

NATURALITY IN REAL HEEGAARD FLOER THEORY

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ABSTRACT. In a previous paper we defined real Heegaard Floer homology, an invariant of three-manifolds equipped with involutions. Here we prove that real Heegaard Floer homology is natural, and that it admits an action of the equivariant mapping class group. We also establish naturality of real link Floer homology and real sutured Floer homology, and give a definition of involutive real Heegaard Floer homology.

CONTENTS

1. Introduction	1
2. Real Heegaard Floer invariants	4
3. Singularities of real functions	19
4. Families of real gradients	25
5. Constructing Heegaard diagrams	38
6. Simplifying moves and loops of real Heegaard diagrams	64
7. Strong real Heegaard Floer invariants have no monodromy	78
8. Naturality of real Heegaard Floer homology	79
9. Involutive real Heegaard Floer homology	106
Appendix A. Resolutions of codimension-2 bifurcations	108
References	126

1. INTRODUCTION

A real manifold (Y, τ) is an oriented smooth manifold together with a smooth involution $\tau : Y \rightarrow Y$ with codimension-two fixed point set. Real Seiberg-Witten theory [TW09, Nak13, Kat22, KMT25, KMT24, Li, BH] assigns invariants to real three- and four-manifolds, by lifting the involution in an anti-linear way to the spinor bundle, and counting invariant solutions to the Seiberg-Witten equations. Among its best-known applications are proofs that cables of the figure-eight are not slice [KPT26, KPT], the existence of an exotic embedding of $\mathbb{RP}^2$ in S^4 [Miy], and of various exotic involutions on four-manifolds [HKM, Bar].

Ordinary (“unreal”) Seiberg-Witten theory has a symplectic counterpart, Heegaard Floer theory, which is more computable and has numerous applications [OS04b, OS04a]. In [GM], the authors developed real Heegaard Floer homology as the symplectic counterpart of real monopole Floer homology. This allowed for the general calculation of the analogues of Miyazawa’s knot invariant; see [GM, Section 7] and [Sri]. Further, using bordered techniques, Lipshitz and Ozsváth [LO] gave an algorithmic procedure for calculating the (hat) real Heegaard Floer homology of any real three-manifold with connected fixed point set. For general real manifolds, the hat version can be computed combinatorially via nice diagrams, à la Sarkar-Wang [SW10], by the work of [BGX].

We expect that real Heegaard Floer theory will also lead to invariants of real four-manifolds that are more amenable to computations than the gauge-theoretic ones. However, for this

one first needs to prove that the real Heegaard Floer groups are functorial under equivariant cobordisms.

In this paper we take the first step towards functoriality, by proving that it holds under equivariant diffeomorphisms (which can be thought of as the identity cobordism with different identifications at the ends). Most of the work involves establishing naturality: whereas in [GM] we only showed that the isomorphism class of real Heegaard Floer homology is an invariant, here we define the group itself as a natural invariant. Specifically, the real Heegaard Floer homology is constructed from a real Heegaard diagram. In [GM] we showed that a sequence of moves between such diagrams induces an isomorphism on Floer homology, and here we show that this isomorphism is independent of the sequence. This gives a transitive system of groups (as in [JTZ21, Definition 1.1]), and its colimit is the natural invariant.

Definition 1.1. A *based real manifold* is a triple (Y, τ, p) , where (Y, τ) is a real manifold and p is a base point on the fixed point locus Y^τ . We let $\mathbf{RMan}_*$ be the category whose objects are closed, connected, based real three-manifolds (Y, τ, p) , and whose morphisms are basepoint-preserving equivariant diffeomorphisms.

If $\mathbf{k}$ is a field, we let $\mathbf{k}\text{-Vect}$ be the category of $\mathbf{k}$ -vector spaces. Similarly, if R is a ring, we let $R\text{-Mod}$ be the category of modules over R . Let $\mathbb{F}$ be the field of two elements.

Theorem 1. For $\circ \in \{-, \infty, +, \wedge\}$, there are functors

$$HFR^\circ : \mathbf{RMan}_* \rightarrow \mathbb{F}[U]\text{-Mod}$$

such that for any triple (Y, τ, p) , the groups $HFR^\circ(Y, \tau, p)$ are isomorphic to the real Heegaard Floer homology groups defined in [GM]. Moreover, isotopic diffeomorphisms induce identical maps on HFR° .

Definition 1.2. Let (Y, τ) be a closed real three-manifold. Let $\text{Diff}(Y, \tau)$ be the group of equivariant diffeomorphisms of Y and let $\text{Diff}_0(Y, \tau)$ be the subgroup consisting of those diffeomorphisms equivariantly isotopic to the identity. Then, the *equivariant mapping class group* of (Y, τ) is the group

$$\text{MCG}(Y, \tau) = \text{Diff}(Y, \tau) / \text{Diff}_0(Y, \tau).$$

For a based real manifold (Y, τ, p) , the *based equivariant mapping class group* is

$$\text{MCG}(Y, \tau, p) = \text{Diff}(Y, \tau, p) / \text{Diff}_0(Y, \tau, p),$$

where all groups involved consist of maps fixing p .

Theorem 1 has the following immediate consequence.

Corollary 1.3. The real Floer homology groups have well-defined actions of $\text{MCG}(Y, \tau, p)$.

There are analogues of Theorem 1 in a few other related settings.

In [Hen], Hendricks defined an invariant of strongly invertible knots called *real knot Floer homology*. In [Xiab] and [Xiaa], Xiao extended this to *real link Floer homology*, an invariant of multi-based generalized strongly invertible links $\mathbb{L} := (L, \mathbf{w}, \mathbf{z})$ with $L \subset Y$ and $\mathbf{w}$ and $\mathbf{z}$ being collections of basepoints. (See [Xiaa, Definition 4.1] for the details.) If k is the number of link components preserved by τ setwise, and l is the number of pairs of link components interchanged by τ , then the real link Floer homology $HFLR^-$ takes the form of a module over $\mathbb{F}[u_1, \dots, u_k, U_1, \dots, U_l]$. There is also a version $\widehat{HFLR}$, which is an $\mathbb{F}$ -vector space. Let $\mathbf{RLink}_*^{k,l}$ be the category of multi-based generalized strongly invertible links with fixed k and l , with morphisms being equivariant diffeomorphisms of (Y, τ) preserving $\mathbb{L}$.

Theorem 2. *There are functors*

$$\begin{aligned} HFLR^- : \mathbf{RLink}_*^{k,l} &\rightarrow \mathbb{F}[u_1, \dots, u_k, U_1, \dots, U_l]\text{-Mod}, \\ \widehat{HFLR} : \mathbf{RLink}_*^{k,l} &\rightarrow \mathbb{F}\text{-Vect}, \end{aligned}$$

that produce, up to isomorphism, the real link Floer homologies from [Xiab]. Moreover, isotopic diffeomorphisms induce identical maps. Hence, the mapping class group of $\mathbb{L}$ acts on $HFLR^-(\mathbb{L})$ and $\widehat{HFLR}(\mathbb{L})$.

There is also an extension of real Heegaard Floer homology to real sutured manifolds. The resulting invariant is called real sutured Floer homology and is developed in [BGX]. Let $\mathbf{RSut}$ be the category of real sutured manifolds.

Theorem 3. *There is a functor*

$$RSFH : \mathbf{RSut} \rightarrow \mathbb{F}\text{-Vect},$$

that produces, up to isomorphism, the real sutured Floer homologies from [BGX]. Isotopic diffeomorphisms induce identical maps, and therefore the mapping class group of a real sutured manifold acts on the real sutured Floer homology.

The proofs of Theorems 1, 2 and 3 follow the same strategy as in the unreal setting, where naturality was proved by Juhász, Thurston, and Zemke in [JTZ21]. We outline below the main steps, emphasizing what is new compared to [JTZ21].

In Section 2 we set up some of the background. Following [GM], we list the real Heegaard moves between real Heegaard diagrams. These suffice to define a weak real Heegaard invariant. We then list the axioms (corresponding to elementary loops of real Heegaard moves) that are needed to upgrade to a strong real Heegaard invariant; that is, to prove naturality. These axioms parallel those that appear in [JTZ21], with one notable exception: We now encounter a *simple trade*, a loop in which a pair of trivial orbit stabilizations are canceled by a single $\mathbb{Z}/2$ -orbit stabilization. (See Figure 2.4.) We also formulate Theorem 8, that real Heegaard Floer homology and its variants are strong Heegaard invariants, and show that it implies Theorems 1, 2 and 3.

In Section 3 we study the singularities of *real functions* $f : Y \rightarrow [-1, 1]$, that is, those that satisfy $f \circ \tau = -f$. We distinguish between critical points that appear in pairs (related by the involution τ) and those that are fixed by τ . The former can be treated as in [JTZ21], whereas the latter require a new analysis. We classify real singularities up to codimension 2. The main novelty (compared to the unreal case) is that in codimension 2 we may encounter an A_4 singularity on the fixed point set.

In Section 4 we go from singularities of functions to those of real gradient flows. We classify bifurcations of such flows up to codimension 2.

In Section 5 we describe how gradient flow bifurcations (of codimension 0, 1 and 2) produce real Heegaard diagrams, real Heegaard moves, and loops of such moves. We draw these explicitly. In particular, we encounter a new move called a *crossover*, coming from the failure of a transversality condition on the fixed point set.

In Section 6 we decompose the real Heegaard moves from Section 5 into simpler ones. For example, the crossover is a combination of a stabilization, some handleslides, and a destabilization. We also decompose the loops of Heegaard moves into those needed in the definition of a strong real Heegaard invariant. The new A_4 singularity is the origin of the simple trade loop.

In Section 7 we prove that having a strong real Heegaard invariant does indeed lead to naturality.

In Section 8 we check that the axioms in the definition of a strong real Heegaard invariant hold for real Heegaard Floer homology (and its link and sutured variants). We thus establish Theorem 8. Showing that the simple trade induces the identity requires analyzing specific

holomorphic curves, and the analysis is somewhat similar (but not the same) as in the proof of handleswap invariance in [JTZ21, Section 9.3]. In the process we give a cylindrical reformulation of real Heegaard Floer theory, in the manner of Lipshitz's reformulation of the unreal theory [Lip06]. This is, of course, closely related to the real bordered theory of [LO].

Finally, in Section 9 we use the naturality for HFR° to construct an involutive version of real Heegaard Floer homology. Recall that in [HM17], Hendricks and the second author used the conjugation involution ι on the Heegaard Floer complex CF° to define involutive Heegaard Floer homology, $HFI^\circ(Y, \mathfrak{s})$, as the homology of the mapping cone of $1 + \iota$. Since then, involutive Heegaard Floer homology has found various topological applications, e.g., to homology cobordism [DHST23, HHSZ] and concordance [DHST21, DKM⁺24]. We can mimic the construction using real Heegaard Floer complexes and define involutive real Heegaard Floer homology, $HFRI^\circ(Y, \tau, \mathfrak{s})$. Naturality is used in showing that the conjugation involution is well-defined up to chain homotopy. We deduce the following.

Theorem 4. *Let (Y, τ) be a real three-manifold and $\mathfrak{s}$ a self-conjugate real Spin^c structure. For $\circ \in \{-, \infty, +, \wedge\}$, the isomorphism class of $HFRI^\circ(Y, \tau, \mathfrak{s})$ is an invariant of the triple $(Y, \tau, \mathfrak{s})$.*

Note on AI use. We used Gemini Deep Think and Claude to produce first drafts of the proofs of Propositions 3.8 and 4.22, Corollary 4.23, Lemmas 5.10, 5.11 and 5.12. The proofs were checked and then rewritten by the authors.

During the revision process we used GPT-5.6 Sol to look for typographical and mathematical errors, which we then fixed.

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2. REAL HEEGAARD FLOER INVARIANTS

2.1. Real sutured manifolds. As in [JTZ21], we will work with sutured manifolds, so that our naturality results will extend to other variants of real Heegaard Floer homology (sutured, knot, and link; see [BGX, Xiab].) We briefly review the relevant objects, closely following [JTZ21, Section 2].

Definition 2.1. A *sutured manifold* (Y, γ) is a compact, oriented 3-manifold Y with boundary, together with a set $\gamma \subset \partial Y$ of pairwise disjoint annuli. Furthermore, the interior of each component of γ contains a homologically nontrivial oriented simple closed curve, called a *suture*. The union of the sutures will be denoted $s(\gamma)$. In addition, every component of $R(\gamma) = \partial Y \setminus \text{Int}(\gamma)$ is oriented. Define $R_+(\gamma)$ (respectively $R_-(\gamma)$) to be the set of those components of $\partial Y \setminus \text{Int}(\gamma)$ whose orientations agree (respectively disagree) with the orientation of ∂Y , or equivalently, whose normal vectors point out of (respectively into) Y . The orientation on $R(\gamma)$ must be coherent with respect to $s(\gamma)$; i.e., if η is a component of $\partial R(\gamma)$ and is given the boundary orientation, then η must represent the same homology class in $H_1(\gamma)$ as some suture.

Remark 2.2. We will always assume our sutured manifolds are *proper*, which is to say that the maps $\pi_0(\gamma) \rightarrow \pi_0(\partial Y)$ and $\pi_0(\partial Y) \rightarrow \pi_0(Y)$ are surjective. We will also assume our manifolds are *balanced*, i.e., $\chi(R_+(\gamma)) = \chi(R_-(\gamma))$.

Definition 2.3. A *real sutured manifold* (Y, γ, τ) is a sutured manifold (Y, γ) equipped with an orientation-preserving involution τ whose fixed point set is codimension-2 and such that τ exchanges R_+ and R_- .

We note that because of the symmetry condition, (Y, γ, τ) is necessarily balanced.

2.2. Real sutured diagrams.

Definition 2.4. Let Σ be a compact oriented surface with boundary. An *attaching set* in Σ is a one-dimensional smooth submanifold $\boldsymbol{\eta} \subset \text{Int}(\Sigma)$ such that each component of $\Sigma \setminus \boldsymbol{\eta}$ contains at least one component of $\partial\Sigma$. The isotopy class of $\boldsymbol{\eta}$ will be denoted $[\boldsymbol{\eta}]$.

Definition 2.5. A sutured manifold (Y, γ) is a *sutured compression body* if either there is an attaching set $\boldsymbol{\eta} \subset R_+(\gamma)$ such that if we compress $R_+(\gamma)$ inside Y along all the components of $\boldsymbol{\eta}$, we get a surface that is isotopic to $R_-(\gamma)$ relative to γ , or there is an attaching set $\boldsymbol{\eta} \subset R_-(\gamma)$ such that if we compress $R_-(\gamma)$ inside Y along all the components of $\boldsymbol{\eta}$, we get a surface that is isotopic to $R_+(\gamma)$ relative to γ . We call $\boldsymbol{\eta}$ an *attaching set* for (Y, γ) .

Given a set of attaching curves $\boldsymbol{\eta}$ in Σ , we define $C(\boldsymbol{\eta})$ to be the sutured compression body obtained by attaching 3-dimensional 1-handles to $\Sigma \times [0, 1]$ along $\boldsymbol{\eta} \times \{0\}$ and $\gamma = \partial\Sigma \times [0, 1]$. We will write $C_{\pm}(\boldsymbol{\eta})$ for the boundary components of $C(\boldsymbol{\eta})$ which do not include the sutures. If $\boldsymbol{\eta}$ and $\boldsymbol{\eta}'$ are two attaching sets in Σ , then we say they are *compression equivalent*, and we write $\boldsymbol{\eta} \sim \boldsymbol{\eta}'$, if there is a diffeomorphism $d: C(\boldsymbol{\eta}) \rightarrow C(\boldsymbol{\eta}')$ such that $d|_{C_-(\boldsymbol{\eta})}$ is the identity. This is an equivalence relation that descends to isotopy classes of attaching sets. So we will write $[\boldsymbol{\eta}] \sim [\boldsymbol{\eta}']$ if $\boldsymbol{\eta} \sim \boldsymbol{\eta}'$.

Definition 2.6. A *real sutured diagram* is a quadruple $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \tau)$, where Σ is a compact oriented surface with boundary, and $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are two attaching sets in Σ with the property that $\tau(\boldsymbol{\alpha}) = \boldsymbol{\beta}$. An *isotopy diagram* is a quadruple $(\Sigma, [\boldsymbol{\alpha}], [\boldsymbol{\beta}], \tau)$, where $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \tau)$ is a real sutured diagram.

Remark 2.7. Following [JTZ21], given isotopy classes $A = [\boldsymbol{\alpha}]$ and $B = [\boldsymbol{\beta}]$, we will usually write $H = (\Sigma, A, B, \tau)$ for an isotopy class of diagrams and $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \tau)$ for a representative of the class.

Definition 2.8. Let (Y, γ, τ) be a real sutured manifold. Then $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \tau)$ is an (*embedded*) *real sutured diagram* of (Y, γ, τ) if

- (1) $\Sigma \subset Y$ is an oriented surface with $\partial\Sigma = s(\gamma)$ as oriented 1-manifolds,
- (2) the components of $\boldsymbol{\alpha}$ bound disjoint disks to the negative side of Σ ,
- (3) if we compress Σ along $\boldsymbol{\alpha}$, we get a surface isotopic to $R_-(\gamma)$ relative to γ .

In short, Σ splits (Y, γ, τ) into two symmetric sutured compression bodies, with attaching sets $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$.

Lemma 2.9. *Every real sutured manifold (Y, γ, τ) admits a real sutured diagram.*

Proof. This follows from [Nag79]. Alternatively, this follows from the Morse theoretic analyses of Section 3 and Section 4. $\square$

Definition 2.10. Let (Y, γ, τ) be a real sutured manifold. A *real Heegaard surface* of (Y, γ, τ) is an oriented surface $\Sigma \subset Y$ such that $\partial\Sigma = s(\gamma)$ and Σ divides (Y, γ, τ) into two sutured compression bodies exchanged by the involution.

2.3. Moves on diagrams and weak Heegaard invariants. Let us introduce a notion of equivalence for isotopy diagrams.

Definition 2.11. We say that the isotopy diagrams (Σ_1, A_1, B_1) and (Σ_2, A_2, B_2) are *real-equivalent* if $\Sigma_1 = \Sigma_2$ and $A_1 \sim A_2$. Note that the symmetry condition forces $B_1 \sim B_2$ as well.

Definition 2.12. (cf. [JTZ21, Definition 2.10]) Let η_1 and η_2 be two disjoint simple closed curves in Σ , and fix an embedded arc a from η_1 to η_2 whose interior is disjoint from $\eta_1 \cup \eta_2$ and from $\partial\Sigma$. Then a regular neighborhood of the graph $\eta_1 \cup a \cup \eta_2$ is a planar surface with three boundary components: one is isotopic to η_1 , the other is isotopic to η_2 , and the third is a new curve η'_1 that we call the curve obtained by *handlesliding η_1 over η_2 along the arc a* .

Suppose $\boldsymbol{\eta}$ and $\boldsymbol{\eta}'$ are two systems of attaching circles. We say that $\boldsymbol{\eta}$ and $\boldsymbol{\eta}'$ are *related by a handleslide* if there are components η_1 and η_2 of $\boldsymbol{\eta}$ and a component η'_1 of $\boldsymbol{\eta}'$ such that η'_1 can be obtained by handle-sliding η_1 over η_2 along some arc whose interior is disjoint from $\boldsymbol{\eta}$, and $\boldsymbol{\eta}' = (\boldsymbol{\eta} \setminus \eta_1) \cup \eta'_1$. If D and D' are isotopy classes of attaching sets, then they are related by a handleslide if they have representatives $\boldsymbol{\eta}$ and $\boldsymbol{\eta}'$, respectively, such that $\boldsymbol{\eta}$ and $\boldsymbol{\eta}'$ are related by a handleslide.

We note that by [JTZ21, Lemma 2.11] we have $\boldsymbol{\eta} \sim \boldsymbol{\eta}'$ if and only if they are related by a sequence of handleslides.

We now recall the basic moves which relate real sutured Heegaard diagrams.

Definition 2.13. We say that the real sutured diagram $(\Sigma_2, \boldsymbol{\alpha}_2, \boldsymbol{\beta}_2, \tau_2)$ is obtained from $(\Sigma_1, \boldsymbol{\alpha}_1, \boldsymbol{\beta}_1, \tau_1)$ by a *real handleslide* if

- (1) $\Sigma_1 = \Sigma_2$,
- (2) $\tau_1 = \tau_2$,
- (3) $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ are related by a handleslide (and necessarily $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_2$ are also related by a handleslide).

In the real setting, there are two different stabilization operations, appearing in both trivial and $\mathbb{Z}/2$ -orbits.

Definition 2.14. The real sutured diagram $(\Sigma_2, \boldsymbol{\alpha}_2, \boldsymbol{\beta}_2, \tau_2)$ is obtained from $(\Sigma_1, \boldsymbol{\alpha}_1, \boldsymbol{\beta}_1, \tau_1)$ by a $\mathbb{Z}/2$ -*stabilization* if

- there is a disk $D \subset \Sigma_1$ and a punctured torus $T \subset \Sigma_2$ such that $\Sigma_1 \setminus D = \Sigma_2 \setminus T$, $D \cap \tau_1(D) = \emptyset$ and $T \cap \tau_2(T) = \emptyset$,
- $\boldsymbol{\alpha}_1 = \boldsymbol{\alpha}_2 \cap (\Sigma_2 \setminus (T \cup \tau_2(T)))$,
- $\boldsymbol{\alpha}_2 \cap T$ and $\boldsymbol{\beta}_2 \cap T$ are simple closed curves that intersect each other transversely in a single point; the same holds for $\boldsymbol{\alpha}_2 \cap \tau_2(T)$ and $\boldsymbol{\beta}_2 \cap \tau_2(T)$.

In the reverse direction, we also say that $(\Sigma_1, \boldsymbol{\alpha}_1, \boldsymbol{\beta}_1, \tau_1)$ is obtained from $(\Sigma_2, \boldsymbol{\alpha}_2, \boldsymbol{\beta}_2, \tau_2)$ by a $\mathbb{Z}/2$ -*destabilization*. See Figure 2.1.

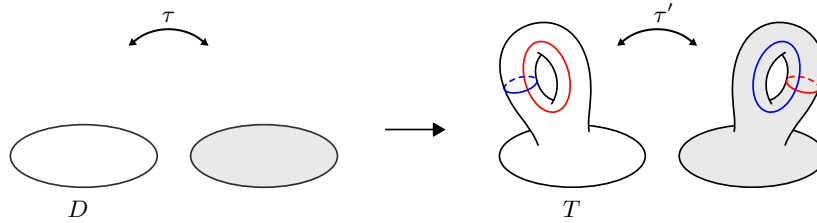


FIGURE 2.1. A (simple) $\mathbb{Z}/2$ -stabilization.

Let H_1 and H_2 be isotopy diagrams. Then H_2 is obtained from H_1 by a $\mathbb{Z}/2$ -(de)stabilization if they have representatives $(\Sigma_2, \alpha_2, \beta_2, \tau_2)$ and $(\Sigma_1, \alpha_1, \beta_1, \tau_1)$, respectively, such that $(\Sigma_2, \alpha_2, \beta_2, \tau_2)$ is obtained from $(\Sigma_1, \alpha_1, \beta_1, \tau_1)$ by a $\mathbb{Z}/2$ -(de)stabilization.

Definition 2.15. The sutured diagram $(\Sigma_2, \alpha_2, \beta_2, \tau_2)$ is obtained from $(\Sigma_1, \alpha_1, \beta_1, \tau_1)$ by a $\{1\}$ -stabilization if

- there is a disk $D \subset \Sigma_1$ which is fixed setwise by τ_1 and a punctured torus $T \subset \Sigma_2$ fixed setwise by τ_2 such that $\Sigma_1 \setminus D = \Sigma_2 \setminus T$,
- $\alpha_1 = \alpha_2 \cap (\Sigma_2 \setminus T)$,
- $\alpha_2 \cap T$ and $\beta_2 \cap T$ are simple closed curves that intersect each other transversely in a single point and are exchanged by the involution.

We also say that $(\Sigma_1, \alpha_1, \beta_1)$ is obtained from $(\Sigma_2, \alpha_2, \beta_2)$ by a $\{1\}$ -destabilization. See Figure 2.2.

Just as before, if H_1 and H_2 are isotopy diagrams, then H_2 is obtained from H_1 by a $\{1\}$ -(de)stabilization if they have representatives $(\Sigma_2, \alpha_2, \beta_2, \tau_2)$ and $(\Sigma_1, \alpha_1, \beta_1, \tau_1)$, respectively, such that $(\Sigma_2, \alpha_2, \beta_2, \tau_2)$ is obtained from $(\Sigma_1, \alpha_1, \beta_1, \tau_1)$ by a $\{1\}$ -(de)stabilization.

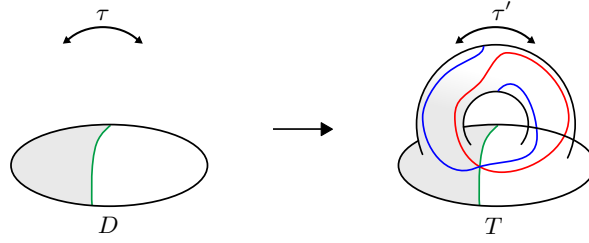


FIGURE 2.2. A (simple) $\{1\}$ -stabilization.

Remark 2.16. We will often write $\mathcal{O}$ -stabilization when we do not want to distinguish between the two orbit types of stabilizations.

If $d: \Sigma \rightarrow \Sigma'$ is an equivariant diffeomorphism of surfaces and C is an isotopy class of attaching sets in Σ , then $d(C)$ is defined as $[d(\eta)]$, where η is an arbitrary attaching set representing C .

Definition 2.17. Given isotopy diagrams $H_1 = (\Sigma_1, A_1, B_1, \tau_1)$ and $H_2 = (\Sigma_2, A_2, B_2, \tau_2)$, a diffeomorphism¹ $d: H_1 \rightarrow H_2$ is an orientation-preserving equivariant diffeomorphism $d: \Sigma_1 \rightarrow \Sigma_2$ such that $d(A_1) = A_2$ and $d(B_1) = B_2$.

Definition 2.18. Let $\mathcal{G}^R$ be the graph whose class of vertices $|\mathcal{G}^R|$ consists of real isotopy diagrams and, for $H_1, H_2 \in |\mathcal{G}^R|$, the edges $\mathcal{G}^R(H_1, H_2)$ can be written as a union

$$\mathcal{G}^R(H_1, H_2) = \mathcal{G}_{\alpha\beta}^R(H_1, H_2) \cup \mathcal{G}_{\{1\}}^R(H_1, H_2) \cup \mathcal{G}_{\mathbb{Z}/2}^R(H_1, H_2) \cup \mathcal{G}_{\text{diff}}^R(H_1, H_2).$$

The set $\mathcal{G}_{\alpha\beta}^R(H_1, H_2)$ consists of a single arrow if H_1 and H_2 are real-equivalent and is empty otherwise. The set $\mathcal{G}_{\{1\}}^R(H_1, H_2)$ consists of a single arrow if H_2 is obtained from H_1 by a $\{1\}$ -stabilization or a $\{1\}$ -destabilization and is empty otherwise. Similarly, set $\mathcal{G}_{\mathbb{Z}/2}^R(H_1, H_2)$ consists of a single arrow if H_2 is obtained from H_1 by a $\mathbb{Z}/2$ -stabilization or a $\mathbb{Z}/2$ -destabilization and is empty otherwise. Finally, $\mathcal{G}_{\text{diff}}^R(H_1, H_2)$ consists of all diffeomorphisms from H_1 to H_2 .

¹We emphasize that we define diffeomorphisms of real Heegaard diagrams to be equivariant; for this reason we will not always include the adjective “equivariant”.

Proposition 2.19. *The isotopy diagrams $H_1, H_2 \in |\mathcal{G}^R|$ can be connected by an oriented path if and only if they define equivariantly diffeomorphic real sutured manifolds. Furthermore, the existence of an unoriented path from H_1 to H_2 implies the existence of an oriented one.*

Proof. This follows from the work of Nagase [Nag79]; also see [GM, Lemma 3.16]. $\square$

Definition 2.20. Let $\mathcal{S}^R$ be a set of diffeomorphism types of real sutured manifolds, and let $\mathcal{C}$ be any category. Let $\mathcal{G}^R(\mathcal{S}^R)$ be the full subgraph of $\mathcal{G}^R$ spanned by those isotopy diagrams H for which $S(H) \in \mathcal{S}^R$. A *weak Heegaard invariant* of $\mathcal{S}^R$ is a morphism of graphs $F: \mathcal{G}^R(\mathcal{S}^R) \rightarrow \mathcal{C}$ such that for every arrow e of $\mathcal{G}^R(\mathcal{S}^R)$ the image $F(e)$ is an isomorphism.

