the untwisted double $Wh^-(K)$ with a negative clasp is simply $-Wh^+(-K)$, so our discussion can also be phrased in terms of Wh^-.

Denote by K^r the knot K with its string orientation reversed. The following surgery description of $\Sigma(Wh(K))$ is useful:

Proposition 6.1. $\Sigma(Wh(K))$ *is the manifold obtained by Dehn surgery on the knot $K \# K^r$ in S^3 with framing $1/2$.*

Proof. The knot $Wh(K)$ may be unknotted by changing one crossing, so as to undo the clasp. A theorem originally due to Montesinos tells us that $\Sigma(Wh(K))$ is given by Dehn surgery on a knot with half-integral framing. An algorithm is described in [19, Lemma 3.1] to obtain a Dehn surgery diagram for the double branched cover of any knot C by first obtaining a Kirby calculus diagram of a disk bounded by C in a blown-up four-ball. The
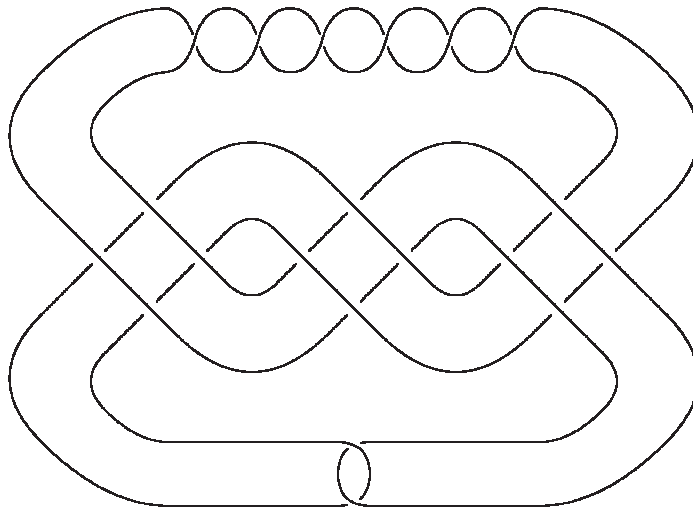
FIGURE 5. **The untwisted Whitehead double of the right handed trefoil, with a positive clasp.**

description of $\Sigma(Wh(K))$ follows from the application of this algorithm to $Wh(K)$. (This is illustrated in Figure 6.)   □

In [31], Rasmussen introduced a set of invariants h_i for a knot $K \subset S^3$, where $i \in \mathbb{Z}$. They are nonnegative integers that encode the information in the maps in the surgery exact triangles for K. According to [31, Section 7.2] and [33, Section 2.2], the correction terms d for manifolds obtained by integral surgery on a knot $K \subset S^3$ can be computed in terms of the h_i's. In particular, the d invariant of the -1 surgery on K is equal to $2h_0$.

Proposition 6.2. *Let $K_{-1/2}$ be the manifold obtained from S^3 by surgery on the knot K with framing $-1/2$. Then $d(K_{-1/2}) = 2h_0(K)$. (Note that $K_{-1/2}$ has a unique Spin^c structure, which we drop from the notation.)*

Proof. Let K_0 and K_{-1} be the results of 0 and -1 surgery on $K \subset S^3$, respectively. Consider the exact triangle with twisted coefficients [21]:

$$(6) \quad \underline{HF}^+(K_0) \to HF^+(S^3)[T, T^{-1}] \xrightarrow{F_W} HF^+(K_{-1})[T, T^{-1}] \xrightarrow{F_{W_0}} \underline{HF}^+(K_0)$$