Definition 2.21. As in [JTZ21] we will consider three different classes.

- (1) $\mathcal{S}_{man}^R$, the set of $[Y(p, \xi), \tau]$, where (Y, p, τ) is a based real manifold and $\xi = (\xi_1, \xi_2, \xi_3)$ is a framing of p so that the restriction $d\tau|_{\text{span}(\xi_1, \xi_2)}$ is orientation reversing (this implies that ξ_3 points in the direction tangent to the fixed point set).
- (2) $\mathcal{S}_{link}^R$, the set of $[Y(L, \mathbf{w}, \mathbf{z}), \tau]$, where (Y, τ) is a real manifold and $L \subset Y$ is a generalized strongly invertible link with at least one pair of $\mathbf{w}$ and $\mathbf{z}$ markings on each component (which are also fixed setwise by τ).
- (3) $\mathcal{S}_{sut}^R$, the set of all real sutured manifolds $[(Y, \gamma, \tau)]$.

Theorem 5 ([GM]). *The functors*

$$\widehat{CFR}, CFR^-, CFR^+, CFR^\infty: \mathcal{G}^R(\mathcal{S}_{man}^R) \rightarrow \mathcal{C}$$

are weak real Heegaard invariants of $\mathcal{S}_{man}^R$ (where the U -action is trivial on $\widehat{CFR}$). Here, $\mathcal{C}$ is the homotopy category of $\mathbb{F}[U]$ -complexes.

There are also variants of real Heegaard Floer homology for knots [Hen], links [Xiab], and sutured manifolds. The invariance results of [Xiab] and [BGX] can also be phrased in this language.

Theorem 6 ([Xiaa]). *The functors*

$$\widehat{CFLR}, CFLR^-, CFLR^+, CFLR^\infty: \mathcal{G}^R(\mathcal{S}_{link}^R) \rightarrow \mathcal{C}$$

are weak real Heegaard invariants of $\mathcal{S}_{link}^R$ (where the polynomial action is trivial on $\widehat{CFLR}$). Here, $\mathcal{C}$ is the homotopy category of curved $\mathbb{F}[u_1, \dots, u_k, U_1, \dots, U_\ell]$ -complexes.

Theorem 7 ([BGX]). *The functor*

$$RSFC: \mathcal{G}^R(\mathcal{S}_{sut}^R) \rightarrow \mathcal{C}$$

is a weak real Heegaard invariant of $\mathcal{S}_{sut}^R$. Here, $\mathcal{C}$ is the homotopy category of $\mathbb{F}$ -complexes.

2.4. Strong Real Heegaard Floer Invariants. We now outline the additional axioms which must be satisfied in order to obtain natural invariants of real manifolds.

Definition 2.22. Let $H_i = (\Sigma_i, [\alpha_i], [\beta_i], \tau_i)$ be real isotopy diagrams for $1 \leq i \leq 4$. A *distinguished rectangle* in $\mathcal{G}^R$ is a subgraph

$$\begin{array}{ccc} H_1 & \xrightarrow{e} & H_2 \\ \downarrow f & & \downarrow g \\ H_3 & \xrightarrow{h} & H_4 \end{array}$$

of $\mathcal{G}^R$ that satisfies one of the following properties:

- (1) Either both e and h are real-equivalences while f and g are both $\mathcal{O}$ -stabilizations.

- (2) Both e and h are real-equivalences while f and g are both equivariant diffeomorphisms. In this case, we necessarily have $\Sigma_1 = \Sigma_2$ and $\Sigma_3 = \Sigma_4$. We require, in addition, that the diffeomorphisms $f: \Sigma_1 \rightarrow \Sigma_3$ and $g: \Sigma_2 \rightarrow \Sigma_4$ are the same.
- (3) The maps e and h are $\mathcal{O}_1$ -stabilizations while f and g are $\mathcal{O}_2$ -stabilizations, such that there are disjoint orbits of disks $\mathcal{O}_1 \times D_1$ and $\mathcal{O}_2 \times D_2$ and disjoint orbits of punctured tori $\mathcal{O}_1 \times T_1, \mathcal{O}_2 \times T_2 \subset \Sigma_4$ satisfying $\Sigma_1 \setminus ((\mathcal{O}_1 \times D_1) \cup (\mathcal{O}_2 \times D_2)) = \Sigma_4 \setminus (\mathcal{O}_1 \times T_1) \cup (\mathcal{O}_2 \times T_2)$, and such that $\Sigma_2 = (\Sigma_1 \setminus (\mathcal{O}_1 \times D_1)) \cup (\mathcal{O}_1 \times T_1)$ and $\Sigma_3 = (\Sigma_1 \setminus (\mathcal{O}_2 \times D_2)) \cup (\mathcal{O}_2 \times T_2)$.
- (4) The maps e and h are $\mathcal{O}$ -stabilizations, while f and g are equivariant diffeomorphisms. Furthermore, there are orbits of disks $(\mathcal{O} \times D) \subset \Sigma_1$ and $(\mathcal{O} \times D') \subset \Sigma_3$ and orbits of punctured tori $T \times \mathcal{O} \subset \Sigma_2$ and $\mathcal{O} \times T' \subset \Sigma_4$ such that $\Sigma_1 \setminus (\mathcal{O} \times D) = \Sigma_2 \setminus (\mathcal{O} \times T)$ and $\Sigma_3 \setminus (\mathcal{O} \times D') = \Sigma_4 \setminus (\mathcal{O} \times T')$, and the diffeomorphisms f, g satisfy $f(\mathcal{O} \times D) = \mathcal{O} \times D'$, $g(\mathcal{O} \times T) = \mathcal{O} \times T'$, and $f|_{\Sigma_1 \setminus (\mathcal{O} \times D)} = g|_{\Sigma_2 \setminus (\mathcal{O} \times T)}$.

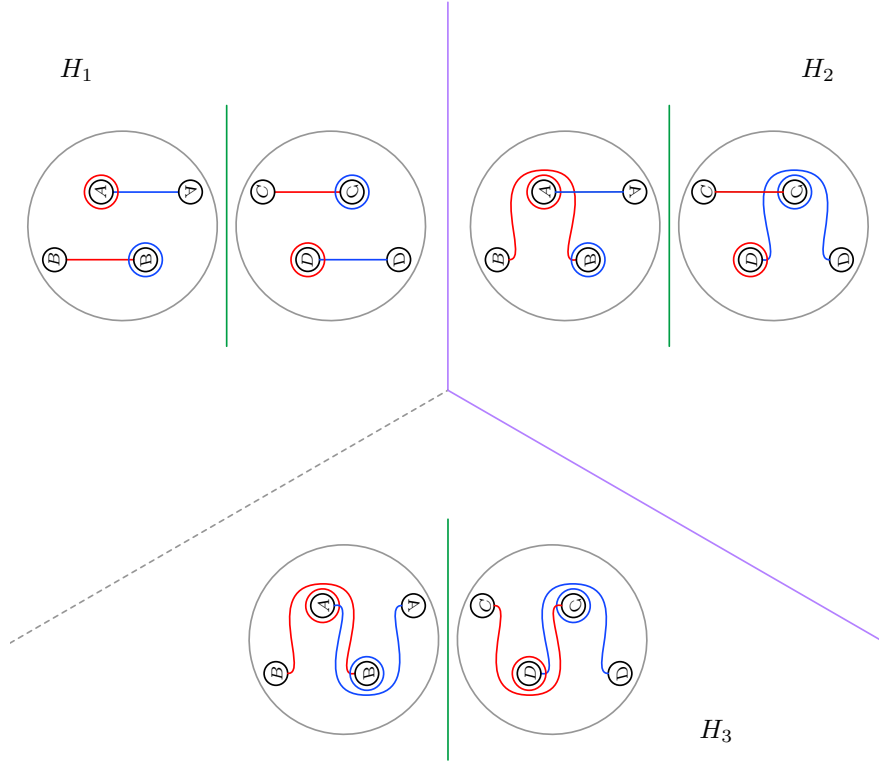
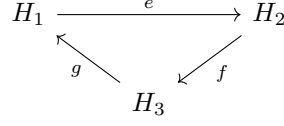


FIGURE 2.3. A simple real handleswap loop. As usual in Heegaard Floer theory, the alpha curves are in red and the beta curves in blue. Also, the fixed point set is in green. The purple lines represent real equivalences, and the dashed line represents a diffeomorphism. See Remark 4.14.

Definition 2.23. A *simple real handleswap* is a subgraph of $\mathcal{G}^R$ of the form

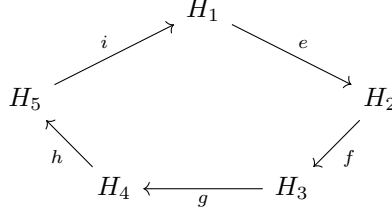


such that

- (1) $H_i = (\Sigma_0 \# \Sigma \# \Sigma_1, [\alpha_i], [\beta_i], \tau)$ are isotopy diagrams for $i \in \{1, 2, 3\}$, where Σ_0 is a genus two surface,
- (2) e and f are real equivalences and g is an equivariant diffeomorphism,
- (3) in the surface $P = (\Sigma_0 \# \Sigma \# \Sigma_1) \setminus (\Sigma)$ (which is a pair of punctured genus two surfaces), the above triangle is conjugate to the triangle in Figure 2.3; i.e., there is a diffeomorphism that throws $P \cap H_i$ onto the pictures in the green circles, sending the α -circles in P to the two red circles, and the β -circles in P to the two blue circles,
- (4) Σ_0 and Σ_1 are exchanged by τ ,
- (5) in Σ , the diagrams H_1 , H_2 , and H_3 are identical.

Remark 2.24. We could have combined the two real handleslides from Definition 2.23 into a single real equivalence. We chose to keep them separate to be in line with [JTZ21, Definition 2.32].

Definition 2.25. A *simple trade* is a subgraph of $\mathcal{G}_R$ of the form



such that

- (1) $H_1 = (\Sigma \# \Sigma_0, [\alpha_1], [\beta_1], \tau_1)$, $H_2 = (\Sigma \# \Sigma_0 \# \Sigma_1, [\alpha_2], [\beta_2], \tau_2)$, $H_i = (\Sigma \# \Sigma_0 \# \Sigma_1 \# \Sigma_2, [\alpha_i], [\beta_i], \tau_i)$ for $i \in \{3, 4\}$, and $H_5 = (\Sigma \# \Sigma_0, [\alpha_5], [\beta_5], \tau_5)$ are isotopy diagrams where Σ_i are genus 1 surfaces,
- (2) e and f are $\{1\}$ -stabilizations, g is a real equivalence (consisting of three real handleslides: first G over F , then F over R , and then R over F), h is an equivariant diffeomorphism, and i is a $\mathbb{Z}/2$ -destabilization,
- (3) in the complement of Σ , the above pentagon is conjugate to the pentagon in Figure 2.4; i.e., there is a diffeomorphism that throws $P \cap H_i$ onto the pictures in the gray circles, sending the α -circles in P to the three red circles, and the β -circles in P to the three blue circles.

Definition 2.26. Let $\mathcal{S}^R$ be a set of equivariant diffeomorphism types of real sutured manifolds. A *strong real Heegaard invariant* of $\mathcal{S}^R$ is a weak Heegaard invariant $F: \mathcal{G}^R(\mathcal{S}^R) \rightarrow \mathcal{C}$ that satisfies the following axioms:

- (1) **Functoriality:** The restrictions of F to $\mathcal{G}_{\alpha\beta}^R(\mathcal{S}^R)$ and $\mathcal{G}_{\text{diff}}^R(\mathcal{S}^R)$ are functors to $\mathcal{C}$. Furthermore, if $e: H_1 \rightarrow H_2$ is an $\mathcal{O}$ -stabilization and $e': H_2 \rightarrow H_1$ is the corresponding $\mathcal{O}$ -destabilization, then $F(e') \sim F(e)^{-1}$.

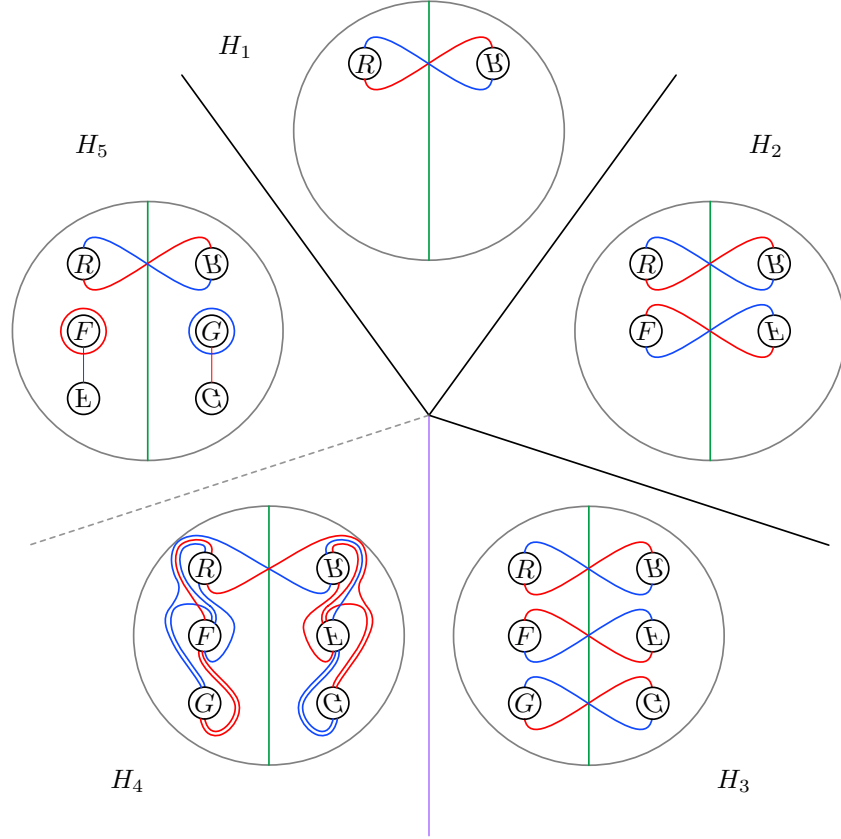


FIGURE 2.4. A simple trade loop. The black lines indicate (de)stabilizations. The letters in the circles indicate which feet of the handles are identified, and how. (We emphasize that for a $\{1\}$ -stabilization, while the attaching circles are symmetric along a vertical axis, the gluing is by reflection in a horizontal axis.) We chose the letters R , F , G because they have no symmetries, so their position fully describes the gluing.

(2) **Commutativity:** For every distinguished rectangle

$$\begin{array}{ccc} H_1 & \xrightarrow{e} & H_2 \\ \downarrow f & & \downarrow g \\ H_3 & \xrightarrow{h} & H_4 \end{array}$$

in $\mathcal{G}^R(\mathcal{S}^R)$, we have $F(g) \circ F(e) \sim F(h) \circ F(f)$.

(3) **Continuity:** If $H \in |\mathcal{G}^R(\mathcal{S}^R)|$ and $e \in \mathcal{G}_{\text{diff}}^R(H, H)$ is a diffeomorphism isotopic to Id_Σ , then $F(e) \sim \text{Id}_{F(H)}$.

(4) **Handleswap invariance:** For every simple handleswap

$$\begin{array}{ccc} H_1 & & \\ \uparrow g & \searrow e & \\ H_3 & \xleftarrow{f} & H_2 \end{array}$$

in $\mathcal{G}^R(\mathcal{S}^R)$, we have $F(g) \circ F(f) \circ F(e) \sim \text{Id}_{F(H_1)}$.

(5) **Trade invariance:** For every simple trade pentagon

$$\begin{array}{ccccc} & & H_1 & & \\ & \nearrow i & & \searrow e & \\ H_5 & & & & H_2 \\ & \nwarrow h & & \swarrow f & \\ & & H_4 & \xleftarrow{g} & H_3 \end{array}$$

in $\mathcal{G}^R(\mathcal{S}^R)$, we have $F(i) \circ F(h) \circ F(g) \circ F(f) \circ F(e) \sim \text{Id}_{F(H_1)}$.

The main result of this paper is the following:

Theorem 8. *The following are strong real Heegaard invariants:*

- (1) *Real sutured Floer homology;*
- (2) *Real Heegaard Floer homology;*
- (3) *Real link Floer homology.*

Definition 2.27. Suppose that H_1 and H_2 are two isotopy diagrams of (Y, γ, τ) with $H_i = (\Sigma_i, A_i, B_i, \tau_i)$, and let $\iota_i : \Sigma_i \rightarrow Y$ be the inclusion for $i \in \{1, 2\}$. Then a diffeomorphism $d : H_1 \rightarrow H_2$ is *isotopic to the identity in Y* if $\iota_2 \circ d : \Sigma_1 \rightarrow Y$ is equivariantly isotopic to $\iota_1 : \Sigma_1 \rightarrow Y$ relative to $s(\gamma)$.

Definition 2.28. Let (Y, γ, τ) be a real sutured manifold. Then $\mathcal{G}_{(Y, \gamma, \tau)}^R$ is the subgraph of $\mathcal{G}^R$ whose vertex set $|\mathcal{G}_{(Y, \gamma, \tau)}^R|$ consists of all isotopy diagrams of (Y, γ, τ) . The set of edges between $H_1, H_2 \in |\mathcal{G}_{(Y, \gamma, \tau)}^R|$ is defined to be

$$\mathcal{G}_{(Y, \gamma, \tau)}^R(H_1, H_2) = \mathcal{G}_{\alpha\beta}^R(H_1, H_2) \cup \mathcal{G}_{\{1\}}^R(H_1, H_2) \cup \mathcal{G}_{\mathbb{Z}/2}^R(H_1, H_2) \cup (\mathcal{G}^R)_{\text{diff}}^0(H_1, H_2),$$

where $(\mathcal{G}^R)_{\text{diff}}^0(H_1, H_2)$ is the set of diffeomorphisms from H_1 to H_2 isotopic to the identity in Y .

In Section 7 we will establish a stronger version of Nagase's result:

Proposition 2.29. *Any two vertices in $\mathcal{G}_{(Y, \gamma, \tau)}^R$ can be connected by an oriented path.*

Definition 2.30. Given a weak real Heegaard invariant $F : \mathcal{G}^R(\mathcal{S}^R) \rightarrow \mathcal{C}$ and an oriented path η in $\mathcal{G}^R(\mathcal{S}^R)$ of the form

$$H_0 \xrightarrow{e_1} H_1 \xrightarrow{e_2} \dots \xrightarrow{e_n} H_n,$$

define $F(\eta)$ to be the isomorphism

$$F(e_n) \circ \dots \circ F(e_1) : F(H_0) \rightarrow F(H_n).$$

Naturality of a real Heegaard invariant means that the equivalence $F(\eta)$ is independent of the choice of path η up to homotopy. For strong real Heegaard invariants, this is indeed the case; in Section 7 we will prove the following.

Theorem 9. *Let $\mathcal{S}^R$ be a set of diffeomorphism types of real sutured manifolds containing $[(Y, \gamma, \tau)]$ and let $F : \mathcal{G}^R(\mathcal{S}^R) \rightarrow \mathcal{C}$ be a strong real Heegaard invariant. Given real isotopy diagrams $H, H' \in |\mathcal{G}_{(Y, \gamma, \tau)}^R|$ and any two oriented paths η and ν in $\mathcal{G}_{(Y, \gamma, \tau)}^R$ from H to H' , we have*

$$F(\eta) = F(\nu).$$

In particular, for any two vertices H and H' , there is a canonical homotopy equivalence between them.

Definition 2.31. If H and H' are isotopy diagrams of (Y, γ, τ) , then let

$$F_{H \rightarrow H'} = F(\eta),$$

where η is an arbitrary oriented path connecting H to H' . By Theorem 9, the map $F_{H \rightarrow H'}$ does not depend on the choice of η up to homotopy.

2.5. Transitive systems of curved chain complexes. Finally, before proceeding to our analysis of the space of real Heegaard diagrams, we recall the notion of a transitive system.

Definition 2.32. Let $\mathcal{C}$ be a category and A a set. Let $\text{Trans}_A(\mathcal{C})$ be the category of transitive systems over $\mathcal{C}$ indexed by A . An object of this category consists of a map $S : A \rightarrow \text{Ob}(\mathcal{C})$ together with a collection of distinguished morphisms $\Psi_{a \rightarrow b} : S(a) \rightarrow S(b)$ for all $a, b \in A$ which satisfy

- (1) $\Psi_{a \rightarrow a} = \text{id}$;
- (2) $\Psi_{b \rightarrow c} \circ \Psi_{a \rightarrow b} = \Psi_{a \rightarrow c}$.

We will at times write objects in this category as $\{S(a)\}_{a \in A}, \{\Psi_{a \rightarrow b}\}_{a, b \in A}$ if we want to emphasize the collection of morphisms; we will write S when the distinguished morphisms are clear from the context.

Definition 2.33. If S_1 and S_2 are two transitive systems over $\mathcal{C}$ indexed by A_1 and A_2 , then a morphism of transitive systems $F : S_1 \rightarrow S_2$ is a collection of morphisms $F_{a_1, a_2} : S_1(a_1) \rightarrow S_2(a_2)$ indexed over $A_1 \times A_2$ satisfying

$$\begin{array}{ccc} S_1(a_1) & \xrightarrow{F_{a_1, a_2}} & S_2(a_2) \\ \Psi_{a_1 \rightarrow b_1} \downarrow & & \downarrow \Psi_{a_2 \rightarrow b_2} \\ S_1(b_1) & \xrightarrow{F_{b_1, b_2}} & S_2(b_2) \end{array}$$

for all pairs $(a_1, b_1), (a_2, b_2) \in A_1 \times A_2$.

The identity map of a transitive system $(\{S(a)\}_{a \in A}, \{\Psi_{a \rightarrow b}\}_{a, b \in A})$ is defined by the diagrams

$$\begin{array}{ccc} S(a) & \xrightarrow{\Psi_{a \rightarrow b}} & S(b) \\ \Psi_{a \rightarrow c} \downarrow & & \downarrow \Psi_{b \rightarrow d} \\ S(c) & \xrightarrow{\Psi_{c \rightarrow d}} & S(d). \end{array}$$

for all $a, b, c, d \in A$. A morphism of transitive systems $F : S_1 \rightarrow S_2$ is called an *isomorphism* if there is a morphism $G : S_2 \rightarrow S_1$ such that $F \circ G$ and $G \circ F$ are the identity morphisms.

Theorem 10. *Let $F : \mathcal{G}^R(\mathcal{S}^R) \rightarrow \mathcal{C}$ be a strong real Heegaard invariant. Then, the map $|\mathcal{G}_{(Y, \gamma, \tau)}^R| \rightarrow \text{Ob}(\mathcal{C})$ given by $H \mapsto F(H)$, together with the collection $\{F_{H \rightarrow H'}\}_{H, H' \in |\mathcal{G}_{(Y, \gamma, \tau)}^R|}$, forms a transitive system. Furthermore, diffeomorphisms of real manifolds induce isomorphisms of transitive systems.*

Proof. Given $H \in |\mathcal{G}_{(Y,\gamma,\tau)}^R|$, the $F_{H \rightarrow H}$ can be represented by $F(\eta)$ for any loop $\eta \in \pi_1(\mathcal{G}_{(Y,\gamma,\tau)}^R, H)$. In particular, we may take η to be the constant map, $*_H$. Therefore, $F_{H \rightarrow H} = F(*_H) = \text{id}_{F(H)}$.

If $H, H', H'' \in |\mathcal{G}_{(Y,\gamma,\tau)}^R|$, let $\eta_{H \rightarrow H''}$, $\eta_{H \rightarrow H'}$, and $\eta_{H' \rightarrow H''}$ be paths from H to H'' , H to H' , and H' to H'' respectively. From Theorem 9 it follows that

$$F_{H \rightarrow H''} = F(\eta_{H \rightarrow H''}) = F(\eta_{H' \rightarrow H''} \circ \eta_{H \rightarrow H'}).$$

Functoriality implies that

$$F(\eta_{H' \rightarrow H''} \circ \eta_{H \rightarrow H'}) = F(\eta_{H' \rightarrow H''}) \circ F(\eta_{H \rightarrow H'}) = F_{H' \rightarrow H''} \circ F_{H \rightarrow H'}.$$

Let $\varphi : (Y_0, \tau_0) \rightarrow (Y_1, \tau_1)$ be a diffeomorphism of real manifolds and let H_0 and H_1 be real isotopy diagrams for (Y_0, τ_0) . Let $d_i = \varphi|_{\Sigma_i}$ for $i \in \{0, 1\}$ so that $H'_i := d_i(H_i)$ are isotopy diagrams for (Y_1, τ_1) (again, $i \in \{0, 1\}$). The diagram

$$\begin{array}{ccc} F(H_0) & \xrightarrow{F(d_0)} & F(H'_0) \\ F_{H_0 \rightarrow H_1} \downarrow & & \downarrow F_{H'_0 \rightarrow H'_1} \\ F(H_1) & \xrightarrow{F(d_1)} & F(H'_1) \end{array}$$

is commutative by the Commutativity Axiom (Definition 2.26 Item 2). In this way, φ induces a morphism $F(\varphi)$ of transitive systems. The map induced by φ^{-1} is easily seen to satisfy $F(\varphi^{-1}) = F(\varphi)^{-1}$. $\square$

2.6. Naturality of the real Heegaard Floer invariants. Equipped with Theorems 8, 9, and 10, we are ready to establish naturality of the real Heegaard Floer invariants. The simplest case is real sutured Floer homology.

Proof of Theorem 3. By Theorem 8, the morphism $F = RSFH$ is a strong real Heegaard invariant of $\mathcal{S}_{sut}^R$. Hence, the groups $F(H)$ and the isomorphisms $F_{H \rightarrow H'}$ form a transitive system by Theorem 10. In the same proof, we showed how a diffeomorphism φ of (Y, γ, τ) induced a morphism of transitive systems.

All that remains is to show that isotopic diffeomorphisms induce identical maps on $RSFH$. Equivalently, we show that for any diffeomorphism $\varphi : (Y, \gamma, \tau) \rightarrow (Y, \gamma, \tau)$ isotopic to $\text{id}_{(Y,\gamma,\tau)}$, we have $F(\varphi) = \text{id}_{RSFH(Y,\gamma,\tau)}$. This is a consequence of Theorem 9. Let H be an isotopy diagram of (Y, γ, τ) , and let $d = \phi|_H$ and $H' = \phi(H)$. Since d is isotopic to the identity, it corresponds to an edge η of $\mathcal{G}_{(Y,\gamma,\tau)}^R$ connecting H and H' . By Theorem 9, $F(\eta) = F_{H \rightarrow H'}$. Hence, the map $F(\varphi)$ induces the identity map of the transitive system.

Following [JTZ21, Remark 1.2], we define the canonical invariant $RSFH(Y, \gamma, \tau)$ to be the colimit of this transitive system. $\square$

The case of the real Heegaard Floer invariants for closed (pointed) three-manifolds is slightly more subtle. In order to associate a real sutured manifold to (Y, τ, p) , we must also choose a framing $\xi = (\xi_1, \xi_2, \xi_3)$ of p so that $\xi_3 \in T_p C$, as in Definition 2.21. Let $\text{RMan}_{*,\xi}$ be the category of real 3-manifolds equipped with framed basepoints like this. In this category, a morphism from $(Y_0, \tau_0, p_0, \xi_0)$ to $(Y_1, \tau_1, p_1, \xi_1)$ is a pointed, equivariant diffeomorphism $\varphi : (Y_0, \tau_0, p_0) \rightarrow (Y_1, \tau_1, p_1)$ satisfying $d\varphi_p(\xi_0) = \xi_1$.

By way of [JTZ21, Definition 2.3], the data (Y, τ, p, ξ) determines a real sutured manifold $(Y(p), \gamma(p, \xi), \tau)$: We replace p with the unit sphere bundle of $T_p Y$ equipped with the suture $\gamma(p, \xi)$ which is the unit circle bundle of the plane $\text{span}(\xi_1, \xi_2)$.

Proposition 2.34. *There is a functor*

$$CFR^\circ : \text{RMan}_{*,\xi} \rightarrow \text{Trans}_{\mathcal{G}_{(Y,\tau,p)}^R}(C),$$

where $\mathcal{C}$ is the homotopy category of chain complexes over the ring $\mathbb{F}[U]$.

Proof. As in the sutured case, Theorems 8 and 9 give rise to a functor $\mathcal{S}_{man}^R \rightarrow \text{Trans}_{\mathcal{G}_{(Y,\tau,p)}^R}(\mathcal{C})$. By precomposing with the functor taking (Y, τ, p, ξ) to $(Y(p), \gamma(p, \xi), \tau)$ we obtain a natural invariant of (Y, τ, p, ξ) , which we denote by $CFR^\circ(Y, \tau, p, \xi)$. $\square$

Remark 2.35. More generally, there is a functor

$$CFR^\circ : \text{RMan}_{*,\vec{\xi}} \rightarrow \text{Trans}_{\mathcal{G}_{(Y,\tau,p)}^R}(\mathcal{C})$$

where $\text{RMan}_{*,\vec{\xi}}$ is the category of multi-pointed real three-manifolds equipped with a multi-framing $\vec{\xi}$ (i.e., a framing for each basepoint) and $\mathcal{C}$ is the homotopy category of curved chain complexes over the ring $\mathbb{F}[U_{\mathbf{w}}] := \mathbb{F}[U_{w_1}, \dots, U_{w_\ell}]$, where $\ell = |\mathbf{w}|$.

In order to remove the framing dependence, we must define canonical isomorphisms

$$\iota_{\xi,\xi'} : CFR^\circ(Y, \tau, \mathbf{w}, \xi) \rightarrow CFR^\circ(Y, \tau, \mathbf{w}, \xi').$$

This is more subtle than in the unreal case. Note that a framing of $T_p Y$ is equivalent to a choice of an *oriented* 2-plane bundle $V \subset T_p Y$. Forgetting the symmetry, the fiber of the forgetful map $\text{Man}_{*,\xi} \rightarrow \text{Man}_*$ is $\widetilde{\text{Gr}}_2(T_p Y)$, the space of *oriented* 2-planes in $T_p Y$, which is diffeomorphic to S^2 . However, in the present situation, the fiber of the forgetful map

$$\text{RMan}_{*,\xi} \rightarrow \text{RMan}_*$$

is not simply connected. Indeed, the fiber is $\widetilde{\text{Gr}}_2^R(T_p Y)$, the subspace of $\widetilde{\text{Gr}}_2(T_p Y)$ consisting of planes V satisfying $d\tau(V) = -V$; we call such planes *anti-invariant*. The map assigning $V \in \widetilde{\text{Gr}}_2^R(T_p Y)$ its positive unit normal vector establishes a homeomorphism $\widetilde{\text{Gr}}_2^R(T_p Y) \rightarrow S^1$. Therefore, we must prove that the monodromy about this loop is trivial.

2.6.1. Equivariant basepoint twist. Let us establish some notation.

Definition 2.36. Fix a based real 3-manifold (Y, τ, p) as well as an anti-invariant plane $V_0 \in \widetilde{\text{Gr}}_2^R(T_p Y)$.

- (1) Define $\text{Diff}^\tau(Y, \tau, p)$ to be the group of τ -equivariant, orientation-preserving diffeomorphisms of Y fixing p , with the C^∞ topology, and let $\text{Diff}_0^\tau(Y, \tau, p)$ be the subgroup of those diffeomorphisms that are equivariantly isotopic to id_Y through elements of $\text{Diff}^\tau(Y, \tau, p)$.
- (2) Let $G \leq GL^+(T_p Y)$ be the centralizer of $d\tau_p$ in the group of orientation-preserving automorphisms of $T_p Y$, and let G_0 be its identity component.

Remark 2.37. We note that G consists of the automorphisms preserving the splitting $T_p Y = \ell_p \oplus N_p$ where $\ell_p := T_p C$ and N_p is the (-1) -eigenspace of $d\tau$. Writing $A = (a, B) \in GL(\ell_p) \times GL(N_p)$, we have $G = \{(a, B) : a \cdot \det B > 0\}$ and $G_0 = \{(a, B) : a > 0, \det B > 0\} \cong \mathbb{R}_{>0} \times GL^+(N_p)$.

For $\chi \in \text{Diff}^\tau(Y, \tau, p)$, the differential $d\chi_p$ lies in the group G of Definition 2.36, so, having fixed $V_0 \in \widetilde{\text{Gr}}_2^R(T_p Y)$, we obtain an evaluation map

$$\text{ev}_{V_0} : \text{Diff}^\tau(Y, \tau, p) \longrightarrow \widetilde{\text{Gr}}_2^R(T_p Y), \quad \text{ev}_{V_0}(\chi) := d\chi_p(V_0).$$

Remark 2.38 (Conventions). We fix once and for all an orientation of ℓ_p near p as well as a positive vector ν_p ; orient N_p by declaring that an ordered basis (v_1, v_2) of N_p is positive if (ν_p, v_1, v_2) is a positive basis of $T_p Y$. Orient $\widetilde{\text{Gr}}_2^R(T_p Y)$ as the unit circle of N_p . We may therefore identify $\pi_1(\widetilde{\text{Gr}}_2^R(T_p Y), V_0)$ with $\mathbb{Z}$.

Definition 2.39. On an open set $U \subset Y$ containing p , fix τ -invariant coordinates $\kappa: U \rightarrow \mathbb{R}^3$ such that $\kappa(p) = 0$, τ acts as $R_\pi = \text{diag}(1, -1, -1)$, the chart is orientation-preserving, and $d\kappa_p$ carries the positive direction of ℓ_p to $+\partial_x$. Here $R_\theta \in SO(3)$ denotes the rotation by angle θ about the x -axis, counterclockwise with respect to the induced orientation of the (y, z) -plane. Fix a smooth function $\lambda: [0, \infty) \rightarrow [0, 1]$ with $\lambda \equiv 1$ on $[0, r_1]$ and $\lambda \equiv 0$ on $[r_2, \infty)$, where $0 < r_1 < r_2 < 1$. For $\alpha \in \mathbb{R}$ define

$$\Delta_\alpha := \begin{cases} \kappa^{-1} \circ (x \mapsto R_{\alpha\lambda(|x|)} x) \circ \kappa & \text{on } U, \\ \text{id}_Y & \text{on } Y \setminus U, \end{cases}$$

and for $n \in \mathbb{Z}$ define the n -fold equivariant basepoint twist

$$\delta^n := \Delta_{2\pi n}, \quad \delta := \delta^1.$$

Since $\lambda \equiv 0$ on $[r_2, \infty)$, the two formulas glue smoothly and Δ_α is supported in $\kappa^{-1}(\overline{B}_{r_2})$; since $\lambda \equiv 1$ on $[0, r_1]$, the map Δ_α agrees with the rotation R_α near p , and in particular δ^n is the identity near p .

We record the following facts about the map $\alpha \mapsto \Delta_\alpha$.

Lemma 2.40. (1) *The map $\alpha \mapsto \Delta_\alpha$ is a smooth homomorphism $\mathbb{R} \rightarrow \text{Diff}^\tau(Y, \tau, p)$; that is, each Δ_α is a τ -equivariant, orientation-preserving diffeomorphism fixing p (and $C \cap U$ pointwise), and $\Delta_{\alpha+\beta} = \Delta_\alpha \circ \Delta_\beta$.*
 (2) *Identifying $T_p Y$ with $\mathbb{R}^3$ via $d\kappa_p$, we have $d(\Delta_\alpha)_p = R_\alpha$. Hence $d(\delta^n)_p = \text{id}_{T_p Y}$, so δ^n is a morphism $(Y, \tau, p, V) \rightarrow (Y, \tau, p, V)$ of $\text{RMan}_{*, \xi}$ for every $V \in \widetilde{\text{Gr}}_2^R(T_p Y)$; moreover $s \mapsto \Delta_{2\pi n s}$ is a path in $\text{Diff}^\tau(Y, \tau, p)$ from id_Y to δ^n , so $\delta^n \in \text{Diff}_0^\tau(Y, \tau, p)$, and $[\text{ev}_{V_0} \circ \Delta_{2\pi n} \cdot] = n \in \pi_1(\widetilde{\text{Gr}}_2^R(T_p Y)) \cong \mathbb{Z}$ for every V_0 .*

Proof. Straightforward. $\square$

Definition 2.41. Let $V_0, V_1 \in \widetilde{\text{Gr}}_2^R(T_p Y)$, and let $\Phi = \{\Phi_s\}_{s \in [0, 1]}$ be an isotopy through elements of $\text{Diff}^\tau(Y, \tau, p)$ whose endpoints satisfy $d(\Phi_0)_p(V_0) = d(\Phi_1)_p(V_0) = V_1$ in the oriented sense. Then $\text{ev}_{V_0} \circ \Phi$ is a loop based at V_1 , and we define the *winding number*

$$w(\Phi) := [\text{ev}_{V_0} \circ \Phi] \in \pi_1(\widetilde{\text{Gr}}_2^R(T_p Y), V_1) \cong \mathbb{Z},$$

the identification with $\mathbb{Z}$ being given by the orientation of Remark 2.38.

The following is the analogue of [JTZ21, Lemma 2.45] in the real setting. It plays the role of the calibration of identifications by basepoint twists.

Lemma 2.42. *Let (Y, τ, p) be a based real 3-manifold, let $V_0, V_1 \in \widetilde{\text{Gr}}_2^R(T_p Y)$, and let $\Phi = \{\Phi_s\}_{s \in [0, 1]}$ be an isotopy through elements of $\text{Diff}^\tau(Y, \tau, p)$ whose endpoints $\phi_0 := \Phi_0$ and $\phi_1 := \Phi_1$ satisfy $d(\phi_0)_p(V_0) = d(\phi_1)_p(V_0) = V_1$ in the oriented sense. Set $w := w(\Phi)$. Then ϕ_0 is isotopic to $\delta^{-w} \circ \phi_1$ through τ -equivariant diffeomorphisms fixing p and carrying V_0 to V_1 in the oriented sense; that is, through morphisms $(Y, \tau, p, V_0) \rightarrow (Y, \tau, p, V_1)$ of $\text{RMan}_{*, \xi}$.*

Proof. Identify $T_p Y$ with $\mathbb{R}^3$ via $d\kappa_p$, so that $d(\Delta_\alpha)_p = R_\alpha$ by Lemma 2.40(2). The map

$$e: \mathbb{R} \longrightarrow \widetilde{\text{Gr}}_2^R(T_p Y), \quad e(\theta) := R_\theta(V_1),$$

is a universal covering with deck group $2\pi\mathbb{Z}$. The path $s \mapsto \text{ev}_{V_0}(\Phi_s) = d(\Phi_s)_p(V_0)$ begins and ends at V_1 ; let $\Theta: [0, 1] \rightarrow \mathbb{R}$ be its unique lift through e with $\Theta(0) = 0$, so that

$$d(\Phi_s)_p(V_0) = R_{\Theta(s)}(V_1) \quad \text{for all } s.$$

Clearly Θ is smooth, and, by Remark 2.38, $w = w(\Phi) = \Theta(1)/2\pi \in \mathbb{Z}$.

Consider the smooth family $g_s := \Delta_{-\Theta(s)} \circ \Phi_s$ in $\text{Diff}^\tau(Y, \tau, p)$. Since $\Phi_s(p) = p$,

$$d(g_s)_p(V_0) = R_{-\Theta(s)}(d(\Phi_s)_p(V_0)) = R_{-\Theta(s)}R_{\Theta(s)}(V_1) = V_1 \quad \text{for all } s,$$

so g is an isotopy through morphisms $(Y, \tau, p, V_0) \rightarrow (Y, \tau, p, V_1)$. Its endpoints are $g_0 = \Delta_0 \circ \phi_0 = \phi_0$ and, since $\Theta(1) = 2\pi w$,

$$g_1 = \Delta_{-2\pi w} \circ \phi_1 = (\Delta_{2\pi})^{-w} \circ \phi_1 = \delta^{-w} \circ \phi_1,$$

as desired. $\square$

Theorem 11. *Fix a real pointed three-manifold (Y, τ, p) as well as a framing ξ for p . Let δ be the associated equivariant basepoint twist. Then, the induced map*

$$CFR^\circ(\delta) : CFR^\circ(Y, \tau, p, \xi) \rightarrow CFR^\circ(Y, \tau, p, \xi)$$

acts by the identity map.

Proof. Intuitively, $CFR^\circ(\delta)$ acts by the identity because we may arrange that it has no effect on the alpha and beta curves. Hence, we may work in a small neighborhood $U \subset \Sigma$ of p , and exhibit a sequence of real Heegaard moves from $\delta(U)$ to U . This can be done entirely with stabilizations and isotopies.

Let $F = CFR^\circ$. Recall that $F(Y, \tau, p, \xi)$ is by definition the transitive system of $\mathbb{F}[U]$ -complexes indexed by the set of real Heegaard diagrams for (Y, τ, p) . The proof amounts to showing that $F(\delta_H) = F_{H \rightarrow \delta(H)}$, or equivalently, that $F_{\delta(H) \rightarrow H} \circ F(\delta_H) = \text{id}_{F(H)}$ for all diagrams H . By the commutativity axiom (Definition 2.26 Item 2),

$$F_{\delta(H) \rightarrow H} \circ F(\delta_H) = F_{H_0 \rightarrow H} \circ (F_{\delta(H_0) \rightarrow H_0} \circ F(\delta_{H_0})) \circ F_{H \rightarrow H_0}$$

for H_0 any other diagram. It follows that $F_{\delta(H) \rightarrow H} \circ F(\delta_H) = \text{id}_{F(H)}$ if and only if $F_{\delta(H_0) \rightarrow H_0} \circ F(\delta_{H_0}) = \text{id}_{F(H_0)}$, and therefore it suffices to prove the claim for any particular diagram H_0 .

Let H_0 be any diagram such that $\delta|_{\Sigma_0}$ is supported on a small disk $D \subset \Sigma_0$ which is disjoint from all attaching curves. Apply δ and consider the surface $\delta(D)$. The generators of $CFR^\circ(\delta(H_0))$ and $CFR^\circ(H_0)$ are identical. Figure 2.5 shows cross-sections of $\delta(\Sigma)$ near p ; we first perform a $\mathbb{Z}/2$ -stabilization as shown in the transition between the first and second columns of the diagram. The remaining columns show an isotopy of the stabilization of $\delta(\Sigma_1)$ to a surface which is obtained from Σ_0 by a $\mathbb{Z}/2$ -stabilization. This sequence of real Heegaard moves taking $\delta(H_0)$ to H_0 consists entirely of (de)stabilizations and diffeomorphisms isotopic to the identity. Hence, the composition $F_{\delta(H_0) \rightarrow H_0} \circ F(\delta_{H_0}) = \text{id}_{F(H_0)}$ as claimed. $\square$

We are at last situated to prove Theorem 1.

Proof of Theorem 1. Let CFR° be any version of real Heegaard Floer homology. Since CFR° is a strong real Heegaard invariant, it determines a transitive system. It remains to show that elements of $\text{Diff}_0^\tau(Y, \tau, p)$ act by the identity.

Let $\xi_0, \xi_1 \in \widetilde{\text{Gr}}_2^R(T_p Y)$ be two framings of p . Let $\phi \in \text{Diff}_0^\tau(Y, \tau, p)$ such that $d\phi_p(\xi_0) = \xi_1$. According to Proposition 2.34, there is an induced map

$$\iota_{\xi_0 \rightarrow \xi_1} := CFR^\circ(\phi) : CFR^\circ(Y, \tau, p, \xi_0) \rightarrow CFR^\circ(Y, \tau, p, \xi_1).$$

This map is independent of ϕ . Indeed, if ψ is another such map, we may choose an equivariant isotopy $\Phi = \{\Phi_s\}_{s \in [0,1]}$ in $\text{Diff}_0^\tau(Y, \tau, p)$ from ϕ to ψ (these exist by the definition of $\text{Diff}_0^\tau(Y, \tau, p)$). For $w = w(\Phi)$, Lemma 2.42 shows that ϕ and $\delta^{-w} \circ \psi$ are isotopic as morphisms $(Y, \tau, p, \xi_0) \rightarrow (Y, \tau, p, \xi_1)$. Hence, by Proposition 2.34 and functoriality,

$$CFR^\circ(\phi) = CFR^\circ(\delta^{-w} \circ \psi) = CFR^\circ(\delta)^{-w} \circ CFR^\circ(\psi) = CFR^\circ(\psi),$$

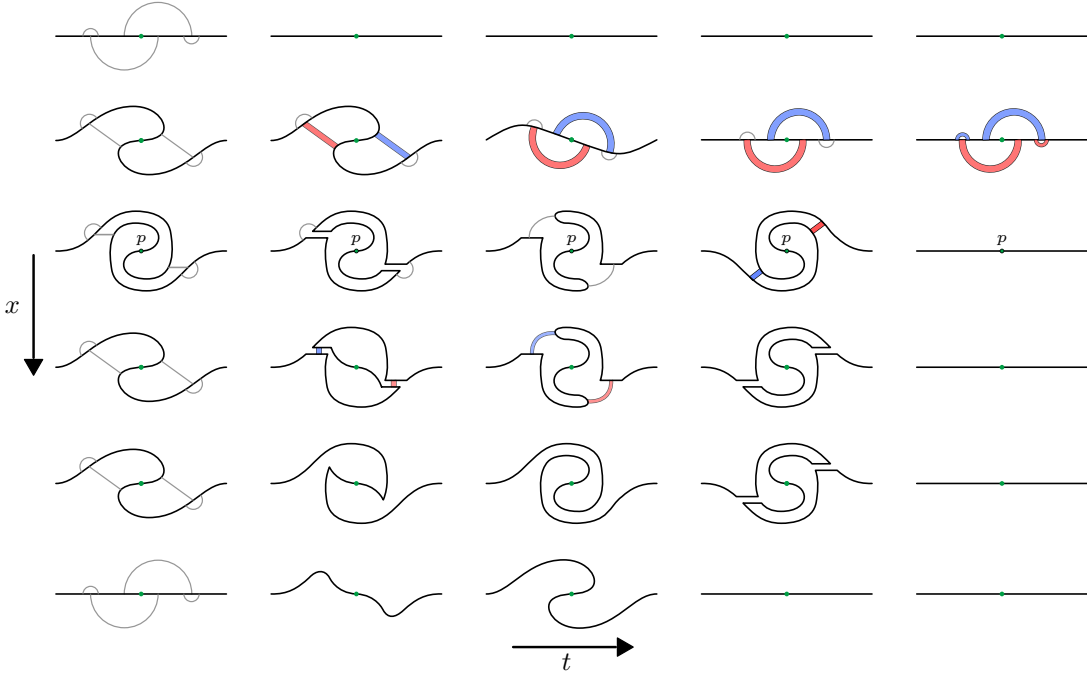


FIGURE 2.5. Here is the equivariant basepoint twist displayed in the coordinates described in Definition 2.39; the vertical direction displays slices of Σ_0 along the x -axis of the image of $\delta(\Sigma_0)$. The involution is given by rotation about the axis normal to the page. Horizontal slices show a sequence of real Heegaard moves relating $\delta(\Sigma_0)$ to Σ_0 . The gray lines are used to keep track of the attaching regions of the bands amid the isotopies. In the transition between the first and second columns we performed a stabilization, which we indicated by the four bands. (Each pair of bands makes up a tube.) The remaining columns depict an isotopy from a stabilization of $\delta(\Sigma_0)$ to a $\mathbb{Z}/2$ -stabilization of Σ_0 .

where the last equality is by Theorem 11. It follows that

$$(\{CFR^\circ(Y, \tau, p, \xi)\}_{\xi \in \widetilde{Gr}^R(T_p Y)}, \{\iota_{\xi_0 \rightarrow \xi_1}\}_{\xi_0, \xi_1 \in \widetilde{Gr}^R(T_p Y)})$$

form a transitive system, which we define to be $CFR^\circ(Y, \tau, p)$. By passing to homology and taking the colimit as in [JTZ21, Remark 1.2], we obtain the desired functor

$$HFR^\circ : \mathbf{RMan}_* \rightarrow \mathbb{F}[U]\text{-Mod}.$$

This completes the proof. $\square$

Remark 2.43. The case of multiple basepoints follows similarly. The only subtlety is that in the case that $|\mathbf{w}| > 1$, we must work in the category of curved complexes. Therefore, the invariant takes the form of a functor

$$CFR^\circ : \mathbf{RMan}_* \rightarrow \text{Trans}_{\mathcal{G}_{(Y, \tau, \mathbf{w})}^R}(\mathcal{C})$$

where $\mathcal{C}$ is the homotopy category of chain complexes over the ring $\mathbb{F}[U_{\mathbf{w}}]$. The curvature is given by $\partial^2 = \sum_{i=1}^n U_i^2$, and in these cases, we may instead work over the quotient ring $\mathbb{F}[U_1, \dots, U_n]/(U_1^2 + \dots + U_n^2)$ and then pass to homology and take the colimit.

Proof of Theorem 2. By Theorem 10, $CFLR^\circ$ defines a functor from the category $\mathcal{S}_{link}^R$ to the category of transitive systems over the homotopy category of curved chain complexes over $\mathbb{F}[u_{\mathbf{z}}, U_{\mathbf{w}}]$. By composing with the functor $(Y, L, \tau, \mathbf{w}, \mathbf{z}) \mapsto (Y(L, \mathbf{w}, \mathbf{z}), \tau)$, we may assign invariants to multi-based generalized strongly invertible links. By quotienting out the curvature and then passing to homology, we obtain the desired invariants $HFLR^\circ$. $\square$

3. SINGULARITIES OF REAL FUNCTIONS

In this section, we recall some basic singularity theory and its equivariant version. We will follow the exposition of [JTZ21, Section 4], as well as [BL, BB, Sie74, GS85, Wal85]

Throughout, (Y, γ, τ) is a real manifold with fixed set C of codimension 2. We allow Y to have boundary, but this will not play a significant role.

Definition 3.1. A *real function* on (Y, τ) is a smooth function $f : Y \rightarrow [-1, 1]$ satisfying $f \circ \tau = -f$. Let $C_R^\infty(Y, \tau)$ denote the space of real functions on (Y, τ) .

Definition 3.2. Let f be a real function on (Y, γ, τ) . We say that an orbit $\mathbb{Z}/2 \cdot p \subset Y$ is a *critical orbit* if $df_{g \cdot p} = 0$ for $g \in \mathbb{Z}/2$. A critical orbit $\mathbb{Z}/2 \cdot p$ is *non-degenerate* if the Hessian of f is non-singular at p (and therefore is non-degenerate at each $g \cdot p$). The *index* of a critical point p , which we denote $\mathcal{I}(p)$, is the number of negative eigenvalues of the Hessian at p .

We note that in the real setting, the index is not preserved by the group action: rather $\mathcal{I}(\tau(p)) = 3 - \mathcal{I}(p)$.

Critical points of real functions come in either $(\mathbb{Z}/2)/\{1\}$ - or $(\mathbb{Z}/2)/(\mathbb{Z}/2)$ -orbits (which we henceforth write as $\mathbb{Z}/2$ - or $\{1\}$ -orbits). Since critical points are generically isolated, analyzing critical points appearing in $\mathbb{Z}/2$ -orbits can be reduced immediately to the non-equivariant setting. Critical points which are fixed by the involution are more subtle and require a careful analysis.

For a real 3-manifold with codimension-2 fixed point set, there are two relevant local models. Near a point which is fixed by the $\mathbb{Z}/2$ -action, there is a neighborhood equivalent to $(\mathbb{R}^3, \tau)$ where $\tau(x, y, z) = (y, x, -z)$. For points on which $\mathbb{Z}/2$ acts freely, the model is simply $(\mathbb{R}^3, \text{id})$.

Definition 3.3. Let $\mathcal{E}_3(\tau)$ be the space of germs at 0 of $\mathbb{Z}/2$ -equivariant functions $f : (\mathbb{R}^3, \tau) \rightarrow (\mathbb{R}, -\text{id})$. We call elements of $\mathcal{E}_3(\tau)$ *real germs*. Let $\mathcal{E}_3(\text{id})$ be the space of germs at 0 of $\mathbb{Z}/2$ -equivariant functions $f : (\mathbb{R}^3, \text{id}) \rightarrow (\mathbb{R}, \text{id})$. Let $\mathcal{D}_3(\mathbb{Z}/2)$ be the group of germs of $\mathbb{Z}/2$ -equivariant diffeomorphisms $g : (\mathbb{R}^3, 0) \rightarrow (\mathbb{R}^3, 0)$. The group $\mathcal{D}_3(\mathbb{Z}/2)$ acts on $\mathcal{E}_3(\rho)$ by $g \cdot f := f \circ g^{-1}$ for $\rho \in \{\text{id}, \tau\}$. Two real germs f and f' are *equivalent* if they lie in the same $\mathcal{D}_3(\mathbb{Z}/2)$ -orbit.

In our setting, we will sometimes refer to equivalence classes of germs of $\mathcal{E}_3(\tau)$ as $\{1\}$ -*orbit singularities* and equivalence classes of germs of $\mathcal{E}_3(\text{id})$ as $\mathbb{Z}/2$ -*orbit singularities*.

In the following subsections we classify singularities of real functions of codimension up to 2 in the space of real functions.

3.1. $\mathbb{Z}/2$ -singularities.

Definition 3.4. A real function $f : Y \rightarrow \mathbb{R}$ is a *real Morse function* if all of its critical orbits are nondegenerate.

If p is a nondegenerate point of index k , the condition $f \circ \tau = -f$ forces $\tau(p)$ to be a critical point of index $3 - k$. In particular, the critical points of real Morse functions always come in $\mathbb{Z}/2$ -orbits. Therefore, the usual Morse lemma implies the following.

Lemma 3.5 (Morse Lemma for $\mathbb{Z}/2$ -critical orbits). *Let $f : Y \rightarrow \mathbb{R}$ be a real function and let $\mathbb{Z}/2 \cdot p$ be a nondegenerate $\mathbb{Z}/2$ -critical orbit. In a neighborhood of p in Y , there is a coordinate*

system in which f has the form

$$f(x) = f(p) + \varepsilon_1 \cdot x_1^2 + \varepsilon_2 \cdot x_2^2 + \varepsilon_3 \cdot x_3^2,$$

where $\varepsilon_i \in \{-1, 1\}$. There is a symmetric coordinate system near $\tau(p)$. The class of such $\mathbb{Z}/2$ -singularities is called $A_1(\mathbb{Z}/2)$.

Lemma 3.6. *Real Morse functions are generic in $C_R^\infty(Y, \gamma, \tau)$.*

Proof. This follows as in [BL], so we only sketch the proof. Let $f \in C_R^\infty(Y, \gamma, \tau)$. We claim that any such f can be made Morse by a small (equivariant) perturbation. In a neighborhood of a point p of C , we may choose coordinates in which $\tau(x, y, z) = (y, x, -z)$. The space of real linear functionals, $\mathcal{A} = \{\phi \in (\mathbb{R}^3)^* : \phi \circ \tau = -\phi\}$ is two dimensional. A standard argument shows that $f|_{\nu(p)} + A$ is Morse for a generic $A \in \mathcal{A}$. By covering C by finitely many balls and using a partition of unity, we may perturb f so that it is a real Morse function in a neighborhood of C . Furthermore, it is clear that we can ensure that the perturbed function is close to f .

We then cover $Y \setminus \nu(C)$ by pairs of disjoint balls $B_i \cup \tau(B_i)$. Each $f|_{B_i}$ can be made Morse by a small perturbation; we apply the symmetric perturbation to $f|_{\tau(B_i)}$. Again, these functions can be stitched together with a partition of unity. The resulting function can be made arbitrarily close to f and is real Morse. $\square$

We turn now to 1- and 2-parameter families of real functions.