We tensor everything with $\mathbb{Q}$ for simplicity, so that (6) is a long exact sequence of modules over the ring $\mathbb{Q}[U, T, T^{-1}]$. The map F_W is induced by the cobordism W corresponding to the -1 surgery on K. We let $y \in H^2(W)$ be a generator and denote by $\mathfrak{t}_i$ the Spin^c structure on W with $c_1(\mathfrak{t}_i) = 2i - 1$. There are maps $F_{W,\mathfrak{t}_i} : HF^+(S^3) \to HF^+(K_{-1})$ associated to each Spin^c structure, and

$$F_W(x) = \sum T^i \cdot F_{W,\mathfrak{t}_i}(x).$$

Since S^3 and K_{-1} are integral homology spheres, $F_{W,\mathfrak{t}_i}$ are graded maps. More precisely, they shift the absolute grading on HF^+ by $(c_1(\mathfrak{t}_i)^2 + 1)/4 = i - i^2$. In particular, the maximal degree shift is zero.
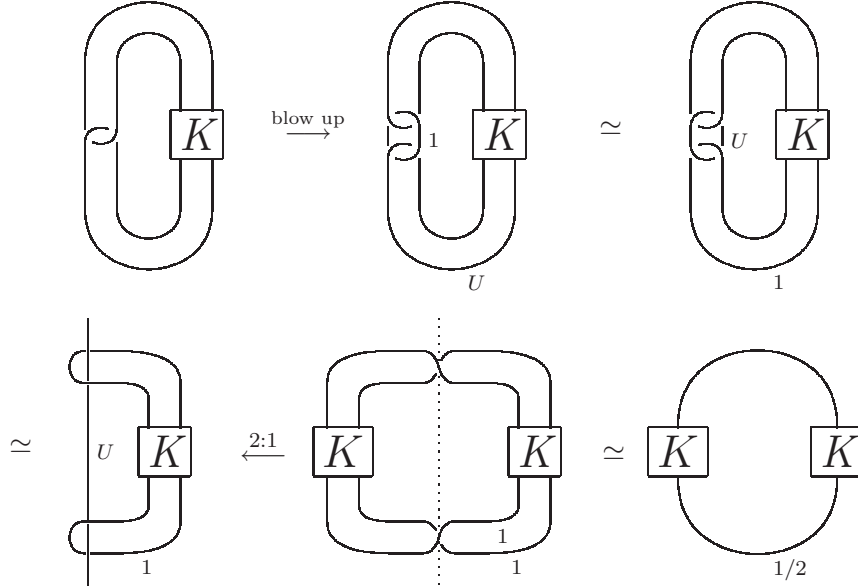
FIGURE 6. **Proof that the double branched cover of $Wh(K)$ is $(K\#K^r)_{1/2}$.** Note that in the last diagram K is traversed twice with different orientations, so that the knot shown is $K\#K^r$ rather than $K\#K$.

For every integral homology sphere Y, the Heegaard Floer homology $HF^+(Y;\mathbb{Q})$ is a $\mathbb{Q}[U]$-module that decomposes as $\mathcal{T}^+_{d(Y)} \oplus HF_{\mathrm{red}}(Y)$. Here HF_{red} is finite dimensional as a $\mathbb{Q}$-vector space, and $\mathcal{T}^+_{d(Y)} \cong \mathbb{Q}[U, U^{-1}]/\mathbb{Q}[U]$ with U reducing the degree by 2 and the bottom-most element in $\mathcal{T}^+_{d(Y)}$ being in absolute degree $d(Y)$. The piece $\mathcal{T}^+_{d(Y)}$ can be described as the image of $HF^\infty(Y)$ into $HF^+(Y)$, or as the image of U^k in $HF^+(Y)$ for k sufficiently large.

The map F_W induces an isomorphism on HF^∞ in each Spin^c structure, cf. [21, Section 10.1]. This fact, together with U-equivariance, implies that F_W must take $HF^+(S^3)[T, T^{-1}] = \mathcal{T}^+_0[T, T^{-1}]$ onto $\mathcal{T}^+_{d(K_{-1})}[T, T^{-1}]$, with a finite dimensional kernel. The number $h_0 = h_0(K)$ is the rank of the kernel of F_W as a $\mathbb{Q}[T, T^{-1}]$ module, which is the span of $\{1, U^{-1}, \ldots, U^{-h_0+1}\}$. Since the degree shift of $F_{W,\mathfrak{t}_0}$ is zero, we must have $d(K_{-1}) = 2h_0$. Let x be a generator of the part of $\mathcal{T}^+_{d(K_{-1})}$ sitting in the bottom-most degree $d(K_{-1})$. For all k we have