Theorem 12. *In a generic 1-parameter family of smooth functions, the only degenerate $\mathbb{Z}/2$ -critical orbits $\mathbb{Z}/2 \cdot p$ that appear have normal form*

$$f(x) = f(p) + \varepsilon_1 \cdot x_1^2 + \varepsilon_2 \cdot x_2^2 + x_3^3,$$

where $\varepsilon_i \in \{-1, 1\}$. We refer to these as $A_2(\mathbb{Z}/2)$ singularities.

In generic 2-parameter families of smooth functions, there also appear degenerate $\mathbb{Z}/2$ -critical orbits $\mathbb{Z}/2 \cdot p$ of the form

$$f(x) = f(p) + \varepsilon_1 \cdot x_1^2 + \varepsilon_2 \cdot x_2^2 + \varepsilon_3 \cdot x_3^4,$$

where $\varepsilon_i \in \{-1, 1\}$. We refer to these as $A_3^\pm(\mathbb{Z}/2)$ singularities (depending on the sign of x_3^4).

Proof. Singularities appearing in $\mathbb{Z}/2$ -orbits behave exactly as individual singularities in the non-equivariant setting. See [Sie74] for an elementary proof. $\square$

3.2. Fixed set singularities. We now turn to the classification of fixed set singularities up to codimension 2. We note that A_1 - and $A_3^\pm$ -singularities cannot appear in $\{1\}$ -orbits.

Let f be a real function with a critical point at the origin. Recall that the *corank* of f is the dimension of the kernel of the Hessian of f . In the non-equivariant setting, the space of functions with corank greater than 1 has codimension greater than two in $C^\infty(\mathbb{R}^n)$. The same is true in the real setting. For simplicity, we specialize to dimension 3.

Lemma 3.7. *Let $f \in C_R^\infty(\mathbb{R}^3)$ have a critical point at the origin (which is fixed by the group action). Then, the Hessian $H_f(0)$ has corank 1 or 3. Furthermore, the space of $f \in C_R^\infty(\mathbb{R}^3)$ with a critical point at the origin with Hessian of corank 3 has codimension 4.*

Proof. Choose coordinates near zero so that $\tau(x_1, x_2, x_3) = (x_2, x_1, -x_3)$. Differentiating the condition

$$f \circ \tau = -f,$$

twice at fixed points yields the relation

$$(d\tau)^\perp \circ H_f \circ (d\tau) = -H_f.$$

In our chosen coordinates, we are therefore interested in matrices satisfying

$$\begin{bmatrix} d & b & -e \\ b & a & -c \\ -e & -c & f \end{bmatrix} = - \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix},$$

i.e. matrices of the form

$$\begin{bmatrix} a & 0 & c \\ 0 & -a & c \\ c & c & 0 \end{bmatrix}.$$

These matrices have eigenvalues $\{0, \pm\sqrt{a^2 + 2c^2}\}$, and therefore have corank either 1 or 3. In particular, these matrices are always singular.

Any real function must satisfy $f_{x_1} + f_{x_2} = 0$. Having a critical point on the fixed set is thus specified by the condition $f_{x_1} = 0 = f_{x_3}$, which is a codimension 2 constraint. The condition that $a = 0 = c$ (i.e. that the Hessian is corank 3) gives two further constraints. This completes the proof. $\square$

In view of Lemma 3.7, if we consider the space of odd functions having a critical point *somewhere* along a 1-dimensional fixed point set, its codimension is $4 - 1 = 3$. In particular, critical points with Hessians of corank 3 do not appear in generic 1- or 2-parameter families of real functions.

We now prove an equivariant splitting lemma, and establish the normal forms for the singularities we will encounter.

Proposition 3.8. *Let $f : (\mathbb{R}^3, 0) \rightarrow (\mathbb{R}, 0)$ be a real germ with a critical point at the origin whose Hessian has corank 1. Then,*

- (1) *there is an equivariant change of coordinates $(x_1, x_2, x_3) \mapsto (\hat{x}_1, \hat{x}_2, \hat{x}_3)$ and a smooth, odd germ $h : (\mathbb{R}, 0) \rightarrow (\mathbb{R}, 0)$ such that*

$$f = \hat{x}_1^2 - \hat{x}_2^2 + h(\hat{x}_3);$$

- (2) *further, if h has finite order $2k + 1$ at the origin², then the coordinates may be chosen so that*

$$f = \hat{x}_1^2 - \hat{x}_2^2 + \hat{x}_3^{2k+1}.$$

The normal form stated in Proposition 3.8 can be obtained formally, just as in [Sie74, Splitting Lemma]. The non-equivariant statement then follows from the fact that any germ of finite codimension is finitely determined (see [Was74], for instance). Rather than formulate and prove an equivariant version of finite determinacy, we give an elementary proof of Proposition 3.8. This is modeled on the proof of the equivariant Morse lemma; cf. [Arn76, Section 2] and [Was69, Lemma 4.1].

We make use of the following well-known fact, for which we refer to [Hig08, Theorem 1.29].

Lemma 3.9 (Principal square root). *There is a neighborhood $\mathcal{U}$ of the identity in $\mathrm{GL}_n(\mathbb{R})$ and a smooth map $\sqrt{\cdot} : \mathcal{U} \rightarrow \mathrm{GL}_n(\mathbb{R})$ with $(\sqrt{N})^2 = N$ and $\sqrt{I} = I$, characterized as the unique square root of N whose spectrum lies in $\{\mathrm{Re} > 0\}$. It satisfies:*

$$\sqrt{PNP^{-1}} = P(\sqrt{N})P^{-1} \quad \text{whenever } N, PNP^{-1} \in \mathcal{U}.$$

²As h is odd, its order is necessarily odd as well.

Proof of Proposition 3.8. It will be more convenient to use coordinates $(u, v, z) = (x_1 + x_2, x_1 - x_2, x_3)$ so that $\tau(u, v, z) = (u, -v, -z)$. Let $W = \mathbb{R}^2 \times 0 \subset \mathbb{R}^3$ and write $w = (u, v)$ for a typical element of W .

In these coordinates, the Hessian is given by

$$H = \begin{bmatrix} 0 & b & c \\ b & 0 & 0 \\ c & 0 & 0 \end{bmatrix}.$$

After a linear change of variables, we may assume that $b = 1$ and $c = 0$. Then, the kernel of the Hessian is the z -axis and its restriction to W is the hyperbolic form

$$H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Therefore, the 2-jet of f is uv .

Let

$$\sigma : W \rightarrow W, \sigma(u, v) := (u, -v),$$

so that $\tau(w, z) = (\sigma(w), -z)$.

Define also $A_0 = \frac{1}{2}H$, so that $w^T A_0 w = uv = x^2 - y^2$ and $\sigma A_0 \sigma = -A_0$.

Let $G(w, z)$ be the gradient of f in the w -directions,

$$G(w, z) := \nabla_w f(w, z) \in \mathbb{R}^2.$$

Note that since f has a critical point at the origin we have $G(0, 0) = 0$. Also, $D_w G(0, 0) = D_w^2 f(0) = H$ is invertible, so the inverse function theorem supplies a unique smooth function $z \mapsto \xi(z)$ with $\xi(0) = 0$ and

$$(1) \quad G(\xi(z), z) = 0$$

for z near 0. The symmetry of f yields the following symmetry of G :

$$G \circ \tau(w, z) = -\sigma \circ G(w, z).$$

It follows that $z \mapsto \sigma(\xi(-z))$ also solves (1) and vanishes at the origin; hence the uniqueness of ξ forces the relation

$$\xi(-z) = \sigma \xi(z).$$

After the equivariant change of coordinates $(w, z) \mapsto (w - \xi(z), z)$, we may assume that $G(0, z) = 0$ for all z . Let us keep the symbol f for the function on the new coordinates.

We define

$$(2) \quad h(z) := f(0, z).$$

Observe that h is odd, $h(-z) = -h(z)$.

Consider the germ $\Phi(w, z) := f(w, z) - h(z)$. By (2), we have $\Phi(0, z) = 0$ and $\nabla_w \Phi(0, z) = 0$ for all z . Taylor's theorem with remainder in the w -variables yields

$$\Phi(w, z) = w^T A(w, z) w, \quad A(w, z) = \int_0^1 (1-s) D_w^2 \Phi(sw, z) ds,$$

which is a smooth family of symmetric matrices. Furthermore, $A(0, 0) = \frac{1}{2}H$ which is precisely A_0 . For points p in a sufficiently small neighborhood of $(0, 0)$, the matrix

$$M(p) = A_0^{-1} A(p)$$

is defined and lies in the neighborhood $\mathcal{U}$ from Lemma 3.9. Let $C(p) = \sqrt{M(p)}$. The symmetry of f implies that:

$$(3) \quad A(\tau p) = -\sigma A(p) \sigma, \quad M(\tau p) = \sigma M(p) \sigma, \quad C(\tau p) = \sigma C(p) \sigma.$$

Here, the relation for $C(p)$ is a consequence of the uniqueness claim of Lemma 3.9.

Notice that the matrix $M = M(p)$ is self-adjoint with respect to A_0 :

$$A_0^{-1} M^T A_0 = A_0^{-1} (A_0^{-1} A)^T A_0 = A_0^{-1} A = M.$$

Since both $C = C(p)$ and its adjoint are square roots of $M(p)$ with spectrum in $\{\text{Re} > 0\}$, the uniqueness part of Lemma 3.9 implies that C is self-adjoint. Therefore,

$$(4) \quad C^T A_0 C = A.$$

Finally, define $\hat{w} = (\hat{u}, \hat{v}) = C(w, z)w$. One can easily verify that this is a diffeomorphism germ. Further, it is equivariant by (3), and $\tau(\hat{w}, z) = (\hat{u}, -\hat{v}, -z)$. Putting everything together,

$$\begin{aligned} f(w, z) &= h(z) + w^T A(w, z)w \\ &= h(z) + w^T C(w, z)^T A_0 C(w, z)w \\ &= h(z) + \hat{w}^T A_0 \hat{w} \\ &= h(z) + \hat{u}\hat{v}. \end{aligned}$$

Taking $\hat{u} \mapsto \hat{x}_1 + \hat{x}_2$ and $\hat{v} \mapsto \hat{x}_1 - \hat{x}_2$ proves part (1) of the proposition.

Part (2) is standard. If h has finite order $2k+1$, we can write $h(z) = z^{2k+1}\rho(z)$ with $\rho(0) \neq 0$. Finally, we define $\hat{z} = z\rho(z)^{1/(2k+1)}$, which yields the desired normal form. $\square$

Following [GS85], we will write $\mathcal{E}_x$ for the ring of smooth germs in a single variable x . As noted in [GS85, Chapter VI, Section 2], the space of odd germs $\mathcal{E}_x(\mathbb{Z}/2)$ is simply

$$\mathcal{E}_x(\mathbb{Z}/2) = \mathcal{E}_u \cdot \{x\},$$

where $u = x^2$. Note that $\mathcal{E}_x(\mathbb{Z}/2)$ is not a ring (the product of two odd germs is not odd) but rather a module over $\mathcal{E}_u$, the ring of even germs.

Let $f(x)$ be an element of $\mathcal{E}_x(\mathbb{Z}/2)$. Write $f(x) = r(u)x$. The *tangent space at f* is the space

$$T(f, \mathbb{Z}/2) := \langle r, u \partial_u r(u) \rangle \cdot \{x\},$$

according to [GS85, Chapter VI, Equation 3.2]. The codimension of f agrees with the codimension of $T(f, \mathbb{Z}/2)$ in $\mathcal{E}_x(\mathbb{Z}/2)$. It is clear that $T(x^{2k+1})$ has codimension k in $\mathcal{E}_x(\mathbb{Z}/2)$, as

$$(5) \quad \dim_{\mathbb{R}}(\mathcal{E}_x(\mathbb{Z}/2)/T(x^{2k+1}, \mathbb{Z}/2)) = \dim_{\mathbb{R}}((\mathbb{R}[x^2] \cdot \{x\})/(x^{2k+1})).$$

Combining the previous results yields the following classification result.

Theorem 13. *In a generic 1-parameter family of smooth functions, the only degenerate $\{1\}$ -critical orbits that appear are isolated, index 1 birth-death singularities and have normal form*

$$f(x) = f(p) + x_1^2 - x_2^2 + x_3^3, \quad \tau(x_1, x_2, x_3) = (x_2, x_1, -x_3),$$

which we refer to as $A_2(\{1\})$ -singularities.

In generic 2-parameter families of smooth functions, there also appear isolated singularities of the form

$$f(x) = f(p) + x_1^2 - x_2^2 + x_3^5, \quad \tau(x_1, x_2, x_3) = (x_2, x_1, -x_3),$$

which we refer to as $A_4(\{1\})$ singularities.

We now analyze the behavior question of real functions near these types of singularities.

Definition 3.10. A *real deformation* of $f \in \mathcal{E}_n(\tau)$ with base $\Lambda = \mathbb{R}^k$ is a smooth map

$$F : \mathbb{R}^n \times \mathbb{R}^k \rightarrow \mathbb{R}$$

such that $F(x, 0) = f(x)$ and $F(\tau(x), \lambda) = -F(x, \lambda)$.

A real deformation $F(x, \lambda)$ of f is called *versal* if every other real deformation $F' : \mathbb{R}^n \times \mathbb{R}^\ell \rightarrow \mathbb{R}$ of f can be represented in the form

$$F'(x, \lambda') = F(g(x, \lambda'), \theta(\lambda')),$$

where g is a germ at zero of an ℓ -parameter family of diffeomorphisms $\mathbb{R}^n \times \mathbb{R}^\ell \rightarrow \mathbb{R}^n$ so $g(x, 0) = x$ and $g(\tau(x), \lambda) = -g(x, \lambda)$ and $\theta : (\mathbb{R}^\ell, 0) \rightarrow (\mathbb{R}^k, 0)$ satisfies $\theta(0) = 0$.

A versal (real) deformation whose base is of minimal dimension is called *miniversal*. There is a useful criterion for determining whether a deformation is miniversal.

Theorem 14 (Theorem 3.3 in [GS85]). *Let $g(x) = R(u)x$, where $u = x^2$, be a germ in $\mathcal{E}(\mathbb{Z}/2)$ and let $G(x, \lambda_1, \dots, \lambda_k) = R(x, \lambda_1, \dots, \lambda_k)x$ be a k -parameter real deformation of g . Then G is a miniversal real deformation if and only if the codimension of $T(g, \mathbb{Z}/2)$ in $\mathcal{E}(\mathbb{Z}/2)$ is k and*

$$\mathcal{E}(\mathbb{Z}/2) = T(g, \mathbb{Z}/2) \oplus \mathbb{R} \left\{ \frac{\partial R}{\partial \lambda_1}, \dots, \frac{\partial R}{\partial \lambda_k} \right\} \cdot x.$$

Miniversal real deformations of $A_2(\{1\})$ - and $A_4(\{1\})$ -singularities have the following forms.

Corollary 3.11. *The $\{1\}$ -orbit singularities A_2 and A_4 have miniversal deformations*

$$f(x_1, x_2, x_3) = x_1^2 - x_2^2 + \lambda_1 x_3 + x_3^3$$

and

$$f(x_1, x_2, x_3) = x_1^2 - x_2^2 + \lambda_1 x_3 + \lambda_2 x_3^3 + x_3^5,$$

respectively.

Proof. By Proposition 3.8, it suffices to consider unfoldings of the odd, one-variable germ $g(x_3) = x_3^{2k+1}$. As noted above, its tangent space is

$$T(g, \mathbb{Z}/2) = \langle u^k \rangle \cdot \{x_3\},$$

which clearly has codimension k in $\mathcal{E}(\mathbb{Z}/2)$. Indeed, its complement is given by

$$\text{span}_{\mathbb{R}} \left(\frac{\partial R}{\partial \lambda_1}, \dots, \frac{\partial R}{\partial \lambda_{k-1}} \right) \cdot x = \text{span}_{\mathbb{R}} (u^{k-1}, \dots, u, 1) \cdot x.$$

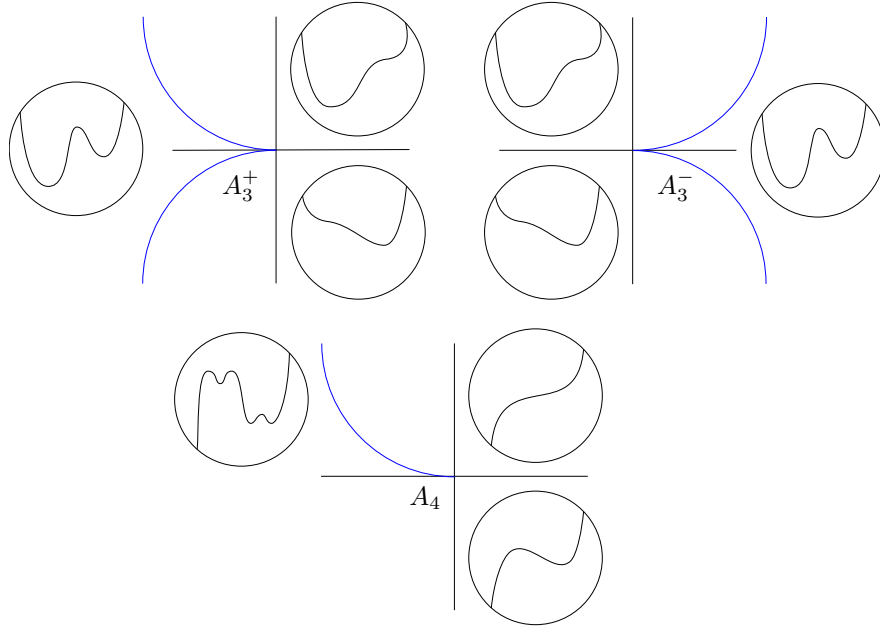
It follows that $G(x, \lambda_0, \dots, \lambda_{k-1}) = (u^k + \lambda_{k-1}u^{k-1} + \dots + \lambda_1 u + \lambda_0) \cdot x$ is a miniversal real deformation. $\square$

In particular, we can explicitly describe the bifurcation diagrams of these singularities. The diagrams for $A_3^\pm$ -singularities are well-known, and in our situation are the same, as these singularities appear in $\mathbb{Z}/2$ -orbits; see Figure 3.1. Let us briefly describe the bifurcation diagram of an $A_4(\{1\})$ -singularity.

Lemma 3.12. *The bifurcation diagram of an $A_4(\{1\})$ -singularity is as in the last frame of Figure 3.1.*

Proof. We consider the real roots of the polynomial $f(x) = 5x^4 + 3\lambda_1 x^2 + \lambda_2$. When $9\lambda_1^2 < 20\lambda_2$, the polynomial $g(y) = 5y^2 + 3\lambda_1 y + \lambda_2$ has no real roots. When $9\lambda_1^2 > 20\lambda_2$, $g(y)$ has two real roots a and b , and therefore $f(x) = g(x^2)$ has either zero, two, or four real roots, depending on the signs of a and b . The product ab is equal to $\lambda_2/5$. When $\lambda_2 < 0$, exactly one root is negative, so $f(x)$ has two real roots. When $\lambda_2 > 0$ the roots have the same sign. The sum $(a + b)$ is equal to $-3\lambda_1/5$; therefore, when $\lambda_1 > 0$ both roots are negative and $f(x)$ has no real roots and when $\lambda_1 < 0$ both roots are positive and $f(x)$ has four real roots. $\square$

In particular, while traversing a loop around an A_4 -singularity, one witnesses a $\{1\}$ -orbit A_2 -singularity when the path crosses the positive λ_1 axis and a second as it crosses the negative λ_1 axis; then, as the path crosses the graph of $\lambda_2 = 9\lambda_1^2/20$, one witnesses a $\mathbb{Z}/2$ -orbit A_2 -singularity.

FIGURE 3.1. $A_3^\pm(\mathbb{Z}/2)$ and $A_4(\{1\})$ bifurcation diagrams for $n = 1$.

4. FAMILIES OF REAL GRADIENTS

We now turn to families of real gradients, following [JTZ21, Section 5]. See also Palis-Takens [PT83], Carneiro-Palis [CP89], Vegter [Veg85], and Bao-Lawson [BL].

Definition 4.1. An *invariant manifold* of a vector field v on Y is a submanifold that is tangent to the vector field at each of its points.

Suppose that p is a singularity of a vector field v on Y . Then, dv_p is an endomorphism of $T_p Y$. The tangent space at p decomposes into three dv_p -invariant subspaces, T^s , T^u , and T^c , with the property that every eigenvalue of $dv_p|_{T^s}$ has negative real part, $dv_p|_{T^u}$ has positive real part, and $dv_p|_{T^c}$ has zero real part.

Definition 4.2. A vector field v has a *hyperbolic singularity* at p if none of the eigenvalues of dv_p are purely imaginary, i.e. if $T^c = 0$.

Theorem 15 (Center Manifold Theorem). *Let v be a C^{r+1} vector field on Y with a singular point at p . Then, v has invariant manifolds $\mathcal{W}^s$, $\mathcal{W}^u$, and $\mathcal{W}^c$ of class C^{r+1} , C^{r+1} , and C^r , respectively, that go through p and are tangent at p to T^s , T^u , and T^c , respectively.*

The manifolds $\mathcal{W}^s$, $\mathcal{W}^u$, and $\mathcal{W}^c$ are called the (strong) stable, (strong) unstable, and center manifolds of the singular point p . The center manifold is usually not unique.

Definition 4.3. A smooth vector field v on Y has a *saddle-node* at p (or a *quasi-hyperbolic singularity of type 1*) if $\dim T^c = 1$ and $v|_{\mathcal{W}^c}$ has the form $v = ax^2\partial_x + O(|x|^3)$ with $a \neq 0$ for some local coordinate x on some center manifold $\mathcal{W}^c$ around p .

If $v^{\bar{\mu}}$ belongs to a 1-parameter family of vector fields $\{v^\mu\}$ and has a saddle-node at p , we say that it *unfolds generically* if there is a center manifold for $\{v^\nu\}$ passing through p at $\mu = \bar{\mu}$ so that the restriction of v^μ has the form

$$v^\mu = (ax^2 + b(\mu - \bar{\mu}))\partial_x + O(|x|^3 + |x(\mu - \bar{\mu})| + |\mu - \bar{\mu}|^2),$$

with $a, b \neq 0$.

Definition 4.4. Let p be a singular point of a vector field v on Y with flow domain $D \subset Y \times \mathbb{R}$ and let $\varphi : D \rightarrow Y$ be the maximal flow of v . The *stable set* of p is

$$W^s(p) = \{y \in Y : \lim_{t \rightarrow \infty} \varphi(y, t) = p\},$$

and the *unstable set* of p is

$$W^u(p) = \{y \in Y : \lim_{t \rightarrow -\infty} \varphi(y, t) = p\}.$$

When p is a hyperbolic singularity of v then both the stable and unstable sets are injectively immersed submanifolds of Y whose tangent spaces agree with T^s and T^u respectively. If p is a saddle-node, $W^s(p)$ and $W^u(p)$ are injectively immersed submanifolds with boundary, denoted $W^{ss}(p)$ and $W^{uu}(p)$ respectively, which we call the strong stable and strong unstable manifolds. The tangent spaces of $W^{ss}(p)$ and $W^{uu}(p)$ at p are equal to T^s and T^u .

Definition 4.5. Let p^μ be a hyperbolic singularity of a 1-parameter family of gradients $X^\mu = (f^\mu, g^\mu)$ with a 1-dimensional stable manifold $W^s(p^\mu)$. Let us write $\gamma_1^\mu \cup \gamma_2^\mu$ for the two components of $W^s(p^\mu) \setminus p^\mu$. (We choose these components consistently for different values of μ , so that they form two smooth families.) For some $t_0 \in \mathbb{R}$ and $\epsilon > 0$ small so that $\epsilon > f^\mu(p^\mu) - t_0 > 0$, consider the slice $(f^\mu)^{-1}(t_0)$. Let $q^\mu = q^\mu(t_0) \in \gamma_1^\mu \cap (f^\mu)^{-1}(t_0)$. Let

$$v_0^\mu = \frac{dq^\mu}{d\mu} \in T_{q^\mu} Y$$

be the velocity of the path traced out by q^μ . Define

$$v^\mu = \lim_{t \rightarrow -\infty} \frac{(d\phi_t^\mu)(v_0^\mu)}{\|(d\phi_t^\mu)(v_0^\mu)\|} \in T_{p^\mu} Y$$

where ϕ_t^μ is the flow of the gradient-like vector field for X^μ . Consider the projection $T_{p^\mu} Y \rightarrow T_{p^\mu} W^u(p^\mu)$ with kernel $T_{p^\mu} W^s(p^\mu)$. Let w^μ be the image of v^μ under this projection. Define the *directional manifold of p^μ relative to γ_1^μ* be

$$D(p^\mu, \gamma_1^\mu) = \left\{ y \in W^u(p^\mu) : \lim_{t \rightarrow \infty} \frac{d}{dt} \left(\frac{\phi(y, t)}{\|\phi(y, t)\|} \right) \in \text{span}(w^\mu) \right\}.$$

See Figure 4.1. The directional manifold of p^μ relative to γ_2^μ is defined similarly.

Said another way, in a 1-parameter family, the manifold $\gamma_1^\mu \subset W^s(p^\mu)$ has some normalized velocity vector in each t -slice. We push this vector to p^μ via the flow of X^μ to obtain a vector v^μ in the tangent space at p^μ . We then project to the unstable directions to get w^μ . The directional manifold is the invariant manifold of v tangent to $\text{span}(w^\mu)$. It can be either zero- or one-dimensional. (In our examples, it will always be the latter.)

Definition 4.6. A vector field v on Y is called *real* if $d\tau(v) = -v$.

4.1. Bifurcations of gradient-like vector fields on 3-manifolds. Given a Riemannian manifold (Y, g) , a gradient vector field $X = \text{grad}_g(f)$ is determined by a smooth function $f : Y \rightarrow \mathbb{R}$ and g according to the relation $g(X, Y) = df(Y)$. Gradient vector fields have no periodic orbits, and their derivative at critical points has only real eigenvalues. We note that if g is a τ -invariant metric and f is a real function on Y , then $X = \text{grad}_g(f)$ is a real vector field:

$$g(d\tau(X), Y) = g(X, d\tau(Y)) = df \circ d\tau(Y) = -df(Y).$$

Definition 4.7. A gradient vector field X on Y is *Morse-Smale* if

- (1) (Hyperbolicity) all singular points of X are hyperbolic;
- (2) (Transversality) all stable and unstable manifolds are transverse.

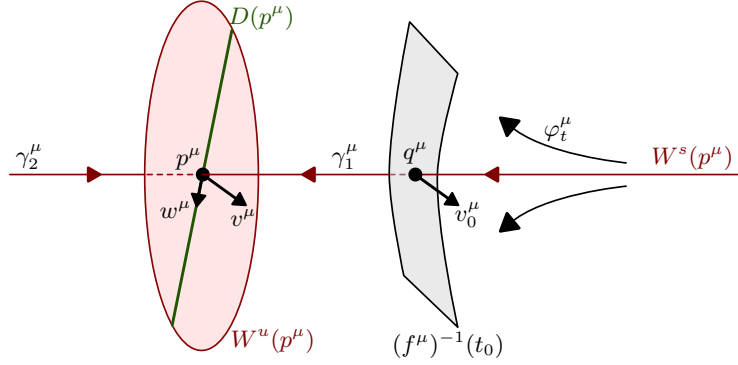


FIGURE 4.1. The directional manifold of a hyperbolic critical point p^μ , as well as the various vectors.

Let $X^g(Y)$ be the set of all gradient vector fields on Y . It is well known that the set of Morse-Smale vector fields on Y is open and dense in $X^g(Y)$. Let $X^g(Y, \tau)$ be the set of real gradient vector fields on Y . We will show that, as in the unreal case, real gradient-like vector fields are generically Morse-Smale.

Before turning to the proof, we introduce some notation and make several observations which will be useful later.

Definition 4.8. Suppose A and B are submanifolds of Y which are invariant under the flow of v . If A and B have a point of tangency, there is necessarily an entire flow line of tangencies, which we call an *orbit of tangency*.

Say γ is an orbit of tangency which is fixed setwise by the involution τ . Note that $C \cap \gamma$ is necessarily a single point, r . Near r , we may choose coordinates (y_1, y_2, y_3) with respect to which $d\tau_r$ is diagonal with eigenvalues $(+1, -1, -1)$. We write E^+ for the positive eigenspace and E^- for the negative eigenspace. Note that E^+ is precisely $T_r C$.

Suppose that $\dim A = 2$ and $B = \tau(A)$. Assume that γ is an orbit of tangency between A and B . Let P be the plane $T_r A = T_r B$. There are two possibilities. Either P is the sum of the positive eigenspace E^+ and $\mathbb{R}\langle v \rangle$, or P is the negative eigenspace E^- . We refer to orbits of tangency of the first kind as *type 1 tangencies* and orbits of the second kind as *type 2 tangencies*.

Lemma 4.9. *Generic real gradient-like vector fields are Morse-Smale.*

Proof. We begin as in [BL]. Suppose γ is an orbit of tangency from p to q . If $\tau(p) \neq q$, we can simply perturb the metric in a small neighborhood of p (and symmetrically in a small neighborhood of q) to ensure that the ascending/descending manifolds of p and q intersect transversely.

If $\tau(p) = q$, we may potentially run into equivariant transversality issues, as modifying the metric near p also alters the metric near q . Note that in this case, $\mathcal{I}(q) = 3 - \mathcal{I}(p)$, and hence it suffices to consider the case that $\mathcal{I}(p) = 2$ and $\mathcal{I}(q) = 1$.

Standardize an equivariant neighborhood of $\gamma \cong D^2 \times I$. Consider the slice $S = D^2 \times \{0\}$ which intersects the fixed point set C . In this slice, $A = W^u(p) \cap S$ and $\tau(A) = W^s(q) \cap S$ appear as curves which are tangent at the point $r = \gamma \cap S$ which is contained in C . Choose one more curve $D \subset S$ such that $\tau(D) = D$ and $T_r D = (T_r C)^\perp$. Notice that curves A and $\tau(A)$ are tangent if and only if A is tangent to C or D at r (these correspond to type 1 and 2 orbits of tangency respectively). Non-equivariant transversality guarantees that generically A intersects both C and D transversely, and hence, A and $\tau(A)$ will generically be transverse. See the central frames of Figure 4.2.

Hence, we may proceed as in the previous case; we simply perturb the metric near p (and symmetrically near q) to ensure that A is transverse to C and D ; doing so will guarantee that $W^u(p)$ and $W^s(q)$ intersect transversely. $\square$

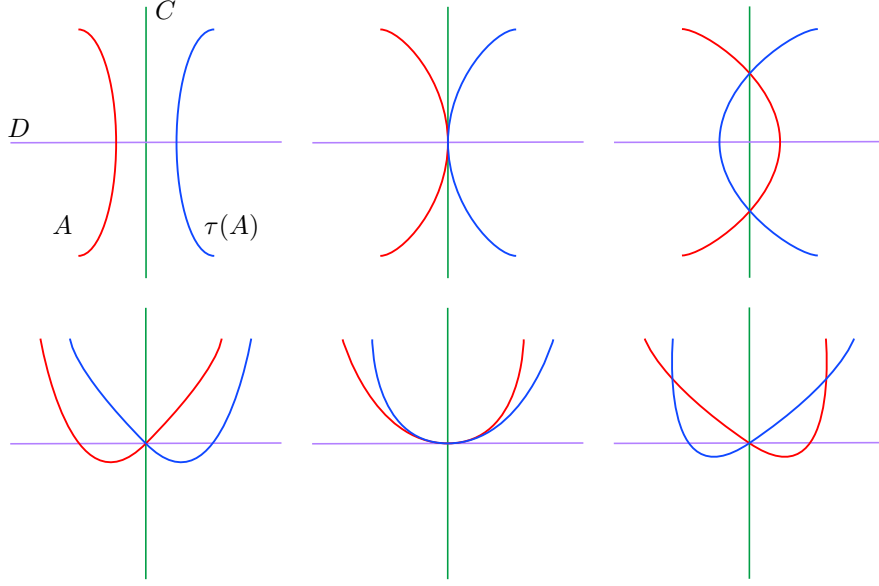


FIGURE 4.2. Two 1-parameter families of equivariant isotopies. The fixed set of the involution, C , is drawn here in green; the involution is given by reflection. The red and blue curves A and $\tau(A)$ represent slices of the stable and unstable manifolds which are interchanged by the involution. In 1-parameter families, these may be tangent to one another at isolated times, as shown in the central figures. The purple line, D , represents an arc intersecting C in a single point p , whose tangent space at p is isomorphic to $(T_p C)^\perp$. The figure illustrates that in 1-parameter families, the two failures of transversality in the equivariant setting correspond either to tangencies between A and C , or to tangencies between A and D (in the non-equivariant setting).

Definition 4.10. A k -parameter family of real gradients on (Y, γ, τ) is a family of pairs (g^μ, f^μ) for $\mu \in \mathbb{R}^k$ where $\{g^\mu\}$ is a family of τ -invariant metrics and $\{f^\mu\}$ is a family of real functions on (Y, γ, τ) . Let $X^\mu := \text{grad}_{g^\mu}(f^\mu)$ for $\mu \in \mathbb{R}^k$.

Both g^μ and f^μ depend smoothly on $(x, \mu) \in Y \times \mathbb{R}^k$ and the set of such pairs is endowed with the Whitney topology. The topological space of such k -parameter families is denoted $X_k^g(Y, \gamma, \tau)$.

Definition 4.11. A parameter value $\bar{\mu} \in \mathbb{R}^k$ is called a *bifurcation value* for $\{X^\mu\}$ in $X_k^g(Y, \gamma, \tau)$ if $X^{\bar{\mu}}$ fails to be Morse-Smale.

At a bifurcation value $\bar{\mu}$, the vector field $X^{\bar{\mu}}$ either has an *orbit of tangency* between stable and unstable manifolds (i.e. an integral curve along which a stable and unstable manifold are tangent) or at least one non-hyperbolic singular point. The *bifurcation set* (or *bifurcation diagram*) of $\{X^\mu\}$ in $X_k^g(Y, \gamma, \tau)$ is the subset of $\mathbb{R}^k$ consisting of all bifurcation values.

4.2. Codimension-1 bifurcations. In codimension 1 families of gradients, we expect tangencies of the various ascending and descending manifolds. However, we require that these tangencies be as generic as possible.

Definition 4.12. [JTZ21, Definition 5.11] Let v be a vector field on a 3-manifold Y . Let A and B be invariant submanifolds of v . Let φ_t be the flow of v and let $t_x := \gamma^{-1}(x)$ for $x \in \gamma$. We say that A and B have a *real quasi-transversal tangency* if their intersection contains a connected integral curve γ (which is not a single point) such that:

- (QT-1) Either $\dim(T_r A + T_r B) = 2$ for all $r \in \gamma$ or, if $B = \tau(A)$ and A (and therefore B) is 1-dimensional, then $\dim(T_{r_0} A + T_{r_0} C) = 2$ for $\gamma \cap C = \{r_0\}$.
- (QT-2) If $\dim A = 2$ and $\dim B = 2$, we impose the following additional conditions. Let S be a smooth 2-dimensional cross-section for v which contains r .
 - (a) If $B \neq \tau(A)$ or $B = \tau(A)$ and γ is a type 1 tangency, take coordinates (x_1, x_2) near r in S where $A \cap S = \{x_2 = 0\}$ and $B \cap S = \{x_2 = F(x_1)\}$ for some smooth function F . Condition (QT-1) forces $F(0) = 0$ and $\partial_{x_1} F(0) = 0$. We require that $\partial_{x_1}^2 F(0) \neq 0$.
 - (b) If $B = \tau(A)$ and γ is a type 2 tangency, we choose local coordinates (x_1, x_2) as above, but require that $\partial_{x_1}^3 F(0) \neq 0$.

Remark 4.13. Real quasi-transversality differs from the usual notion in two ways. First, in a 1-parameter family a 1-dimensional stable (or unstable) manifold A will generically intersect the fixed point set a finite number of times. At such an intersection, A and $\tau(A)$ are necessarily tangent to one another, though they are both 1-dimensional. Therefore, we impose a condition on $\dim(T_{r_0} A + T_{r_0} C)$ rather than on $\dim(T_{r_0} A + T_{r_0} B)$.

Second, in the case that γ is a type 2 tangency, F is generically cubic, rather than quadratic. This can be seen as follows. Since γ is type 2, the manifold A can be expressed as the graph of a function $F : \mathbb{R}\langle v \rangle \oplus E^- \rightarrow \mathbb{R}$ with $F(0) = 0$ and $\partial_{x_1} F(0) = 0$. In the slice S , we can choose coordinates (y_1, y_2) so that $A \cap S = \{y_2 = F(y_1)\}$; the symmetry implies that $B \cap S = \{y_2 = F(-y_1)\}$. Fix coordinates

$$\begin{aligned} x_1 &= y_1 \\ x_2 &= y_2 - F(y_1). \end{aligned}$$

In these coordinates, $A \cap S = \{x_2 = 0\}$ and $B \cap S = \{x_2 = F(-x_1) - F(x_1)\}$. Let $H(x_1) = F(-x_1) - F(x_1)$. Expanding $F(x_1) = ax_1^2 + bx_1^3 + O(x_1^4)$, we see:

$$H(x_1) = (ax_1^2 - bx_1^3 + O(x_1^4)) - (ax_1^2 + bx_1^3 + O(x_1^4)) = -2bx_1^3 + O(x_1^4).$$

For this reason, we cannot expect the version of (QT-2) stated in [JTZ21] to hold in all cases in the real setting.

In the equivariant setting, we of course consider *orbits of orbits of tangencies*. To avoid this awkward language, we will use the term $\{1\}$ -orbits of tangencies to refer to orbits of tangency which are fixed by the involution and $\mathbb{Z}/2$ -orbits of tangency to refer to pairs of orbits of tangency which are exchanged by the involution.

Remark 4.14. In Figures 4.3 through 4.10, we provide schematics for the various codimension 1 and 2 bifurcations we will encounter. We have color-coded these figures for the sake of readability. In each figure, we have drawn the fixed point set of Y as a horizontal green line. Gray lines indicate orbits corresponding to transverse intersections between stable and unstable manifolds; these can occur in three situations: between the unstable manifold of an index 1 critical point and the stable manifold of an index 2 critical point; between the unstable manifold of an index 1 critical point and the stable manifold of an index 1-2 saddle node; or between the unstable

manifold of an index 1-2 saddle node and the stable manifold of an index 2 critical point. Red (respectively blue) lines correspond to quasi-transverse orbits of tangencies between stable and unstable manifolds of index 1 (respectively index 2) critical points. Finally, lines corresponding to real quasi-transverse orbits of tangency will be colored green, emphasizing that these flow lines intersect the fixed point set.

Lemma 4.15. *A generic 1-parameter family X^t of real gradient-like vector fields is Morse-Smale at all but finitely values of t , where exactly one of the following phenomena can occur:*

- (1) X^t has a single non-hyperbolic $\mathbb{Z}/2$ -singularity;
- (2) X^t has a single non-hyperbolic $\{1\}$ -singularity;
- (3) X^t has a single $\mathbb{Z}/2$ -orbit real quasi-transversal tangency;
- (4) X^t has a single $\{1\}$ -orbit real quasi-transversal tangency.

Proof. With regard to singularities, the generic occurrence of only (1) and (2) follows from Theorem 12 and Theorem 13. Case (3) involving orbits of tangency appearing in $\mathbb{Z}/2$ -orbits can easily be reduced to the unreal case. Hence, it suffices to consider $\{1\}$ -orbits between critical points.

First, suppose that γ is an orbit of tangency between the ascending manifold of an index 1 critical point p and the descending manifold of an index 2 critical point q . Note that it must be the case that $q = \tau(p)$. As in Lemma 4.9, we choose a standard neighborhood and consider now families of curves A_t and $\tau(A_t)$ for $t \in [-1, 1]$ appearing in a slice S containing the fixed point set C . Let D be a curve in S with $\tau(D) = D$ and $T_\tau D = (T_\tau C)^\perp$. Again, see Figure 4.2. In a generic 1-parameter family, A_t will generically be transverse to both C and D at all but finitely many times t (with isolated tangencies) and hence the same holds for the families A_t and $\tau(A_t)$. See Figure 4.3.

Next, we consider the case that γ is an orbit of tangency between the descending manifold of an index 1 critical point p and the ascending manifold of an index 2 critical point q (which must be equal to $\tau(p)$). First, note that $W^s(p_t)$ and $W^u(q_t)$ are tangent precisely when $W^s(p_t)$ intersects C . In a 1-parameter family, at isolated parameters t the manifold $W^s(p_t)$ will generically intersect C in a single point. $\square$

Remark 4.16. In the unreal setting, quasi-transversal orbits can be assumed to satisfy certain non-degeneracy conditions. These are denoted (ND-1)-(ND-4) in [JTZ21] (similar conditions are also discussed in [Veg85] and [CP89]). As we will ultimately pass to isotopy diagrams, these conditions do not play any substantial role. The only condition which is relevant to us is (ND-2), which states that, generically, the contracting eigenvalues of the linear part of $X^{\bar{t}}$ at a critical point $p_1^{\bar{t}}$ of index 1 are distinct. This implies that there is a unique 1-dimensional manifold $W^{ss}(p_1^{\bar{t}}) \subset W^s(p_1^{\bar{t}})$ so that $T_{p_1^{\bar{t}}} W^{ss}(p_1^{\bar{t}})$ is the eigenspace of $L_{p_1^{\bar{t}}} X^{p_1^{\bar{t}}}$ corresponding to the strongest contracting eigenvalue. This condition is still generic in the equivariant setting.

As in the unreal case, non-hyperbolic singularities translate to (de)stabilizations of Heegaard diagrams and flows between critical points of the same index translate to handle slides. However, as noted above, in the real case, we can also have flows from critical points of index 2 to their index 1 images under the action of the involution. This codimension-1 bifurcation gives rise to a new move of real Heegaard diagrams, which we call a *crossover*. See Definition 5.18. While index 0-1 and index 2-3 saddle nodes do appear in the real setting, the analysis in these cases follows directly from [JTZ21] since these only appear in $\mathbb{Z}/2$ -orbits.

4.3. Codimension-2 bifurcations. We now turn to 2-parameter families of gradients. Codimension-1 bifurcations appear in open and dense families in these situations, and form smooth curves in

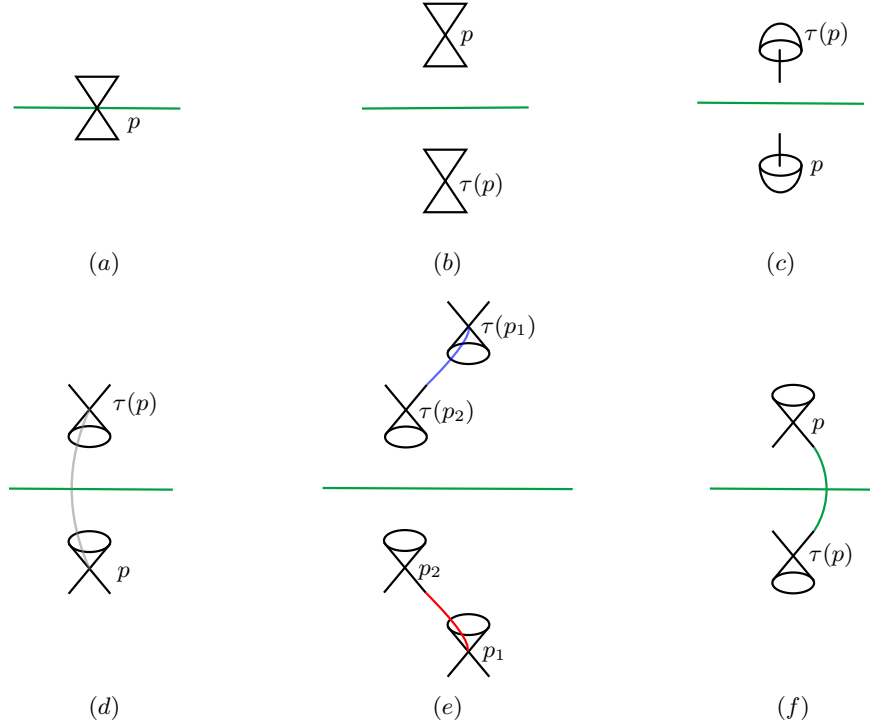


FIGURE 4.3. Some possible bifurcations appearing in 1-parameter families of real gradient vector fields. The gradient flow always goes upwards. (a) an index 1-2 birth-death in a $\{1\}$ -orbit; (b) an index 1-2 birth-death in a $\mathbb{Z}/2$ -orbit; (c) an index 0-1 birth-death and a symmetric 2-3 birth-death; (d) a quasi-transversal orbit of tangency from an index 1 critical point to an index 2 critical point (these can appear in both $\{1\}$ and $\mathbb{Z}/2$ -orbits); (e) a $\mathbb{Z}/2$ -orbit of tangency between index 1 critical points p_1 and p_2 , as well as between their index 2 images $\tau(p_1)$ and $\tau(p_2)$; (f) a quasi-transversal orbit of tangency from an index 2 critical point p and its image $\tau(p)$.

the parameter space $\mathbb{R}^2$. Moreover, for a generic $\{X^\mu\} \in X_2^g(Y, \gamma, \tau)$, at isolated points $\bar{\mu}$, we have the following additional bifurcations. Throughout, $\mathcal{O} \in \{\{1\}, \mathbb{Z}/2\}$.

- (A) The vector field X^μ has exactly two real quasi-transversal $\mathcal{O}$ -orbits of tangency and all singularities are hyperbolic. There are three possible configurations: two $\mathbb{Z}/2$ -orbits, two $\{1\}$ -orbits, or one of each.
- (B) The vector field X^μ has exactly one non-hyperbolic critical orbit, which is an orbit of saddle-nodes, and exactly one real quasi-transversal $\mathcal{O}$ -orbit of tangency. There are several possible configurations.
- (C) The vector field X^μ has exactly two non-hyperbolic critical orbits, which are orbits of saddle-nodes, and all stable and unstable manifolds intersect transversely.
- (D) The vector field X^μ has exactly one codimension 2 non-hyperbolic critical orbit (i.e. a $A_3(\mathbb{Z}/2)$ singularity or an $A_4(\{1\})$ singularity).
- (E) All singularities of X^μ are hyperbolic and there is a violation of the quasi-transversality condition.

As in [JTZ21], we will largely ignore bifurcations corresponding to isotopies, as ultimately, we will be interested in real Heegaard diagrams only up to isotopy.

Next, we describe the bifurcation sets in $\mathbb{R}^2$ near a parameter $\bar{\mu}$ corresponding to one of the codimension 2 bifurcations (A)-(E) described above. We will also consider *secondary bifurcations*, in which the singularities contributing to the failures of the transversality and hyperbolicity conditions interact with other singularities, giving rise to additional codimension-1 strata.

Type (A) secondary bifurcations involve two simultaneous real quasi-transversal $\mathcal{O}$ -orbits of tangency. Necessarily, these bifurcations involve index 1 and 2 critical points. We shall largely ignore orbits of tangency involving the unstable manifold of index 1 critical points and the stable manifold of an index 2 critical point, because these correspond to isotopies of attaching curves and we ultimately pass to isotopy diagrams.

Throughout this section, critical points of index 1 will be written as p^μ (perhaps with some decoration) and so the index 2 points will be written $\tau(p^\mu)$.

- (A1) There are four hyperbolic orbits $(\mathbb{Z}/2) \cdot p_1^\mu$, $(\mathbb{Z}/2) \cdot p_2^\mu$, $(\mathbb{Z}/2) \cdot p_3^\mu$, and $(\mathbb{Z}/2) \cdot p_4^\mu$ such that the orbits of tangencies are contained in $(\mathbb{Z}/2) \cdot (W^u(p_1^\mu) \cap W^s(p_2^\mu))$ and $(\mathbb{Z}/2) \cdot (W^u(p_3^\mu) \cap W^s(p_4^\mu))$. Each critical point is of index 1 or 2; the orbits are not necessarily distinct. Generically, the bifurcation diagram consists of two smooth curves which intersect transversely in a point. Such bifurcations are shown schematically in Figure 4.4.
- (A2) There are three hyperbolic singular orbits $(\mathbb{Z}/2) \cdot p_1^\mu$, $(\mathbb{Z}/2) \cdot p_2^\mu$, and $(\mathbb{Z}/2) \cdot p_3^\mu$ of X^μ so that the orbits of tangency are contained in $(\mathbb{Z}/2) \cdot (W^u(p_1^\mu) \cap W^s(p_2^\mu))$ and $(\mathbb{Z}/2) \cdot (W^u(p_2^\mu) \cap W^s(p_3^\mu))$. Again, see Figure 4.4 for a schematic.
- (A3) There are hyperbolic orbits $(\mathbb{Z}/2) \cdot p_1^\mu$ and $(\mathbb{Z}/2) \cdot p_2^\mu$ of X^μ which each have a $\{1\}$ -orbit of tangency from $W^u(\tau(p_i))$ to $W^s(p_i)$ for $i \in \{1, 2\}$; both manifolds are 1-dimensional. We allow the case that $p_1^\mu = p_2^\mu$, in which case $W^u(\tau(p_i))$ and $W^s(p_i)$ each intersect C twice. Generically, the bifurcation diagram consists of two smooth curves which intersect transversely at $\bar{\mu}$. See Figure 4.5.
- (A4) There are three hyperbolic orbits $(\mathbb{Z}/2) \cdot p_1^\mu$, $(\mathbb{Z}/2) \cdot p_2^\mu$, $(\mathbb{Z}/2) \cdot p_3^\mu$ of X^μ such that there is a $\{1\}$ -orbit of tangency from $W^u(\tau(p_1))$ to $W^s(p_1)$ as well as a $\mathbb{Z}/2$ -orbit of tangency from $W^u(p_2)$ to $W^s(p_3)$ (and symmetrically for $\tau(p_1)$ and $\tau(p_2)$). We allow $p_1^\mu = p_2^\mu$ or $p_1^\mu = p_3^\mu$. The possible configurations are shown in Figure 4.5. In case (A4c), secondary bifurcations may be present due to tangencies between $W^s(p_2^\mu)$ and $W^u(\tau(p_2^\mu))$ if $W^u(p_1^\mu) \cap W^s(p_2^\mu) \neq \emptyset$. See Figure 4.5.

Type (B) secondary bifurcations involve a single saddle-node orbit and a single real quasi-transversal $\mathcal{O}$ -orbit of tangency. All singularities have index 1 and 2. Unlike the previous case, saddle-nodes can appear in both $\mathbb{Z}/2$ and $\{1\}$ -orbits, which makes the analysis of secondary bifurcations in this case somewhat more involved.

- (B1) The vector field X^μ has a saddle-node orbit $(\mathbb{Z}/2) \cdot p^\mu$ and one quasi-transverse $\mathbb{Z}/2$ -orbit of tangency between $(\mathbb{Z}/2) \cdot W^u(p_1^\mu)$ and $(\mathbb{Z}/2) \cdot W^u(p_2^\mu)$, where p_1^μ and p_2^μ are hyperbolic saddle points of index 1. The bifurcation curve consists of two curves intersecting transversely at $\bar{\mu}$. See Figure 4.6.
- (B2) The vector field X^μ has a saddle-node orbit $(\mathbb{Z}/2) \cdot p^\mu$ and a hyperbolic saddle-orbit $(\mathbb{Z}/2) \cdot \bar{p}^\mu$ and a single orbit of tangency between $W^{ss}(\bar{p}^\mu)$ and $W^u(\bar{p}^\mu)$ (and a symmetric orbit between $W^{uu}(\tau(\bar{p}^\mu))$ and $W^s(\tau(\bar{p}^\mu))$). Secondary bifurcations arise due to the occurrence of tangencies in between $W^u(p_*^\mu)$ and the unstable manifolds of index 2 critical points. If there are s such tangencies, the bifurcation diagram consists of $s + 3$ codimension-1 strata meeting at $\bar{\mu}$. See Figure 4.6.

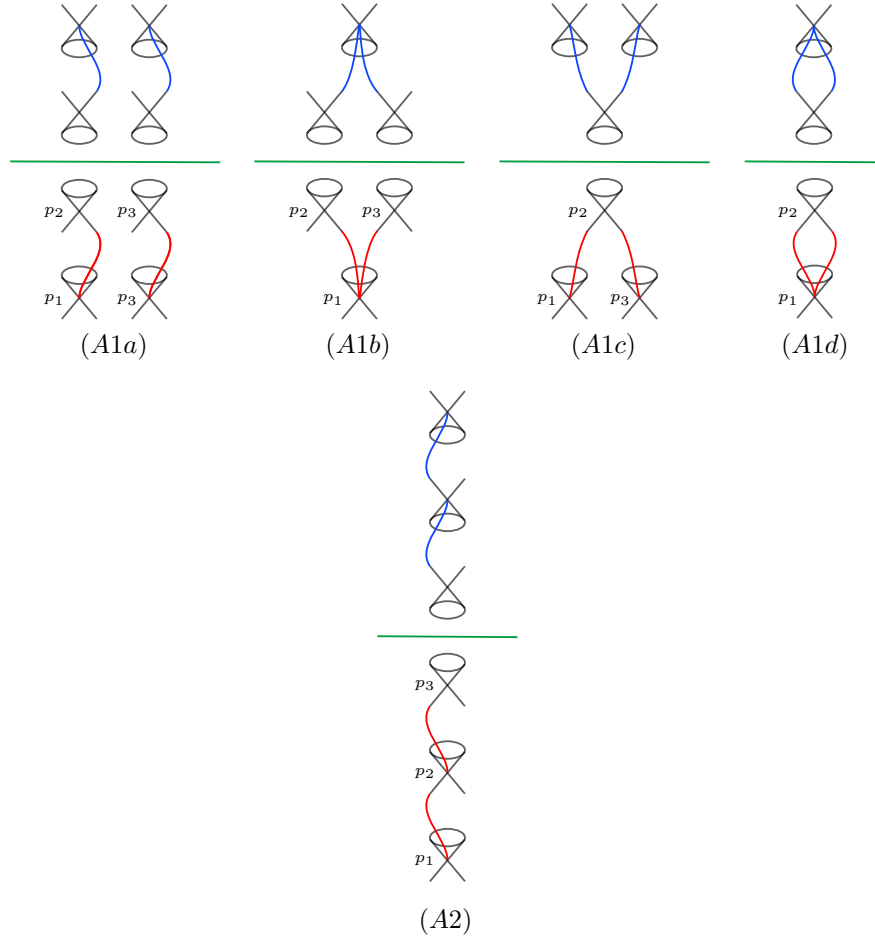


FIGURE 4.4. Codimension 2 bifurcations of types (A1) and (A2), which involve simultaneous orbits of tangency between a one-dimensional and a two-dimensional submanifold.

- (B3) The vector field $X^{\bar{\mu}}$ has a saddle-node orbit $(\mathbb{Z}/2) \cdot p^{\bar{\mu}}$ and a hyperbolic orbit $(\mathbb{Z}/2) \cdot p^{\bar{\mu}}$ with a single orbit of tangency between $W^{ss}(p^{\bar{\mu}})$ and $W^u(p^{\bar{\mu}})$ (and a symmetric orbit between $W^{uu}(\tau(p^{\bar{\mu}}))$ and $W^s(\tau(\bar{p}^{\bar{\mu}}))$). The bifurcation diagram consists of three codimension-1 strata meeting at $\bar{\mu}$. See Figure 4.7 (where $s = 1$).
- (B4) The vector field $X^{\bar{\mu}}$ has a saddle-node orbit $(\mathbb{Z}/2) \cdot p^{\bar{\mu}}$ and one quasi-transverse $\mathbb{Z}/2$ -orbit of tangency between $W^s(q^{\bar{\mu}})$ and $W^u(\tau(q^{\bar{\mu}}))$, where $q^{\bar{\mu}}$ is a hyperbolic singularity of index 1. The bifurcation curve consists of two curves intersecting transversely at $\bar{\mu}$. See Figure 4.7.
- (B5) The vector field $X^{\bar{\mu}}$ has a saddle-node orbit $(\mathbb{Z}/2) \cdot p^{\bar{\mu}}$ with a single orbit of tangency between $W^{uu}(p^{\bar{\mu}})$ and $W^{ss}(\tau(p^{\bar{\mu}}))$ (this occurs when $W^{uu}(p^{\bar{\mu}})$ intersects C). The bifurcation diagram consists of three codimension-1 strata meeting at $\bar{\mu}$. See Figure 4.7.