$$F_W(U^{-h_0-k}) = P(T)U^{-k}x + \text{ lower degree terms,}$$

where $P(T)$ is a fixed polynomial in T. Since the cobordism W is negative definite, each of the Spin^c structures $\mathfrak{t}_i$ (in particular $\mathfrak{t}_0$ and $\mathfrak{t}_1$) induces an isomorphism on HF^∞. It follows that $P(T) = aT + b$ with $a, b \neq 0$.

The conclusion is that for any element z in the image of F_W (which is also the kernel of F_{W_0}), the part of z sitting in highest degree is always a multiple of $P(T)$. Note that $P(T)$ is not invertible in $\mathbb{Q}[T, T^{-1}]$.

Now let $K' \subset K_{-1}$ be the core of the solid torus glued in for the -1 surgery. Then -1 surgery on $K' \subset K_{-1}$ produces the manifold $K_{-1/2}$, while zero surgery on $K' \subset K_{-1}$ gives K_0 again.

We consider the exact triangle with twisted coefficients for the -1 surgery on $K' \subset K_{-1}$:

$$(7) \qquad \cdots \to \underline{HF}^+(K_0) \xrightarrow{F_Z} HF^+(K_{-1})[T, T^{-1}] \xrightarrow{F_{W'}} HF^+(K_{-1/2})[T, T^{-1}] \to \cdots$$

The map F_Z is induced by a cobordism Z from K_0 to K_{-1}. Drawing inspiration from the arguments in [26, Lemma 2.9] and [28, Lemma 4.5], we compose Z with the cobordism W_0 from K_{-1} to K_0 that appeared in (6). Then $W_0 \circ Z$ has an alternative factorization as the two-handle addition from K_0 to $K_0 \# (S^2 \times S^1)$, followed by another cobordism in which the generator of $H_1(S^2 \times S^1)$ becomes null-homologous. This implies that $F_{W_0} \circ F_Z = F_{W_0 \cup Z} = 0$. (Alternatively, one could observe that $W_0 \cup Z$ contains a sphere of square zero representing a nontrivial relative homology class. Arguments as in [14, Lemma 7.1] then show that the corresponding cobordism map is zero.)

It follows from here that the image of F_Z (which is also the kernel of $F_{W'}$) lies in the kernel of F_{W_0}. Hence every element in the kernel of $F_{W'}$ must have its highest degree part a multiple of $P(T)$. In particular, since the bottom-most generator x of $\mathcal{T}^+_{d(K_{-1})}$ is not of this form, we must have $F_{W'}(x) \neq 0$.

Note that K_{-1} and $K_{-1/2}$ are integral homology spheres, and $F_{W'}$ is a sum of graded maps as before, with the maximal shift in absolute grading being zero. The map $F_{W'}$ must take $\mathcal{T}^+_{d(K_{-1})}[T, T^{-1}]$ to $\mathcal{T}^+_{d(K_{-1/2})}[T, T^{-1}]$. Also, W' is a negative definite cobordism, which means that $F_{W'}(x)$ must have its highest degree part in grading $2h_0 = \text{degree}(x)$. On the other hand, we have $U \cdot F_{W'}(x) = F_{W'}(Ux) = 0$, hence $F_{W'}(x)$ lives in the bottom-most degree $d(K_{-1/2})$. We conclude that $d(K_{-1/2}) = 2h_0$. □

Applying this to the mirror image of K we get:

Corollary 6.3. *Let $K_{1/2}$ be the manifold obtained from S^3 by surgery on the knot K with framing $1/2$. Then $d(K_{1/2}) = -2h_0(-K)$.*

Proof of Theorem 1.5. Proposition 6.1 and Corollary 6.3 imply that $\delta(Wh(K)) = -4h_0(-(K \# K^r))$. Since h_0 is always nonnegative, we must have $\delta(Wh(K)) \leq 0$. Proposition 7.7 in [31] says that $h_0(K) > 0$ whenever $\tau(K) < 0$. Applying this to $-(K \# K^r)$ and using the fact that $\tau(-(K \# K^r)) = -2\tau(K)$, we get that $\delta(Wh(K)) < 0$ for $\tau(K) > 0$.

According to [24], alternating knots are perfect in the sense of [31, Definition 6.1], i.e. their knot Floer homology is supported on one diagonal. For a perfect knot K, $h_0(K) = \max\{\lceil -\tau(K)/2 \rceil, 0\}$ by [31, Corollary 7.2]. The property of being perfect is preserved under taking connected sums by [31, Corollary 6.2]. Also, knot Floer homology is insensitive to orientation reversal of the knot. Therefore, if K is alternating (or, more generally, perfect) then $-(K \# K^r)$ is perfect and $\delta(Wh(K)) = -4 \max\{\tau(K), 0\}$. □

Proof of Corollary 1.8. The knot $K_1 = T(2, 2m + 1)$ is alternating and has $\tau(K_1) = m > 0$. By Theorem 1.5 we have $\delta(Wh(K_1)) = -4m$. On the other hand, by writing down a Seifert matrix it is easy to see that the signature of the Whitehead double of any knot is zero, so $\sigma'(Wh(K_1)) = 0$. The results of Livingston and Naik from [17], together with the fact that the Thurston-Bennequin number $TB(K_1) = 2m - 1$ is positive (see for example [29]), imply that $\tau(Wh(K_1)) = s'(Wh(K_1)) = 1$. The same results hold for $K_2 = T(2, 2n + 1)$, and the corollary follows from the additivity properties of the four invariants. □

The fact that $\delta(Wh(T(2, 2m + 1))) = -4m$ shows that $\delta(Wh(K))$ can be any nonpositive multiple of four. In contrast to this, $\sigma'(Wh(K)) = 0$ for all K and $s'(Wh(K)), \tau(Wh(K)) \in$

$\{0, \pm 1\}$ because their absolute values are bounded above by the unknotting number of $Wh(K)$.

Proof of Corollary 1.9. Note that every topologically slice knot has $\sigma = 0$, and therefore $\delta \equiv 0 \pmod 4$ by (1). We claim that the homomorphism $\psi = (\tau, \delta/4) : \mathcal{C}_{ts} \to \mathbb{Z} \oplus \mathbb{Z}$ is surjective. Indeed, we have $\psi(Wh(T(2,3))) = (1,-1)$ and $\psi(Wh(T(2,5))) = (1,-2)$, and these values span $\mathbb{Z} \oplus \mathbb{Z}$. $\qquad\square$

References

[1] S. Akbulut and R. Matveyev, *Exotic structures and adjunction inequality,* Turkish J. Math. **21** (1997), no. 1, 47-53.

[2] D. Bar-Natan, *On Khovanov's categorification of the Jones polynomial,* Algebr. Geom. Topol. **2** (2002), 337-370.

[3] G. Burde and H. Zieschang, *Knots,* 2nd ed., de Gruyter Studies in Mathematics **5**, Walter de Gruyter & Co., Berlin (2003).

[4] H. Endo, *Linear independence of topologically slice knots in the smooth cobordism group,* Topology Appl. **63** (1995), no. 3, 257-262.

[5] J. C. Cha and C. Livingston, *Table of knot invariants,* online at `http://www.indiana.edu/~knotinfo`

[6] J. Hoste and M. Thistlethwaite, *Knotscape,* online at `http://dowker.math.utk.edu/knotscape.html`

[7] M. Freedman and F. Quinn, *Topology of 4-manifolds,* Princeton Mathematical Series **39**, Princeton University Press, Princeton, NJ (1990).

[8] C. Gordon and R. Litherland, *On the signature of a link,* Invent. Math. **47** (1978), no. 1, 53-69.

[9] C. Gordon, R. Litherland, and K. Murasugi, *Signatures of covering links,* Canad. J. Math. **33** (1981), no. 2, 381-394.