Remark 4.17. We note that bifurcations (B4) and (B5) are analogous to bifurcations (B1) and (B3), respectively, though they involve crossovers (see Definition 5.18), rather than handleslides. Bifurcation (B2) has no codimension 2 analogue for crossovers.

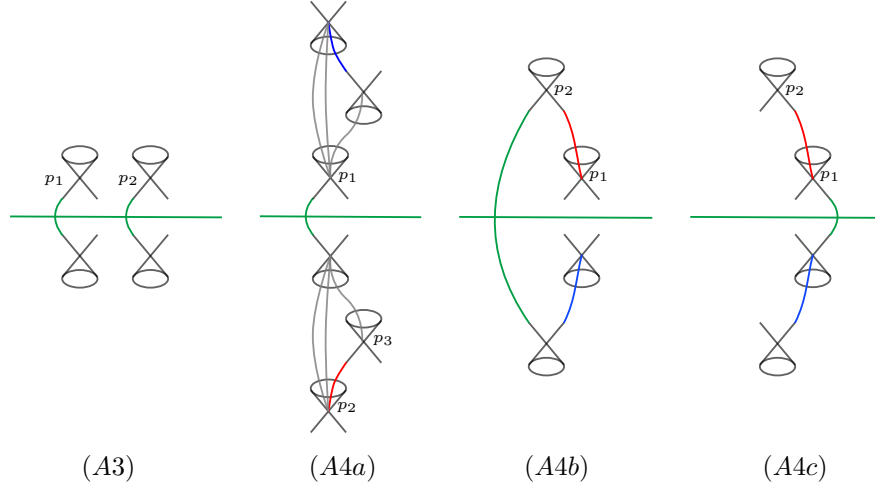


FIGURE 4.5. Codimension 2 bifurcations of types (A3) and (A4) that consist of simultaneous orbits of tangency involving at least one 1-dimensional sub-manifold.

Type (C) bifurcations consist of two simultaneous saddle-node orbits. These, of course, can appear in several configurations.

- (C) The vector field $X^{\bar{\mu}}$ has two saddle-node orbits $(\mathbb{Z}/2) \cdot p_1$ and $(\mathbb{Z}/2) \cdot p_2$. The bifurcation diagram consists of two curves corresponding to $(\mathbb{Z}/2) \cdot p_1$ and $(\mathbb{Z}/2) \cdot p_2$ which intersect transversely. When at least one of $\{p_1, p_2\}$ is not fixed by τ , we assume that $f(p_1) < f(p_2)$; in the case that both p_1 and p_2 are fixed by τ , necessarily $f(p_1) = f(p_2) = 0$. See Figure 4.8.

Type (D) bifurcations correspond to (real) codimension 2 failures of the Morse condition. In the real setting, there are two possibilities.

- (D1) In a neighborhood of $\bar{\mu}$, the bifurcation diagram consists of values μ for which X^μ (and therefore f^μ) has a degenerate critical point near $(\mathbb{Z}/2) \cdot p$. For an open and dense class of families $\{f^\mu\}$ for which $\text{grad}(f^\mu)$ has an $A_3^\pm$ singularity appearing in a $\mathbb{Z}/2$ -orbit, there are μ -dependent local coordinates (x, y, z) in which

$$f^\mu(x, y, z) = \pm x^4 + \mu_1 x^2 + \mu_2 x \pm y^2 \pm z^2.$$

The bifurcation diagram is the familiar cusp. See the left frame of Figure 4.9.

- (D2) In a neighborhood of $\bar{\mu}$, the bifurcation diagram consists of values μ for which X^μ (and therefore f^μ) has a degenerate critical point near p . For an open and dense class of families $\{f^\mu\}$ for which $\text{grad}(f^\mu)$ has an A_4 singularity appearing in a trivial orbit, there are μ -dependent local coordinates (x, y, z) in which

$$f^\mu(x, y, z) = x^5 + \mu_1 x^3 + \mu_2 x + y^2 - z^2.$$

See the right frame of Figure 4.9.

Finally, we consider failures of real quasi-transversality.

- (E1) There is a violation of (QT-1), i.e. there is a parameter value $\bar{\mu}$ and hyperbolic orbits $(\mathbb{Z}/2) \cdot p_1$ and $(\mathbb{Z}/2) \cdot p_2$ for which there is an orbit of tangency between $W^s(p_1^{\bar{\mu}})$ and $W^u(\tau(p_2^{\bar{\mu}}))$ which are both 1-dimensional, and, symmetrically, there is an orbit of tangency between $W^u(\tau(p_1^{\bar{\mu}}))$ and $W^s(p_2^{\bar{\mu}})$ (which is also 1-dimensional). Here, $p_1^{\bar{\mu}}$ and

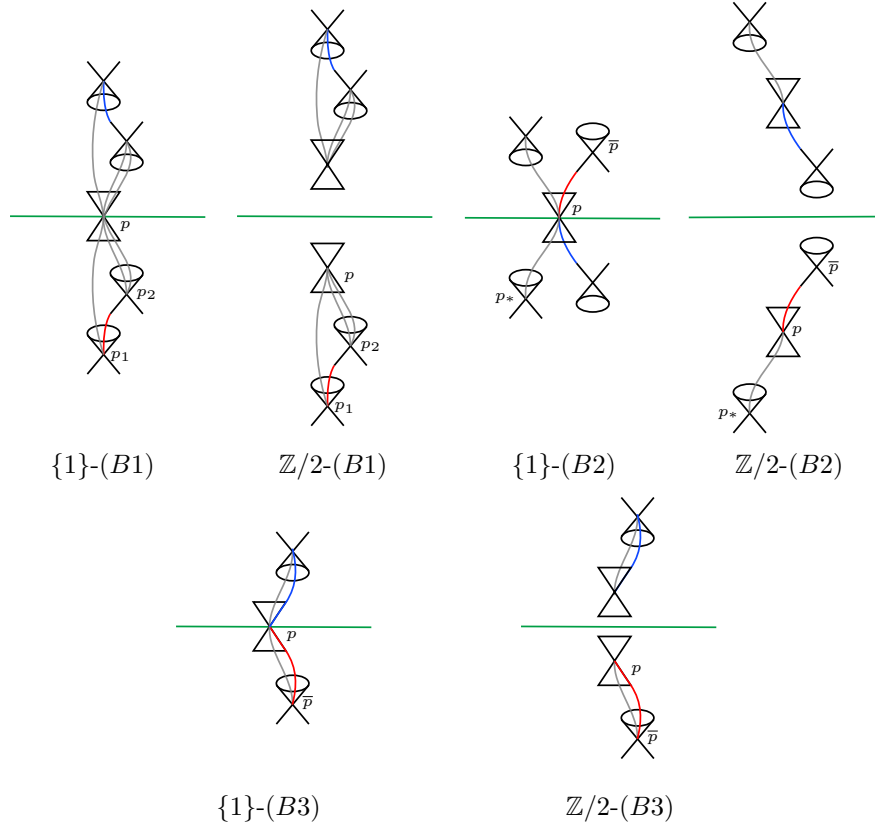


FIGURE 4.6. Codimension-2 bifurcations of type (B) involving pairs of quasi-transversal orbits of tangency.

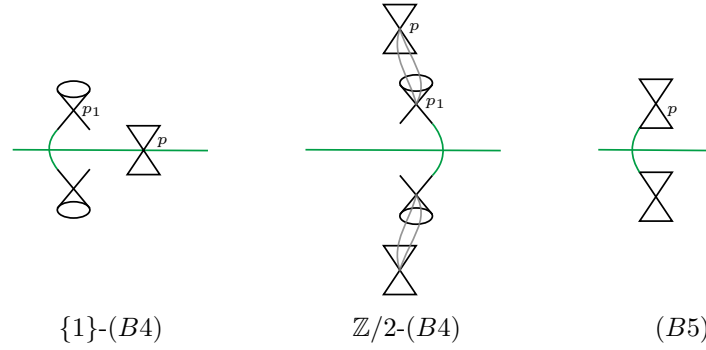


FIGURE 4.7. Codimension-2 bifurcations of type (B) involving a single real quasi-transversal orbit of tangency.

p_2^μ are critical points of index 1. There are, of course, two possible configurations, depending on whether or not $p_1 = p_2$. Secondary bifurcations appear in the presence of orbits of tangency between $(\mathbb{Z}/2) \cdot W^s(\tau(p_2^\mu))$ and a stable manifold of dimension 2

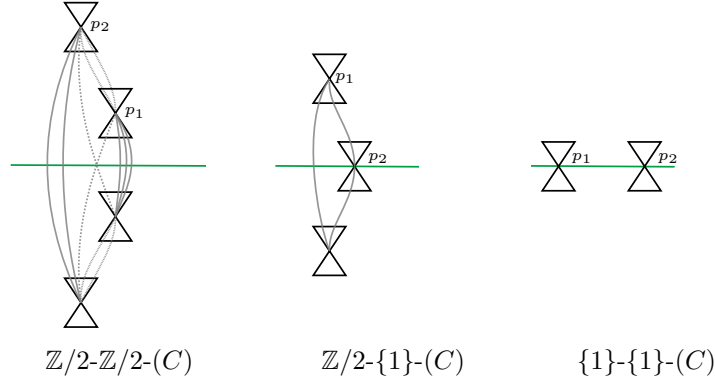


FIGURE 4.8. Codimension-2 bifurcations of type (C): these come in three varieties, depending on the orbit types of the saddle nodes. In the first frame, we have varied the shading to emphasize the different kinds of transverse orbits that may appear.

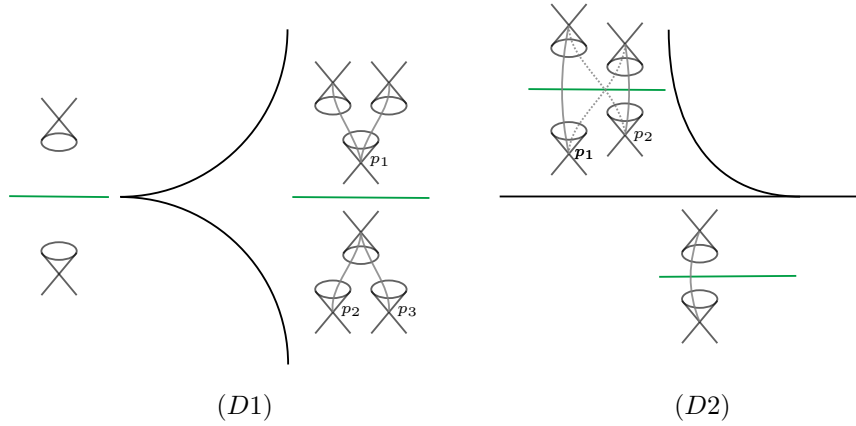


FIGURE 4.9. Codimension-2 bifurcations of type (D) involving a real singularity of codimension 2.

intersecting $(\mathbb{Z}/2) \cdot W^u(p_1^\mu)$ or between $(\mathbb{Z}/2) \cdot W^u(p_1^\mu)$ and a stable manifold of dimension 2 intersecting $(\mathbb{Z}/2) \cdot W^u(\tau(p_2^\mu))$. See Figure 4.10.

- (E2) There is a hyperbolic orbit $(\mathbb{Z}/2) \cdot p^\mu$ of $X^{\bar{\mu}}$ which has a $\{1\}$ -orbit of tangency γ between $W^u(\tau(p^\mu))$ and $W^s(p^\mu)$ as well as a quasi-transverse orbit of tangency between the directional manifold $D(p^\mu, \gamma)$ and the unstable manifold of an index 2 critical point. See Figure 4.10.

Remark 4.18. In 2-parameter families, we will usually not need to consider both directional manifolds. In cases in which γ is clear from the context, we will write $D(p^\mu)$ and suppress γ from the notation.

4.4. Real sutured functions and gradient-like vector fields. Let (Y, γ, τ) be a real sutured 3-manifold.

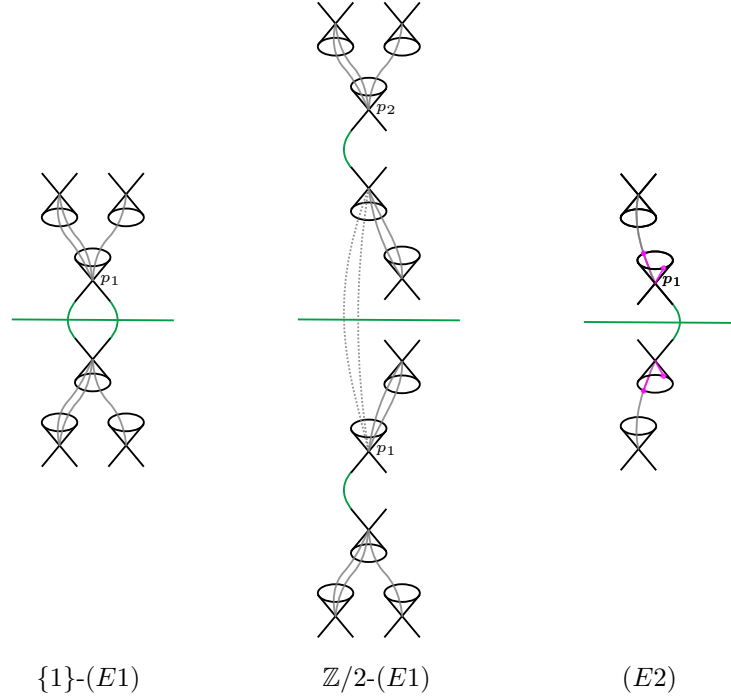


FIGURE 4.10. Type (E) codimension-2 bifurcations. In the third frame, we have drawn the directional manifold in pink.

Definition 4.19. A *real sutured function* on (Y, γ, τ) is a smooth function $f : Y \rightarrow [-1, 1]$ which satisfies:

- (1) $f^{-1}(\pm 1) = R_{\pm}(\gamma)$ and $f^{-1}(0) \supset s(\gamma)$;
- (2) f has no critical points along $R(\gamma)$;
- (3) $f|_{\gamma}$ has no critical points;
- (4) $f \circ \tau = -f$.

For such a function f , we write $C(f)$ for the set of critical points of f . Note that $C(f)$ consists of points rather than orbits. Necessarily, $C(f)$ lies in the interior of Y .

Definition 4.20. Given a real sutured function f on (Y, γ, τ) , we say that a vector field v on Y is a *real gradient-like vector field* of f if

- (1) $v(f) > 0$ on $Y \setminus C(f)$,
- (2) $C(f)$ has a neighborhood U so that $v|_U = \text{grad}_g(f|_U)$ for some τ -invariant Riemannian metric g on U .

Let $\mathcal{FV}(Y, \gamma, \tau)$ be the space of pairs (f, v) where f is a real function on (Y, γ, τ) and v is a real gradient-like vector field for f . The space $\mathcal{FV}(Y, \gamma, \tau)$ is given the C^∞ -topology. We write $\mathcal{FV}_0(Y, \gamma, \tau)$ for the subspace of Morse-Smale pairs.

Definition 4.21. We say that $(f, v) \in \mathcal{FV}(Y, \gamma, \tau)$ is *codimension-1* if $(f, v) \notin \mathcal{FV}_0(Y, \gamma, \tau)$, but v appears as $X^{\bar{\mu}}$ for some 1-parameter family $\{X^\mu\} \in X_1^g(Y, \gamma, \tau)$ which is generic in the sense of Section 4.2. Let $\mathcal{FV}_1(Y, \gamma, \tau)$ be the space of codimension-1 pairs and $\mathcal{FV}_{\leq 1}(Y, \gamma, \tau)$ be the union $\mathcal{FV}_0(Y, \gamma, \tau) \cup \mathcal{FV}_1(Y, \gamma, \tau)$.

Likewise, we say $(f, v) \in \mathcal{FV}(Y, \gamma, \tau)$ is *codimension-2* if $(f, v) \notin \mathcal{FV}_{\leq 1}(Y, \gamma, \tau)$, but v appears as $X^{\bar{\mu}}$ for some 2-parameter family $\{X^\mu\} \in X_2^g(Y, \gamma, \tau)$ which is generic in the sense of

Section 4.3. We write $\mathcal{FV}_2(Y, \gamma, \tau)$ for the space of codimension-2 pairs and $\mathcal{FV}_{\leq 2}(Y, \gamma, \tau)$ for the union $\mathcal{FV}_0(Y, \gamma, \tau) \cup \mathcal{FV}_1(Y, \gamma, \tau) \cup \mathcal{FV}_2(Y, \gamma, \tau)$.

Proposition 4.22. *Let $(f, v) \in \mathcal{FV}(Y, \gamma, \tau)$. Then, the space $G(f, v)^\tau$ of τ -invariant Riemannian metrics on Y for which $v = \text{grad}_g(f)$ is non-empty and contractible.*

Proof. This is the real analogue of [JTZ21, Proposition 5.19]. By that result, there exists a metric g' (not necessarily τ -invariant) such that $v = \text{grad}_g(f)$. We construct a τ -invariant one by averaging:

$$g = \frac{1}{2}(g' + \tau^*g').$$

We verify the gradient condition $g(v, \cdot) = df$:

$$\begin{aligned} g(v, Y) &= \frac{1}{2}(g'(v, Y) + (\tau^*g')(v, Y)) \\ &= \frac{1}{2}(df(Y) + g'(d\tau(v), d\tau(Y))). \end{aligned}$$

Since v is a real vector field, $d\tau(v) = -v$. Thus,

$$\begin{aligned} g(v, Y) &= \frac{1}{2}(df(Y) - g'(v, d\tau(Y))) \\ &= \frac{1}{2}(df(Y) - df(d\tau(Y))). \end{aligned}$$

Since f is a real function, $f \circ \tau = -f$, which implies $df \circ d\tau = -df$. We deduce that $g(v, Y) = df(Y)$. Hence, $v = \text{grad}_g(f)$, and $G(f, v)^\tau$ is non-empty.

To show contractibility, observe that the space of all τ -invariant Riemannian metrics is convex. The condition $g(v, \cdot) = df$ defines an affine subspace in the space of symmetric tensors. It follows that $G(f, v)^\tau$ is convex, and thus contractible. $\square$

Corollary 4.23. *The space $\mathcal{FV}(Y, \gamma, \tau)$ is weakly contractible.*

Proof. This follows just as in [JTZ21, Corollary 5.20]. Let $\mathcal{F}(Y, \gamma, \tau)$ be the space of real sutured functions and $\mathcal{G}(Y, \gamma, \tau)$ be the space of τ -invariant Riemannian metrics. Both spaces are convex and thus contractible. Therefore, their product $\mathcal{F}(Y, \gamma, \tau) \times \mathcal{G}(Y, \gamma, \tau)$ is contractible.

There is a Serre fibration

$$\pi : \mathcal{F}(Y, \gamma, \tau) \times \mathcal{G}(Y, \gamma, \tau) \rightarrow \mathcal{FV}(Y, \gamma, \tau)$$

defined by $\pi(f, g) = (f, \text{grad}_g(f))$. By Proposition 4.22, the map π is surjective and has contractible fibers.

We consider the long exact sequence of homotopy groups associated with the fibration π . Since the total space $\mathcal{F}(Y, \gamma, \tau) \times \mathcal{G}(Y, \gamma, \tau)$ and the fibers $G(f, v)^\tau$ are contractible, the homotopy groups of the base space $\mathcal{FV}(Y, \gamma, \tau)$ also vanish. Therefore, $\mathcal{FV}(Y, \gamma, \tau)$ is weakly contractible. $\square$

5. CONSTRUCTING HEEGAARD DIAGRAMS

This section follows [JTZ21, Section 6], translating generic singularities of Section 4.2 and Section 4.3 to real Heegaard diagrams: generic gradients produce real Heegaard diagrams, codimension-1 singularities give real Heegaard moves between them, and codimension-2 singularities give loops of moves.

Definition 5.1. Let $(f, v) \in \mathcal{FV}_{\leq 2}(Y, \gamma, \tau)$. We define a partition $C(f) = C_{01}(f) \cup C_{23}(f) \cup C_{fix}(f)$ as follows: a point $p \in C(f)$ belongs to $C_{01}(f)$ if one of the following holds:

- (1) $p \in C_0(f) \cup C_1(f)$;

- (2) p is an index 0-1 birth-death appearing in a $\mathbb{Z}/2$ -orbit;
- (3) p is an index 1-2 birth-death appearing in a $\mathbb{Z}/2$ -orbit, (f, v) is codimension-2 of type (B1), and $\mathcal{I}(p_1) = \mathcal{I}(p_2) = 1$;
- (4) p is an index 1-2 birth-death appearing in a $\mathbb{Z}/2$ -orbit, (f, v) is codimension-2 of type (B2) or (B3), and $\mathcal{I}(\bar{p}) = 1$;
- (5) p is an index 1-2 birth-death appearing in a $\mathbb{Z}/2$ -orbit, $(f, v) \in \mathcal{FV}_2(Y, \gamma, \tau)$ is of type (C), and q is another birth-death appearing in a $\mathbb{Z}/2$ -orbit with $q \neq \tau(p)$ and $f(p) < f(q)$;
- (6) (f, v) is codimension-2 of type (D1), and p is an index 1-0-1, 0-1-0, or a 1-2-1 birth-death-birth.
- (7) If p is an index 1-2 birth-death appearing in a $\mathbb{Z}/2$ -orbit and (f, v) is of codimension-1 of type (NH), p may be placed in either $C_{01}(f, v)$ or $C_{23}(f, v)$.

A point $p \in C(f)$ belongs to C_{fix} if one of the following holds:

- (1) p is an index 1-2 birth-death appearing in a trivial orbit.
- (2) (f, v) is codimension-2 of type (D2), and p is an A_4 singularity.

The remaining points are placed in $C_{23}(f)$.

Definition 5.2. A pair $(f, v) \in \mathcal{FV}_{\leq 2}(Y, \gamma, \tau)$ is *separable* if:

- (1) $C_{fix}(f) = \emptyset$;
- (2) it does not involve a tangency of any 1-dimensional strong manifolds, i.e., is not codimension-2 of type (A3), (A4), (B4), (B5), (E1), or (E2);
- (3) if it is codimension-2 of type (C), has birth-death singularities at $(\mathbb{Z}/2) \cdot p$ and $(\mathbb{Z}/2) \cdot q$, and at least one of $(\mathbb{Z}/2) \cdot p$ and $(\mathbb{Z}/2) \cdot q$ is a $\mathbb{Z}/2$ -orbit, then $f(p) \neq f(q)$.

Definition 5.3. Let $(f, v) \in \mathcal{FV}(Y, \gamma, \tau)$. We say that a properly embedded surface Σ separates (f, v) if

- (1) $\Sigma \pitchfork v$;
- (2) $Y = Y_- \cup Y_+$ so that $Y_- \cap Y_+ = \Sigma$;
- (3) $C_{01}(f, v) \subset Y_-$ and $C_{23}(f, v) \subset Y_+$.

Let $\Sigma(f, v)$ be the set of separating surfaces for (f, v) and let $\Sigma^R(f, v)$ be the set of real-invariant separating surfaces for (f, v) , i.e. surfaces Σ such that $\tau(\Sigma) = \Sigma$ and τ is orientation-reversing on Σ .

Remark 5.4. We emphasize that gradients with degenerate critical points on the fixed point set C cannot be separated by real surfaces. Indeed, if Σ is real and separating, it necessarily contains C . Therefore, if (f, v) has a degenerate critical point on C , the condition that $\Sigma \pitchfork v$ necessarily fails.

Definition 5.5. Let $(f, v) \in \mathcal{FV}_{\leq 2}(Y, \gamma, \tau)$ be separable. Then, define $\Gamma_{01}(f, v)$ to be the CW complex:

$$\Gamma_{01}(f, v) = \bigcup_{p \in C_{01}(f, v)} W^s(p).$$

We define $\Gamma_{23}(f, v) = \tau(\Gamma_{01}(f, v))$.

As $\Gamma_{01}(f, v)$ and $\Gamma_{23}(f, v)$ are symmetric, we will always work with the former, and simply write $\Gamma(f, v)$.

Lemma 5.6. *If $(f, v) \in \mathcal{FV}_{\leq 2}(Y, \gamma, \tau)$ is a separable pair, then the space $\Sigma^R(f, v)$ is non-empty and contractible.*

Proof. The corresponding fact for $\Sigma_{\pm}(f, v)$ is [JTZ21, Proposition 6.7]. The existence of such surfaces is constructive: one may simply take the boundary of a small regular neighborhood of

$\Gamma(f, v)$. To see the space is contractible, they note that if (f, v) is separable and $\Sigma \in \Sigma^R(f, v)$, then Σ intersects every flow line in a point. It follows that any other separating surface Σ' determines a homeomorphism $d_{\Sigma, \Sigma'} : \Sigma \rightarrow \mathbb{R}$ where $d_{\Sigma, \Sigma'}(x)$ is the time t such that $\phi_t(x) \in \Sigma'$ (here, ϕ_s is the flow of v). This establishes a homeomorphism $\Sigma(f, v) \cong C^\infty(\Sigma)$, and $C^\infty(\Sigma)$ is clearly contractible.

In the real setting, we can employ a similar strategy. To prove existence, as above we let Σ_0 be the boundary of a small regular neighborhood of $\Gamma(v, f)$. The surface Σ_0 is not invariant, but by reparametrizing v , we can ensure that $\tau(\Sigma_0)$ is equal to $\phi_1(\Sigma_0)$, where ϕ_t is the flow of v . It will follow that $\phi_{1/2}(\Sigma_0)$ is an invariant separating surface.

More precisely, let $W = Y \setminus (N(\Gamma(v, f)) \cup \tau(N(\Gamma(v, f))))$. Since Σ_0 is separating, so is $\tau(\Sigma_0)$, and therefore, W is diffeomorphic to a product. Furthermore, we have a projection

$$\pi_{\Sigma_0} : W \rightarrow \Sigma_0$$

by assigning to a point x the unique point in Σ_0 which is contained in the same flow line. Define $T : \Sigma_0 \rightarrow \mathbb{R}_{\geq 0}$ by the condition that $\phi_{T(x)}(x) \in \tau(\Sigma_0)$. Extend T to a function $\tilde{T} : W \rightarrow \mathbb{R}_{> 0}$ by requiring that $\tilde{T}$ is constant on flow lines: i.e., define $\tilde{T} = T \circ \pi_{\Sigma_0}$. Note that if $x \in \Sigma_0$ flows to a point y in $\tau(\Sigma_0)$, then

$$\tau(x) = \tau(\phi_{-T(x)}(y)) = \phi_{T(x)}(\tau(y)).$$

It follows that $T(\tau(x)) = T(x)$, and hence, $\tilde{T}$ is τ -invariant as well.

Therefore, we can define a vector field $w_x = T(x) \cdot v_x$, with flow ψ_t . As we have only rescaled v , separating surfaces for w are also separating surfaces for v . Now, for any $x \in \Sigma_0$, we have that $\psi_1(x) = \tau(x)$, since

$$\psi_1(x) = \phi_{T(x)}(x) \in \tau(\Sigma_0).$$

We define $\Sigma := \psi_{1/2}(\Sigma_0)$. Then Σ is τ -invariant:

$$\begin{aligned} \tau(\Sigma) &= \tau(\psi_{1/2}(\Sigma_0)) \\ &= \psi_{-1/2}(\tau(\Sigma_0)) \\ &= \psi_{-1/2}(\psi_1(\Sigma_0)) \\ &= \psi_{1/2}(\Sigma_0) = \Sigma. \end{aligned}$$

Hence, Σ is a separating surface for w , and therefore one for v as well.

The second claim follows by an identical argument as in the unreal case. If Σ' were another separating surface, then the homeomorphism $d_{\Sigma, \Sigma'}$ is a real-invariant map $\Sigma \rightarrow \mathbb{R}$, and gives a homeomorphism from $\Sigma^R(f, v)$ to the space of real invariant functions on Σ , i.e. $C_R^\infty(\Sigma)$, which is contractible. $\square$

5.1. Codimension-0. Given a separable pair $(f, v) \in \mathcal{FV}_{\leq 2}(Y, \gamma, \tau)$, we have seen that there is a contractible space of real Heegaard splittings. Of course, the process is reversible: given a real Heegaard diagram $(\Sigma, \alpha, \beta, \tau)$, there is an essentially unique pair $(f, v) \in \mathcal{FV}_0(Y, \gamma, \tau)$ which realizes it. This follows exactly as in [JTZ21, Section 6.2], so our treatment will be terse.