[10] L. Kauffman, *Open books, branched covers and knot periodicity,* Topology **13** (1974), 143-160.

[11] A. Kawauchi, *A survey of knot theory,* Birkhäuser Verlag, Basel (1996).

[12] M. Khovanov, *Patterns in knot cohomology I,* Experiment. Math. **12** (2003), no. 3, 365-374.

[13] *Problems in low-dimensional topology,* edited by Rob Kirby, AMS/IP Stud. Adv. Math. **2.2**, Geometric topology (Athens, GA, 1993), 35-473, Amer. Math. Soc., Providence, RI (1997).

[14] P. Kronheimer, T. Mrowka, P. Ozsváth and Z. Szabó, *Monopoles and lens space surgeries,* Annals of Math., to appear.

[15] E.-S. Lee, *Khovanov invariants for alternating links,* preprint (2002), math.GT/0210213.

[16] C. Livingston, *Computations of the Ozsváth-Szabó knot concordance invariant,* Geom. Topol. **8** (2004), 735-742.

[17] C. Livingston and S. Naik, *Ozsvath-Szabo and Rasmussen invariants of doubled knots,* preprint, math.GT/0505361.

[18] J. Montesinos and W. Whitten, *Constructions of two-fold branched covering spaces,* Pacific J. Math. **125** (1986), no. 2, 415-446.

[19] B. Owens, *Unknotting information from Heegaard Floer homology,* preprint, math.GT/0506485.

[20] B. Owens and S. Strle, *Rational homology spheres and the four-ball genus of knots,* math.GT/0308073, to appear in Adv. Math.

[21] P. Ozsváth and Z. Szabó, *Holomorphic disks and three-manifold invariants: properties and applications,* Annals of Math. **159** (2004), no. 3, 1159-1245.

[22] P. Ozsváth and Z. Szabó, *Absolutely graded Floer homologies and intersection forms for four-manifolds with boundary,* Adv. Math. **173** (2003), 179-261.

[23] P. Ozsváth and Z. Szabó, *On the Floer homology of plumbed three-manifolds,* Geom. Topol. **7** (2003), 185-224.

[24] P. Ozsváth and Z. Szabó, *Heegaard Floer homology and alternating knots,* Geom. Topol. **7** (2003), 225-254.

[25] P. Ozsváth and Z. Szabó, *Knot Floer homology and the four-ball genus,* Geom. Topol. **7** (2003), 615-639.

[26] P. Ozsváth and Z. Szabó, *Knot Floer homology, genus bounds, and mutation,* Topology Appl. **141** (2004), no. 1-3, 59-85.

[27] P. Ozsváth and Z. Szabó, *On the Heegaard Floer homology of branched double-covers,* Adv. Math. **194** (2005), 1-33.

[28] P. Ozsváth and Z. Szabó, *Knots with unknotting number one and Heegaard Floer homology,* preprint (2004), math.GT/0401426.

[29] L. Ng, *Maximal Thurston-Bennequin number of two-bridge links,* Algebr. Geom. Topol. **1** (2001), 427-434.

[30] O. Plamenevskaya, *Bounds for the Thurston-Bennequin number from Floer homology,* Algebr. Geom. Topol. **4** (2004), 399-406.

[31] J. Rasmussen, *Floer homology and knot complements,* Ph. D. Thesis, Harvard University, math.GT/0306378.

[32] J. Rasmussen, *Khovanov homology and the slice genus,* preprint (2004), math.GT/0402131.

[33] J. Rasmussen, *Lens space surgeries and a conjecture of Goda and Teragaito,* Geom. Topol. **8** (2004), 1013-1031.

[34] D. Rolfsen, *Knots and links,* 2nd ed., Mathematics Lecture Series, **7**, Publish or Perish, Houston (1990).

[35] L. Rudolph, *An obstruction to sliceness via contact geometry and "classical" gauge theory,* Invent. Math. **119** (1995), no. 1, 155-163.

[36] A. Shumakovitch, *KhoHo pari package,* online at `http://www.geometrie.ch/KhoHo`

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