Given a separable pair $(f, v) \in \mathcal{FV}_{\leq 2}(Y, \gamma, \tau)$, we can choose a separating real Heegaard diagram $\Sigma \in \Sigma^R(f, v)$. If (f, v) is Morse-Smale, for each $p \in C_1(f)$, we have that $W^u(p) \cap \Sigma$ is an embedded circle, as is $\tau(p) \in C_2(f)$. These curves all intersect transversely.

Definition 5.7. Say $(f, v) \in \mathcal{FV}_0(Y, \gamma, \tau)$ and $\Sigma \in \Sigma^R(f, v)$. Then, we define $H(f, v, \Sigma) = (\Sigma, \alpha, \beta, \tau|_\Sigma)$ to be the Heegaard diagram by taking the alpha curves to be $W^u(p) \cap \Sigma$ for $p \in C_1(f)$ and the beta curves to be $W^s(q) \cap \Sigma$ for $q \in C_2(f)$. Of course, the alpha curves are interchanged with the beta curves under $\tau|_\Sigma$.

Remark 5.8. We will usually abuse notation, and write τ for the involution on Σ rather than τ_Σ .

Of course, $|C_1(f)|$ may be larger than the genus of Σ , in which case we obtain an *overcomplete diagram* [JTZ21, Definition 6.13]. These diagrams specify handle decompositions of (Y, γ, τ) with extra 0- and 3-handles. To obtain a Heegaard diagram in the traditional sense, one starts with the graph $\Gamma_-(f, v)$, removes the vertices $p \in C_1(f)$, and chooses a spanning tree T of $\Gamma_-(f, v)$. We define $H(f, v, \Sigma, T)$ to be the Heegaard diagram obtained by choosing the alpha curves to be $W^u(p) \cap \Sigma$ for $p \in C_1(f)$ and such that $W^s(p)$ is not an edge of T ; the symmetry determines the beta curves. Different choices of T give rise to real-equivalent diagrams.

The process can be reversed: given a real diagram $(\Sigma, \alpha, \beta, \tau)$, standard Morse theory allows one to construct a pair $(f, v) \in \mathcal{FV}_0(Y, \gamma, \tau)$ which satisfies $H(f, v, \Sigma) = (\Sigma, \alpha, \beta, \tau)$. For more details, see [JTZ21, Section 6.2]; the discussion there can be applied *mutatis mutandis* in the real setting.

5.2. Codimension-1. In this subsection, we prove equivariant analogues of the various isotopy extension lemmas of [JTZ21, Section 6.3].

Lemma 5.9 (Symmetric Isotopy Extension, cf. Lemma 6.19 in [JTZ21]). *Suppose $\{\mathcal{H}_t = (\Sigma_t, \alpha_t, \beta_t, \tau_t) : t \in I\}$ is a smooth 1-parameter family of real Heegaard diagrams in (Y, τ) such that $\alpha_t \pitchfork \beta_t$. Then there exists a $\mathbb{Z}/2$ -equivariant isotopy $d_t : Y \times I \rightarrow Y$ (i.e., $d_t \circ \tau = \tau \circ (d_t \times \text{id})$) such that $d_t(\mathcal{H}_0) = \mathcal{H}_t$. The space of such equivariant isotopies is contractible.*

Proof. This follows just as in [JTZ21, Lemma 6.19]: we construct an isotopy by integrating a time-dependent vector field on $Y \times I$. We consider the submanifolds $\Sigma_* = \bigcup_{t \in I} \Sigma_t \times \{t\}$, $\alpha_* = \bigcup_{t \in I} \alpha_t \times \{t\}$, and $\beta_* = \bigcup_{t \in I} \beta_t \times \{t\}$. Let $\tilde{\tau}(x, t) = (\tau(x), t)$ be the involution on $Y \times I$. We seek a vector field ν on $Y \times I$ which is $\tilde{\tau}$ -invariant ($d\tilde{\tau}(\nu) = \nu$), positively transverse to the horizontal foliation $\mathcal{F}$ (e.g., $dt(\nu) = 1$), and tangent to the submanifolds Σ_* , α_* , and β_* .

We construct ν by successive extensions, ensuring invariance using the averaging technique: $\nu = (\omega + d\tilde{\tau}(\omega))/2$. We first define ν on $T(\alpha_* \cap \beta_*)$ transverse to $\mathcal{F}$, which can clearly be chosen to be invariant. Then, we extend ν over $T\alpha_*$ in any way so that it is positively transverse to $\mathcal{F}$; ν is extended over $T\beta_*$ by pushing forward the vector field on $T\alpha_*$. We then extend to $T\Sigma_*$: since Σ_* is invariant, we can choose any extension and then apply the averaging trick. Finally, extend ν to $Y \times I$ invariantly.

The flow of the invariant vector field ν generates the required equivariant isotopy D . The space of such invariant vector fields is convex, hence contractible. $\square$

Lemma 5.10 (Symmetric version of Lemma 6.20 in [JTZ21]). *Let $\{\mathcal{H}_t\}$ and $\{\mathcal{H}'_t\}$ be 1-parameter families of real Heegaard diagrams (possibly overcomplete) connecting $\mathcal{H}_0$ and $\mathcal{H}_1$. If the two families are homotopic relative to their endpoints through real diagrams, then the induced equivariant diffeomorphisms $d_1, d'_1 : Y \rightarrow Y$ are equivariantly isotopic through equivariant diffeomorphisms mapping $\mathcal{H}_0$ to $\mathcal{H}_1$.*

Proof. Let $\mathcal{H}_{t,s}$ be the homotopy on $I \times I$. Equip $Y \times I \times I$ with the involution $\hat{\tau}(x, t, s) = (\tau(x), t, s)$. We construct a $\hat{\tau}$ -invariant vector field ν such that $\nu(t) = 1, \nu(s) = 0$, and ν is tangent to the submanifolds defined by $\mathcal{H}_{t,s}$. This construction follows the generalization of Lemma 5.9 to the 2-parameter setting, using the averaging technique to ensure invariance.

The flow of ν defines a 1-parameter family of equivariant diffeomorphisms $g_s : Y \rightarrow Y$, obtained by flowing from $t = 0$ to $t = 1$ at fixed s . By construction, $g_s(\mathcal{H}_0) = \mathcal{H}_1$. We know that g_0 is equivariantly isotopic to d_1 , and g_1 to d'_1 . The family $\{g_s\}$ provides the required equivariant isotopy. $\square$

Lemma 5.11 (Symmetric version of Lemma 6.21 in [JTZ21]). *Let $\{(f_t, v_t) \in \mathcal{FV}_0(Y, \tau) : t \in I\}$ be a family of real gradient-like vector fields. Let $\Sigma_i \in \Sigma^R(f_i, v_i)$. Let T^0 be a spanning*

tree for $\Gamma(f_0, v_0)$. Note that $\Gamma(f_t, v_t)$ is an isotopy which takes T^0 to a spanning tree T^1 for $\Gamma(f_1, v_1)$. Let $\mathcal{H}_0, \mathcal{H}_1$ be the corresponding real diagrams. Then there is an induced equivariant diffeomorphism $d : \mathcal{H}_0 \rightarrow \mathcal{H}_1$, equivariantly isotopic to the identity. The space of such equivariant diffeomorphisms is path-connected.

Proof. We first show there exists a smooth family of symmetric surfaces $\{\Sigma_t\}$ connecting Σ_0 and Σ_1 such that $\Sigma_t \in \Sigma^R(f_t, v_t)$. This follows from the fact that the fiber $\Sigma^R(f, v)$ is contractible by Lemma 5.6. We can adapt the patching argument from the original Lemma 6.21 in [JTZ21]: when interpolating between two symmetric surfaces, the difference function is anti-invariant, ensuring that the interpolated path consists of symmetric surfaces.

This yields a smooth family of symmetric diagrams $\{\mathcal{H}_t\}$. Applying Lemma 5.9, we obtain an equivariant isotopy $D = (d_t)$ realizing this family. The end point $d = d_1$ is the desired equivariant diffeomorphism.

If $\{\Sigma'_t\}$ is another such family, we construct a homotopy of symmetric surfaces $\Sigma_{t,u}$ connecting them, utilizing the convexity of the fibers $\Sigma^R(f_t, v_t)$. This yields a homotopy of symmetric diagrams $\mathcal{H}_{t,u}$. Applying Lemma 5.10 shows that the induced diffeomorphisms are equivariantly isotopic. $\square$

Lemma 5.12 (Symmetric version of Lemma 6.23 in [JTZ21]). *Let $\{(f_t, v_t) \in \mathcal{FV}_1(Y, \gamma, \tau) : t \in I\}$ be a 1-parameter family of separable gradients. Let $\Sigma_i \in \Sigma^R(f_i, v_i)$ be real Heegaard surfaces for $i \in \{0, 1\}$. This family induces an equivariant diffeomorphism $d : \Sigma_0 \rightarrow \Sigma_1$ that is well-defined up to equivariant isotopy.*

Proof. This follows just as in the proof of Lemma 5.11. $\square$

Lemma 5.13 (Symmetric version of Lemma 6.24 in [JTZ21]). *Let $A : I \rightarrow \mathcal{FV}_0^r(Y)$ be a loop based at (f_0, v_0) . Let $\mathcal{H}$ be a symmetric diagram for (f_0, v_0) . The loop A induces an equivariant diffeomorphism $d : \mathcal{H} \rightarrow \mathcal{H}$. If A is null-homotopic in $\mathcal{FV}_0^r(Y)$, then d is equivariantly isotopic to $\text{id}_{\mathcal{H}}$.*

Proof. Let $L : I \times I \rightarrow \mathcal{FV}_0^r(Y)$ be a null-homotopy of A . Since the fiber $\Sigma^R(f, v)$ is contractible, we can lift L to a 2-parameter family of symmetric surfaces $\Sigma_{t,u}$, which is constant equal to the base surface Σ when $t \in \{0, 1\}$ or $u = 1$.

This induces a family of symmetric diagrams $\mathcal{H}_{t,u}$, providing a homotopy between the path $P_0 = \{\mathcal{H}_{t,0}\}$ (induced by A) and the constant path $P_1 = \{\mathcal{H}_{t,1}\}$.

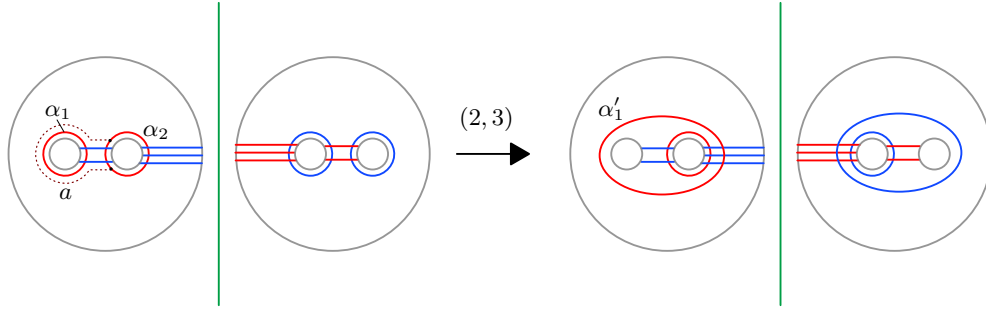
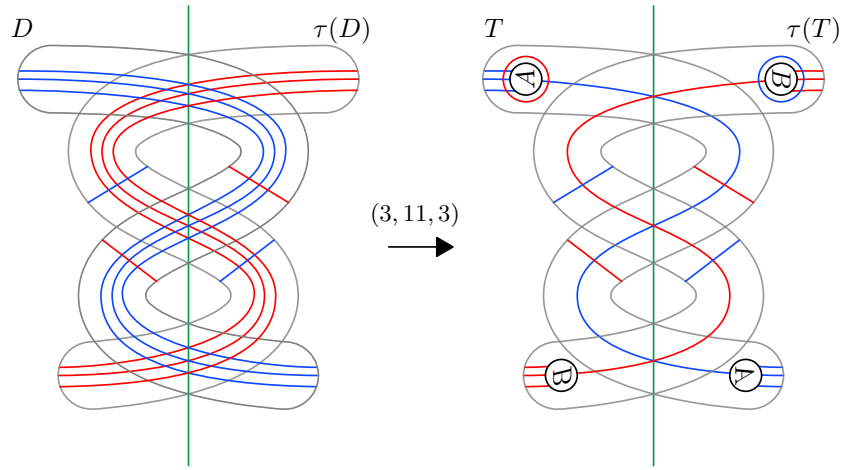
Let d_0, d_1 be the induced equivariant diffeomorphisms. We have $d_0 = d$ and d_1 is induced by the constant path, so d_1 is equivariantly isotopic to $\text{id}_{\mathcal{H}}$. By Lemma 5.10, d_0 and d_1 are equivariantly isotopic. Therefore, d is equivariantly isotopic to $\text{id}_{\mathcal{H}}$. $\square$

Next, we translate the codimension 1 bifurcations to moves between real Heegaard diagrams. There are moves appearing in both $\mathbb{Z}/2$ and trivial orbits.

Definition 5.14. A real Heegaard diagram $(\Sigma', \alpha', \beta', \tau')$ is obtained from $(\Sigma, \alpha, \beta, \tau)$ by a *generalized real handleslide of type (m, n)* if there are curves $\alpha_1, \alpha_2 \in \alpha$, a curve $\alpha'_1 \in \alpha'$, and an embedded arc $a \in \Sigma$ so that

- (1) $\alpha'_1 \setminus \alpha' = \alpha \setminus \alpha_1$;
- (2) $\partial a \subset \alpha_2$ and its interior is disjoint from α ;
- (3) there is a regular neighborhood N of $\alpha_2 \cup a$ so that $\partial N = \alpha_1 \cup \alpha'_1 \cup \tilde{\alpha}_2$, where $\tilde{\alpha}_2$ is a small parallel pushoff of α_2 , and the interior of N is disjoint from $\alpha \cup \alpha' \setminus \{\alpha_2\}$ and
- (4) if $\alpha_2 \setminus \partial a = \alpha_2^0 \cup \alpha_2^1$ so that $\alpha_2^0 \cup a$ is parallel to α_1 and $\alpha_2^1 \cup a$ is parallel to α'_1 , then $|\alpha_2^0 \cap \beta| = m$ and $|\alpha_2^1 \cap \beta| = n$.

Symmetric conditions hold for the beta curves. See Figure 5.1.


 FIGURE 5.1. A generalized real handleslide of type $(m, n) = (2, 3)$.

 FIGURE 5.2. A $\mathbb{Z}/2$ -stabilization of type $(k, \ell, c) = (3, 11, 3)$.

Definition 5.15. A real Heegaard diagram $(\Sigma', \alpha', \beta', \tau')$ is obtained from $(\Sigma, \alpha, \beta, \tau)$ by a *generalized $\mathbb{Z}/2$ -stabilization of type (k, ℓ, c)* if there is a disk $D \subset \Sigma$ and a punctured torus $T \subset \Sigma'$ as well as curves $\alpha \in \alpha'$ and $\beta \in \beta'$ so that:

- (1) $D \neq \tau'(D)$ and the intersection of ∂D and $\tau(\partial D)$ is transverse. Similarly for T and $\tau'(T)$;
- (2) $\Sigma \setminus D = \Sigma' \setminus T$;
- (3) $\alpha \setminus D = \alpha' \setminus T$, $\beta \setminus D = \beta' \setminus T$;
- (4) $\alpha \cap D$ and $\beta \cap D$ consist of ℓ and k arcs, respectively, and each component of $\alpha \cap D$ intersects each component of $\beta \cap D$ transversely in a single point;
- (5) α and β are contained in T and they intersect in a single point;
- (6) $(\alpha \setminus \alpha) \cap T$ consists of ℓ parallel arcs, each intersecting β transversely;
- (7) $(\beta \setminus \beta) \cap T$ consists of k parallel arcs, each intersecting α transversely;
- (8) the intersection $|D \cap \tau(D)|$ consists of c components;
- (9) for each component of $\alpha \cap D$, there is a corresponding component of $\alpha' \cap T$ with the same endpoints, and similarly for the beta curves.

These of course impose symmetric relations on $\tau(D)$, $\tau'(T)$, $\tau'(\alpha)$, and $\tau'(\beta)$. See Figure 5.2.

Remark 5.16. We emphasize that D and $\tau(D)$ are generally not disjoint (similarly for T and $\tau'(T)$); it is for this reason we need to keep track of a third parameter, c . See Figure 5.2 for an example.

We also note that this operation technically depends on the choice of disk D ; by replacing D and T with $\tau(D)$ and $\tau'(T)$, the same description above would specify a $\mathbb{Z}/2$ -stabilization of type (ℓ, k, c) rather than of type (k, ℓ, c) . For the sake of consistency, we will always label the 1-handle of T by A and $\tau'(T)$ by B (or, in the case of multiple $\mathbb{Z}/2$ -stabilization, the label for the T handle will always come first alphabetically). Therefore, k denotes the number of beta curves which crosses over the tube labeled by A .

Now, let us turn to stabilizations appearing in trivial orbits.

Definition 5.17. A real Heegaard diagram $(\Sigma', \alpha', \beta', \tau')$ is obtained from $(\Sigma, \alpha, \beta, \tau)$ by a *generalized $\{1\}$ -stabilization of type (k)* if there is a disk $D \subset \Sigma$ and a punctured torus $T \subset \Sigma'$ as well as curves $\alpha \in \alpha'$ and $\beta \in \beta'$ so that:

- (1) $D = \tau(D)$, $T = \tau'(T)$, $\tau'(\alpha) = \beta$;
- (2) $\Sigma \setminus D = \Sigma' \setminus T$;
- (3) $\alpha \setminus D = \alpha' \setminus T$ and $\beta \setminus D = \beta' \setminus T$;
- (4) $\alpha \cap D$ consists of k arcs and each component of $\alpha \cap D$ intersects each component of $\beta \cap D$ transversely in a single point;
- (5) α and β are contained in T and intersect in a single point;
- (6) $(\alpha \setminus \alpha) \cap T$ consists of k parallel arcs, each intersecting β transversely;
- (7) for each component of $\alpha \cap D$, there is a corresponding component of $\alpha' \cap T$ with the same endpoints, and similarly for the beta curves.

These, of course, impose symmetric relations the beta curves. See Figure 5.3 for an example.

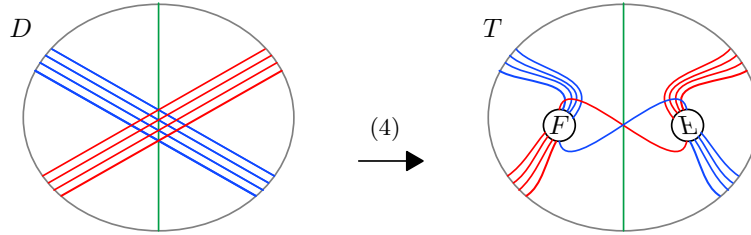


FIGURE 5.3. A generalized $\{1\}$ -stabilization of type (4).

Definition 5.18. A real Heegaard diagram $(\Sigma', \alpha', \beta', \tau')$ is obtained from $(\Sigma, \alpha, \beta, \tau)$ by a *crossover of type (k, ℓ)* if there are twice-punctured disks $P \subset \Sigma$ and $P' \subset \Sigma'$ as well as distinguished curves $\alpha \in \alpha$, $\alpha' \in \alpha'$ such that:

- (1) $\Sigma \setminus P = \Sigma' \setminus P'$;
- (2) $\alpha \setminus P = \alpha' \setminus P'$, $\beta \setminus P = \beta' \setminus P'$;
- (3) P and P' are fixed setwise by τ and τ' (and therefore necessarily intersect C in an arc);
- (4) $\partial P = S \cup A \cup B$ where S is fixed set-wise by τ , A and B are exchanged, and α and $\beta = \tau(\alpha)$ are push-offs of A and B (similarly for $\partial P' = S' \cup A' \cup B'$);
- (5) $(\alpha \setminus \alpha) \cap P$ consists of two collections of parallel arcs from S to B avoiding C of size k and ℓ respectively, each intersecting β transversely once. The arcs $(\beta \setminus \beta) \cap P$ are symmetric;

- (6) $(\alpha' \setminus \alpha) \cap P'$ consists of two collections of parallel arcs from S' to B' of size k and ℓ , respectively, each intersecting C transversely once, and intersecting $\beta' = \tau(\alpha')$ transversely once. The arcs $(\beta' \setminus \beta) \cap P'$ are symmetric;
- (7) for each arc of $(\alpha \setminus \alpha) \cap P$, there is a corresponding component of $(\alpha' \setminus \alpha') \cap P'$ with the same endpoints, and similarly for the beta curves.

See Figure 5.4 for an example.

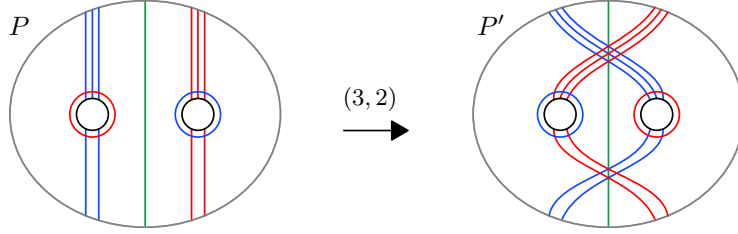


FIGURE 5.4. A crossover of type $(3, 2)$.

Proposition 5.19. *Let $\{(f_t, v_t)\}_{t \in [-1, 1]}$ be a generic 1-parameter family of real functions and real gradient-like vector fields on (Y, γ, τ) that has a bifurcation at $t = 0$. If the bifurcation at $t = 0$ is not an index 1-2 birth-death nor a real quasi-transversal orbit of tangency of 1-dimensional submanifolds, then there is a real separating surface $\Sigma \in \Sigma^R(f_0, v_0)$. There is an $\epsilon > 0$ so that $\Sigma \pitchfork v_t$ for all $|t| < \epsilon$. Moreover, for all $x \in (-\epsilon, 0)$ and $y \in (0, \epsilon)$ the following hold.*

- (1) $\Sigma \in \Sigma^R(f_x, v_x) \cap \Sigma^R(f_y, v_y)$;
- (2) If the bifurcation consists of an index 0-1 birth-death and an index 2-3 birth-death (which always occur together), the two diagrams differ by adding or removing a redundant alpha curve and its corresponding beta curve;
- (3) If the bifurcation is a $\mathbb{Z}/2$ -orbit of tangency between $W^u(p)$ and $W^s(q)$ for $p \in C_1(f)$ and $q \in C_2(f)$, the diagrams are related by an isotopy creating or canceling two symmetric pairs of intersection points away from C ;
- (4) If the bifurcation is a $\{1\}$ -orbit of tangency between $W^u(p)$ and $W^s(\tau(p))$ for $p \in C_1(f)$ and $\tau(p) \in C_2(f)$, the diagrams are related either by an isotopy creating or canceling a pair of intersection points on C or the diagrams are related by an isotopy in which a symmetric pair of intersection points are created or canceled near an intersection point on C ;
- (5) If the bifurcation is a $\mathbb{Z}/2$ -orbit of tangency between $\mathbb{Z}/2 \cdot W^u(p)$ and $\mathbb{Z}/2 \cdot W^s(q)$ for p and q in $C_1(f)$, the diagrams are related by a generalized real handleslide.

If the bifurcation is an index 1-2 birth-death, then $\Sigma \in \Sigma^R(f_x, v_x)$ and there is a surface $\Sigma' \in \Sigma^R(f_y, v_y)$ such that diagrams $(\Sigma, \alpha, \beta) = H(f_x, v_x, \Sigma)$ and $(\Sigma', \alpha', \beta') = H(f_y, v_y, \Sigma')$ are related by either a generalized $\mathbb{Z}/2$ -stabilization of type (k, ℓ, c) if there are ℓ flows from index 1 critical points into one of the degenerate critical points p (and therefore symmetrically ℓ flows from $\tau(p)$ to index 2 critical points), k flows from p to critical points of index 2 (and symmetric flows from index 1 critical points to $\tau(p)$), and c flow lines between p and $\tau(p)$ or a generalized $\{1\}$ -stabilization of type (m) if there are m flows from index 1 critical points to p (and m symmetric flows from p to index 2 critical points). Index 1-2 deaths are similar, with the roles of x and y reversed.

If the bifurcation is a real quasi-transversal orbit of tangency from $W^u(\tau(p))$ to $W^s(p)$ for a critical point p of index 1, then $\Sigma \in \Sigma^R(f_x, v_x)$ and there is a surface $\Sigma' \in \Sigma^R(f_y, v_y)$ such that

the diagrams $(\Sigma, \alpha, \beta) = H(f_x, v_x, \Sigma)$ and $(\Sigma', \alpha', \beta') = H(f_y, v_y, \Sigma')$ are related by a crossover of type (k, ℓ) .

Proof. Much of this follows the same strategy as [JTZ21, Proposition 6.28], so we sketch the argument, pointing out where adaptations must be made in the symmetric setting.

In Case (2), we consider the case of a birth, so that f_0 has a degenerate critical orbit $\{p, \tau(p)\}$ which splits into orbits $\{p_0^t, \tau(p_0^t)\}$ and $\{p_1^t, \tau(p_1^t)\}$ for $p_0^t \in C_0(f)$ and $p_1^t \in C_1(f)$ and $t > 0$. For ϵ small, $\Sigma \cap v_t$ for all $|t| < \epsilon$ and p_0^t and p_1^t are contained in Y_- while $\tau(p_0^t)$ and $\tau(p_1^t)$ are contained in Y_+ . The attaching set (α', β') contains exactly one more α - and beta curve than (α, β) , and the remaining curves are isotopic. The new α -circle is isotopic to $W^{uu}(p_0) \cap \Sigma$ and the new β -circle is isotopic to $W^{ss}(\tau(p_0)) \cap \Sigma$.

In Cases (3) and (4), we simply consider $(\Sigma_t, \alpha_t, \beta_t, \tau_t) = H(f_t, v_t, \Sigma)$ for $t \in [x, y]$. At $t = 0$, we see a generic tangency. Tangencies in $\mathbb{Z}/2$ -orbits behave exactly as in the unreal case, and tangencies in $\{1\}$ -orbits as described in Section 4.2.

Similarly, in Case (5), we fix p and q in $C_1(f_0)$. For all $|t| < \epsilon$, the surface Σ is contained in $\Sigma^R(f_t, v_t)$. At $t = 0$, the union of the alpha curves $W^u(p) \cap \Sigma$ and $W^u(q) \cap \Sigma$ has a regular neighborhood which is a pair of pants. The boundaries are isotopic to the α -attaching curves before and after a generalized (m, n) -handleslide. The symmetric slide of beta curves occurs simultaneously.

We turn now to index 1-2 birth-death bifurcations. We will address the case of births. Suppose that p is a degenerate critical point appearing in a $\mathbb{Z}/2$ -orbit which splits into $p_0^t \in C_1(f_t)$ and $p_1^t \in C_2(f_t)$; we will write q for the symmetric degenerate critical point which splits into $q_0^t \in C_1(f_t)$ and $q_1^t \in C_2(f_t)$. Let us place p in $C_{01}(f_0, v_0)$ and q in $C_{23}(f_0, v_0)$. In this case, (f_0, v_0) is separable, so we may choose an invariant splitting surface Σ . For small $\epsilon > 0$, we have that p_0^t and p_1^t are both contained in Y_- and q_0^t and q_1^t are both contained in Y_+ . The unstable manifold $W^u(q_0^y)$ intersects Σ in two points, as does the stable manifold $W^s(p_1^y)$. Define

$$\Sigma' = \partial(Y_- \cup N(W^u(p_0^y)) \cup W^s(q_1^y)).$$

It is easy to see that the attaching sets for Σ' are given exactly by the description of generalized stabilizations, as in [JTZ21]. In the case of $\mathbb{Z}/2$ -orbits of index 1-2 deaths, we simply reverse the roles of x and y .

Next, we consider the case that p is a degenerate critical point which is fixed by the involution. As noted in Remark 5.16, the gradient (f_0, v_0) is separable in the unreal setting (as in [JTZ21, Definition 6.1]), but not separable in the sense of Definition 5.2 (that is, not separable by a real surface). Fix $\epsilon > 0$ small and choose $-\epsilon < x < 0$ and $0 < y < \epsilon$. Both (f_x, v_x) and (f_y, v_y) are separable, so we can choose real separating surfaces $\Sigma_x \in \Sigma^R(f_x, v_x)$ and $\Sigma_y \in \Sigma^R(f_y, v_y)$. Unlike the previous case, there is no clear way to construct Σ_y from Σ_x . Nevertheless, we can choose an equivariant diffeomorphism of Y which takes $\Sigma_x \setminus N(p)$ to $\Sigma_y \setminus N(p)$. In this way, we can construct separating surfaces which differ by a $\{1\}$ -stabilization. Alternatively, we can choose an unreal splitting surface $\tilde{\Sigma}_x$, and construct $\tilde{\Sigma}_y$ as in the unreal case (or as we did for $\mathbb{Z}/2$ -stabilizations). Then, as in the proof of Lemma 5.6, we can reparametrize v and obtain real splitting surfaces Σ_x and Σ_y from $\tilde{\Sigma}_x$ and $\tilde{\Sigma}_y$ by pushing them forward by the gradient flow.

Finally, we consider the case of a real quasi-transversal orbit of tangency γ from $W^u(\tau(p_0))$ to $W^s(p_0)$ for a nondegenerate critical point p^t of index 1 (see Figure 4.3 (f)). (We emphasize that, in the unreal case, such an orbit fails the quasi-transversal condition.) Define $\tilde{\Sigma}_0$ to be the boundary of a small regular neighborhood of

$$\bigcup \{W^s(q) : q \in C_0(f_0) \cup C_1(f_0) \setminus \{p_0\}\}.$$

This surface is not invariant, nor is it separating, as the index 1 critical point p^t is contained in Y_+ .

Fix $\epsilon > 0$ small, and some $x \in (-\epsilon, 0)$. The manifold $W^s(p^x)$ intersects $\tilde{\Sigma}_0$ in a pair of points a^x and b^x . Let γ_1^x and γ_2^x be the associated flow lines. As t varies from $-\epsilon$ to 0, one of these points (a^x , say) traces out an arc $\tilde{\lambda}_1 := a^t$ for $t \in (-\epsilon, 0)$ in $\tilde{\Sigma}_0$; likewise, for $t \in (0, \epsilon)$ we have a path $\tilde{\lambda}_2 := a^t$ in $\tilde{\Sigma}_0$. Provided ϵ is sufficiently small, we may assume that $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ do not intersect the stable or unstable manifolds of any critical points other than the unstable manifolds of $\tau(p_0)$.

Define a separating surface

$$\tilde{\Sigma}_x = \partial(Y_- \cup N(W^s(p^x))).$$

The neighborhood $N(W^s(p^x))$ intersects Σ in two disks, D_{a^x} and D_{b^x} . Pick a neighborhood $N(\tilde{\lambda}_1)$ which contains D_{a^x} (shrinking $N(W^s(p^x))$ if necessary). By construction, we have that $\tilde{\Sigma}_x = (\tilde{\Sigma}_0 \setminus (D_{a^x} \cup D_{b^x})) \cup \tilde{A}$, where $\tilde{A}$ is a tube (annulus) around $W^s(p^x)$.

In a similar manner, for $y \in (0, \epsilon)$, define

$$\tilde{\Sigma}_y = \partial(Y_- \cup N(W^s(p^y))),$$

and so that $\tilde{\Sigma}_y = (\tilde{\Sigma}_0 \setminus (D_{a^y} \cup D_{b^y})) \cup \tilde{A}'$, where $\tilde{A}'$ is a tube around $W^s(p^y)$.

Once again, we obtain invariant surfaces Σ_x and Σ_y as in the proof of Lemma 5.6, by pushing $\tilde{\Sigma}_x$ and $\tilde{\Sigma}_y$ forward under the flow of v_t (reparametrizing v_t if necessary). Similarly, the annuli $\tilde{A}$ and $\tilde{A}'$ can be pushed forward to annuli A and A' in Σ_x and Σ_y . In the symmetric surfaces, the paths λ_1 and λ_2 limit to a point in C , and can be completed to a single connected path λ in Σ_x and Σ_y . A neighborhood of this path meets both A and A' : define the first pair of pants P to be the union of this neighborhood together with A and $\tau(A)$ and the second P' to be a neighborhood of λ together with A' and $\tau(A')$. See Figure 5.5.

Let $\alpha_x := \Sigma_x \cap W^u(p^\mu)$ and $\beta_x = \tau(\alpha_x)$. Define α_y and β_y similarly. The directional manifold $D(p^\mu, \gamma_1^x) \subset W^u(p^\mu)$ intersects α_x in a pair of points; in particular, this gives a partition of the intersection points of α_x with β into sets of size k and ℓ . The situation is symmetric for Σ_y . In particular, the diagrams $H(f_x, v_x, \Sigma_x)$ and $H(f_y, v_y, \Sigma_y)$ differ by a crossover of type (k, ℓ) . $\square$

Remark 5.20. A real Heegaard splitting of (Y, γ, τ) determines a framing of the fixed set C . The main theorem of [Nag79] states that any two real Heegaard splittings which determine the same framing on C are $\mathbb{Z}/2$ -stably diffeomorphic. It is not hard to see that $\{1\}$ -stabilizations change the framing on C by ± 1 . We have defined $\{1\}$ -stabilization of type k to include both the positive and negative $\{1\}$ -stabilizations we considered in [GM], and exchanging the roles of α and β changes the sign of the stabilization.

Essentially the same is true in 2-parameter families near codimension-1 bifurcations.

Proposition 5.21. *Let $\{(f_\mu, v_\mu) \in \mathcal{FV}(Y, \gamma, \tau) : \mu \in \mathbb{R}^2\}$ be a generic 2-parameter family with a codimension-1 bifurcation at $\mu = 0$. Let S be the (one-dimensional) stratum of the bifurcation passing through the origin. Then, there is an $\epsilon > 0$ so that $D_\epsilon^2 \setminus S$ consists of two components C_1 and C_2 , and for every $x \in C_1$ and $y \in C_2$, the same conclusion holds as in Proposition 5.19.*

Proof. This follows from essentially the same proof as Proposition 5.19. $\square$

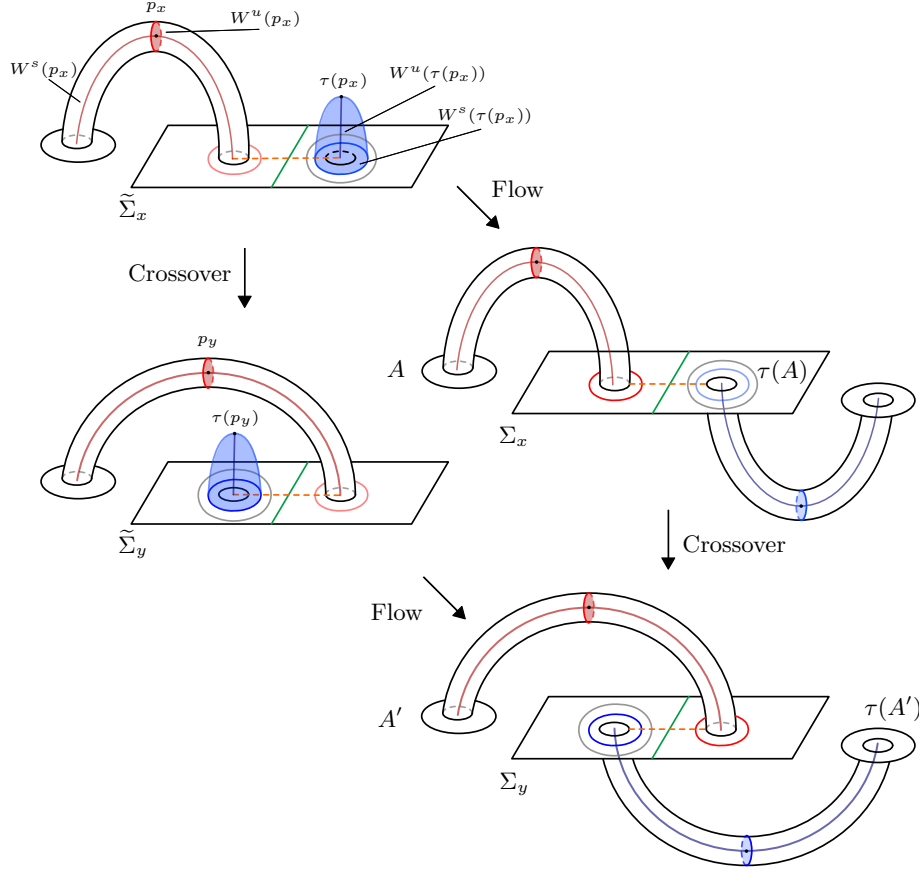


FIGURE 5.5. The construction of the two invariant splitting surfaces on either side of a bifurcation associated to a real quasi-transversal orbit of tangency.

Definition 5.22. Let $H_i = (\Sigma_i, [\alpha_i], [\beta_i], \tau_i)$ be real isotopy diagrams for $1 \leq i \leq 4$. A *generalized distinguished rectangle* is a graph

$$\begin{array}{ccc} H_1 & \xrightarrow{e} & H_2 \\ \downarrow f & & \downarrow g \\ H_3 & \xrightarrow{h} & H_4 \end{array}$$

satisfying any of the conditions (1)-(4) of Definition 2.22 with “simple $\mathcal{O}$ -stabilization” replaced with “generalized $\mathcal{O}$ -stabilization”, and allowing for one extra possibility:

- (5) Both e and h are crossovers of type (k, ℓ) while f and g are both real handleslides. In this case, $\Sigma_1 = \Sigma_3$ and $\Sigma_2 = \Sigma_4$ and f and g are required to be the same map.

We also allow overcomplete diagrams.

Proposition 5.23. Suppose that $\{(f_\mu, v_\mu) \in \mathcal{FV}_{\leq 1}(Y, \gamma, \tau) : \mu \in \mathbb{R}^2\}$ is a generic 2-parameter family and let

$$V_1 = \{\mu \in \mathbb{R}^2 : (f_\mu, v_\mu) \in \mathcal{FV}_1(Y, \gamma, \tau)\}$$

be the codimension-1 bifurcation set. Let a be an arc in V_1 with endpoints μ_0 and μ_1 and let R be a small rectangular neighborhood of a . Let b_0 and b_1 be the boundary arcs of R which

are transverse to V_1 and oriented so that they have the same intersection sign with V_1 . Let $\partial b_i = y_i - x_i$ for $i \in \{0, 1\}$. As this family is generic, we may assume that at x_i and y_i the gradients (f_{x_i}, v_{x_i}) and (f_{y_i}, v_{y_i}) are real separable; fix such separating surfaces $\Sigma_i \in \Sigma^R(f_{x_i}, v_{x_i})$.

Suppose we are given surfaces $\Sigma'_i \in \Sigma^R(f_{y_i}, v_{y_i})$ for $i \in \{0, 1\}$ such that:

- (1) Σ'_i is obtained from Σ_i by an $\mathcal{O}$ -stabilization if (f_{μ_i}, v_{μ_i}) is an index 1-2 birth,
- (2) Σ'_i is obtained from Σ_i by a crossover if (f_{μ_i}, v_{μ_i}) is a non-quasi-transversal orbit of tangency,
- (3) $\Sigma_i = \Sigma'_i$ otherwise.

Then the isotopy diagrams $H_1 = [H(f_{x_0}, v_{x_0}, \Sigma_0)]$, $H_2 = [H(f_{y_0}, v_{y_0}, \Sigma'_0)]$, $H_3 = [H(f_{x_1}, v_{x_1}, \Sigma_1)]$, and $H_4 = [H(f_{y_1}, v_{y_1}, \Sigma'_1)]$, fit into a generalized distinguished rectangle

$$\begin{array}{ccc} H_1 & \xrightarrow{e} & H_2 \\ \downarrow f & & \downarrow g \\ H_3 & \xrightarrow{h} & H_4 \end{array}$$

of type (2) if (f_{μ_i}, v_{μ_i}) is a handleslide, of type (4) if (f_{μ_i}, v_{μ_i}) is an index 1-2 birth-death, or type (5) if (f_{μ_i}, v_{μ_i}) is a non-quasi-transversal orbit of tangency (compare to Definition 5.22). For other types of bifurcations, we have a rectangle with arrows e and h representing either the identity or a simultaneous index 0-1/ index 2-3 birth-death, and $f = g$ a diffeomorphism.

Proof. In the cases that the component of V_1 containing a is not a $\{1\}$ -stabilization or a crossover, this follows exactly as in [JTZ21, Proposition 6.32]; the gradients along a are separable, and it is straightforward to verify that the relevant diagrams fit into distinguished rectangles.

In the case that a corresponds to a $\{1\}$ -stabilization, separability fails. Still, we can apply the same strategy as before. Fix parametrizations of the arcs $b_i = b_i(t)$ so that $b_i(t) = \mu_i$. For $\epsilon > 0$ sufficiently small, $(f_{b_i(-\epsilon)}, v_{b_i(-\epsilon)})$ and $(f_{b_i(\epsilon)}, v_{b_i(\epsilon)})$ are both real separable, and we can choose real separating surfaces $\Sigma_{b_i(\pm\epsilon)}$ so that $\Sigma_{b_i(\epsilon)}$ is obtained from $\Sigma_{b_i(-\epsilon)}$ by attaching a tube around the descending manifold of the new index 1 critical point which is born at μ_i . The previous lemma furnishes diffeomorphisms $\Sigma_{y_i} \cong \Sigma_{b_i(-\epsilon)}$ and $\Sigma_{x_i} \cong \Sigma_{b_i(\epsilon)}$. The same argument appearing in [JTZ21, Proposition 6.32] can be applied to show that this gives rise to a generalized distinguished rectangle.

The same strategy applies to crossovers. □

Before moving to codimension-2 bifurcations, we note that Juhász-Thurston-Zemke show that one may pass from over-complete diagrams to unreal diagrams “without altering the relationships of the diagrams before and after the bifurcation in an essential way”. Their strategy holds in our case without change, since the behavior of index 2-3 birth-deaths is dictated by the behavior of index 0-1 birth-deaths by the symmetry.

As in the codimension-0 case, sequences of real Heegaard moves can be translated to paths of real gradients. This also follows just as in [JTZ21, Section 6.5], so we will not discuss it.

5.3. Codimension-2. In this section, we turn to loops of real Heegaard moves. We shall consider generic 2-parameter families of real gradients (f_μ, v_μ) on (Y, γ, τ) with a codimension-2 singularity at $\mu = 0$. The bifurcation set will be denoted S . For $\epsilon > 0$ sufficiently small, $(S \cap D_\epsilon^2) \setminus \{0\}$ is a disjoint union of strata $S_1, \dots, S_r$ which each have one boundary component at 0 and the other on $S_\epsilon^1 = \partial D_\epsilon^2$. The regions between these strata will be called chambers $C_1, \dots, C_r$, labeled so that C_i lies between S_{i-1} and S_i (here, we define $S_0 = S_r$).

Remark 5.24. Following [JTZ21, Section 6.6], we will color-code these strata: a stratum S_i will be colored purple if it corresponds to a real equivalence, black if it corresponds to a birth-death, green if it corresponds to a crossover, and dashed gray if it corresponds to a diffeomorphism.

Definition 5.25. (cf. [JTZ21, Definition 6.36]) Say $\{(f_\mu, v_\mu) : \mu \in \mathbb{R}^2\}$ is a generic 2-parameter family so that $(f_0, v_0) \in \mathcal{FV}_2(Y, \gamma, \tau)$. For $\epsilon > 0$ as above, the *link of the bifurcation at 0* is an embedded polygonal curve $P \in D_\epsilon^2$ such that

- (1) the bifurcation value 0 lives in the interior of P ;
- (2) $P \pitchfork S$ and $|S_i \cap P| = 1$ for all i ;
- (3) each C_i contains exactly 1 or 2 vertices of P .

A *surface-enhanced link* of the bifurcation at p is a link P together with a choice of surface $\Sigma_\mu \in \Sigma^R(f_\mu, v_\mu)$ for each vertex μ of P .

If C_i contains a single vertex of P , we denote it by μ_i and write a_i for the edge of P which intersects S_i . If C_i contains two vertices, they are denoted μ_i and μ'_i ordered so that $\partial a_i = \mu'_i - \mu_i$. In the case that P is minimal, its vertices are $\mu_1, \dots, \mu_r$ and its edges are $a_1, \dots, a_r$.

In the case that P is not minimal, i.e., there is a chamber C_i which contains two vertices of P , we will draw an extra ray in C_i separating these two vertices (as in [JTZ21]). We will label the edge between them by a'_i . On either side of the ray, we will choose two *different* splitting surfaces which are related by a diffeomorphism. This is used in [JTZ21] in the cases of simultaneous stabilizations and handleswaps, as separability fails. In our setting, this failure is much more frequent: indeed, we will need to include diffeomorphism strata any time there is a bifurcation that involves a $\{1\}$ -stabilization or a crossover. For this reason, our links will usually not be minimal (see [JTZ21, Definition 6.36]). We will not emphasize this, departing from the convention in [JTZ21].

Theorem 16. Suppose $\{(f_\mu, v_\mu) : \mu \in \mathbb{R}^2\}$ is a generic 2-parameter family so that $(f_0, v_0) \in \mathcal{FV}_2(Y, \gamma, \tau)$. For all $\epsilon > 0$ sufficiently small, there is a surface-enhanced link $P \subset D_\epsilon^2$ of the bifurcation at 0 so that the following hold:

- (1) For $i \in \{1, \dots, r\}$, there is a point $x_i \in \partial a_i$ so that $\Sigma_{x_i} \pitchfork v_\mu$ for all $\mu \in a_i$;
- (2) Consecutive isotopy diagrams H_i and H_{i+1} are related by the move corresponding to the type of stratum S_i ;
- (3) Each edge a'_i induces a diffeomorphism $d_i : H_i \rightarrow H'_i$ isotopic to the identity as in Lemma 5.11.

The surface-enhanced links for the bifurcations discussed in Section 4.3 are described in the proof.

Proof. As in [JTZ21, Theorem 6.37], we ignore strata corresponding to index 0-1/ index 2-3 birth-deaths as well as those corresponding to tangencies between unstable manifolds of index 1 critical points and stable manifolds of index 2 critical points.

In the cases that (f_0, v_0) is separable, all surfaces can be constructed from some $\Sigma \in \Sigma^R(f_0, v_0)$. When (f_0, v_0) is not separable, we will construct the surfaces on either side of stabilization and crossover strata as in the proof of Proposition 5.19. In these cases, for each stratum S_i , we will pick a small arc $a^{i \rightarrow i+1}$ transverse to S_i and add vertices $\partial a^{i \rightarrow i+1} = \{\mu_i, \mu'_{i+1}\}$ to the link of the singularity. Since the gradient is not separable, the Heegaard splittings Σ_{μ_i} and $\Sigma_{\mu'_i}$ need not be identical, but are simply diffeomorphic according to Lemma 5.11. Therefore, in these cases we are forced to add in a diffeomorphism stratum relating the two diagrams. Since these strata appear so frequently, we will not always draw them explicitly in the accompanying figures.

Type (A) bifurcations: Bifurcations of type (A) involve simultaneous orbits of tangency, i.e. simultaneous handleslides and crossovers. Bifurcations of type (A1) and (A2) only involve real handleslides, and therefore necessarily appear in $\mathbb{Z}/2$ -orbits, and the analysis is identical to that of [JTZ21, Theorem 6.37]. For this reason, we will focus on describing the links for bifurcations of types (A3) and (A4) and their corresponding Heegaard diagrams. When we refer

to directional manifolds, we will typically not specify which flowline we are considering, as it should be clear from the context. The only bifurcation which involves both directional manifolds of the same critical point is (A3) and in this case we will take care in distinguishing the two.

(A3): Suppose p_1^μ and p_2^μ are critical points (not necessarily distinct) of index 1 and there is a $\{1\}$ -orbit of tangency γ_1^0 from $\tau(p_1^0)$ to p_1^0 as well as one γ_2^0 from $\tau(p_2^0)$ to p_2^0 . The bifurcation diagram S consists of two smooth curves which intersect transversely at 0. Suppose there are $(k_1 + \ell_1)$ and $(k_2 + \ell_2)$ flows of index 2 critical points into p_1^μ and p_2^μ , respectively (recall that the flow lines into p_1^μ are partitioned into two kinds by $D(p_1^\mu, \gamma_1^\mu)$ and flows into p_2^μ are partitioned by $D(p_2^\mu, \gamma_2^\mu)$). The strata S_1 and S_3 correspond to a crossover of type (k_1, ℓ_1) while S_2 and S_4 correspond to crossovers of type (k_2, ℓ_2) . (See Definition 5.18 and Figure 5.4 for an example of a crossover.) Note that when $p_1^\mu = p_2^\mu$, the flows γ_1^0 and γ_2^0 are still distinct, and we make use of both directional manifolds to produce the two different partitions.

The link of the singularity consists of eight vertices, $\mu_i, \mu'_i \in C_i$ for $i \in \{1, 2, 3, 4\}$ obtained as follows. Choose short arcs $a^{i \rightarrow i+1}$ from C_i to C_{i+1} transverse to S_i (here, $i, i+1$ are taken to be elements of $\mathbb{Z}/4$). We define the vertices of the link by $\partial a^{i \rightarrow i+1} = \{\mu_i, \mu'_{i+1}\}$. As in the proof of Proposition 5.19, we construct Heegaard diagrams H_i, H'_i for $i \in \{1, 2, 3, 4\}$ so that H_i and H'_{i+1} differ by a crossover; the type of crossover is determined by the directional manifold of p_1^μ for $\mu \in a^{1 \rightarrow 2}$ and p_2^μ for $\mu \in a^{4 \rightarrow 1}$. See Figure 5.6.

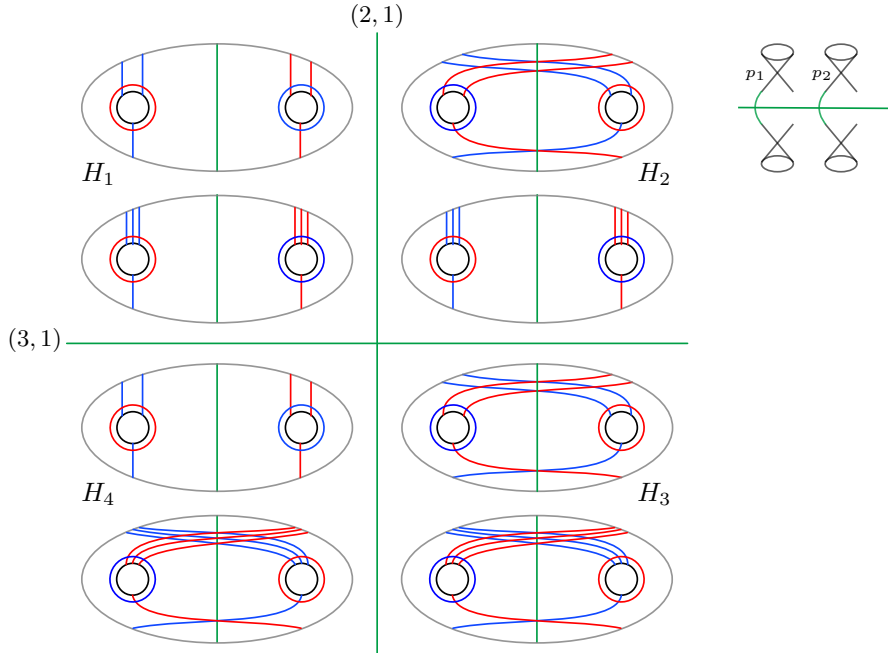


FIGURE 5.6. Commuting crossovers, i.e. the link of a bifurcation of type (A3) with $(k_1, \ell_1) = (2, 1)$ and $(k_2, \ell_2) = (3, 1)$. We see two commuting crossovers.

(A4a): Here, p_1^μ is a hyperbolic singularity of index 1 with a $\{1\}$ -orbit of tangency from $\tau(p_1^0)$ to p_1^0 ; additionally, p_2^μ and p_3^μ are hyperbolic singularities of index 1 distinct from p_1^μ , and there is an orbit of tangency from p_2^0 to p_3^0 . Assume there are $k + \ell$ flows out of p_1^μ to index 2 critical points. The bifurcation diagram consists of two smooth curves intersecting at the origin. The Heegaard diagrams are constructed just as in case (A3), though things are slightly simpler in this context; we need only choose six vertices, as we can take Σ_1 and Σ_4 to be a common

surface (likewise for Σ_2 and Σ_3); we choose small arcs $a^{1 \rightarrow 2}$ and $a^{3 \rightarrow 4}$ transverse to S_1 and S_3 ; these arcs determine four Heegaard splittings H_1, H'_2, H_3, H'_4 and we define $H_2 := H_3$ and $H_4 := H_1$ and choose diffeomorphisms relating H_1 with H'_4 and H_3 with H'_2 . The stratum S_4 is a crossover of type (k, ℓ) , where the partition of the flows out of p_1^μ to index 2 critical points comes from the directional manifolds of p_1^μ for $\mu \in a^{2 \rightarrow 3}$, and S_2 is a crossover of type $(k+1, \ell)$, with the partition coming from the directional manifold of p_1^μ for $\mu \in a^{4 \rightarrow 1}$. See Figure 5.7.

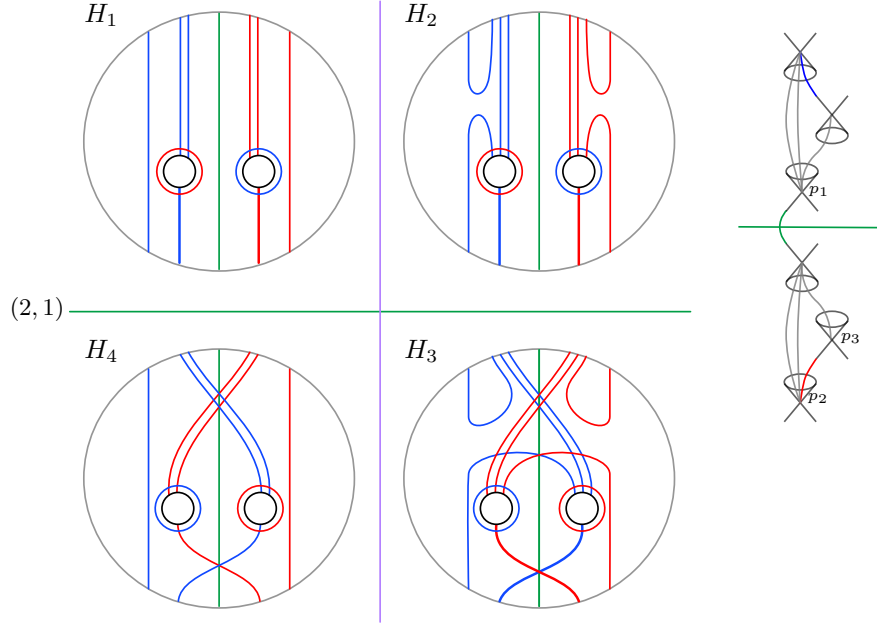


FIGURE 5.7. A commutation of a real handleslide and a crossover, or the link of a bifurcation of type (A4a).

(A4b): In this case, p_1^0 is a hyperbolic singularity of index 1 with a $\{1\}$ -orbit of tangency from $\tau(p_1^0)$ to p_1^0 ; additionally, p_2^μ and p_3^μ are hyperbolic singularities of index 1 (not necessarily distinct from p_1^0). The bifurcation diagram consists of two smooth curves intersecting transversely at 0; the strata S_1 and S_3 correspond to crossovers while strata S_2 and S_4 are real handleslides. The Heegaard diagrams are constructed just as in case (A4a). H_1 and H_4 differ by a real handleslide, as do H_2 and H_3 , while H_1 and H'_2 differ by a crossover, as do H_3 and H'_4 . Recall that in a crossover, there are two distinguished tubes (annuli) which are neighborhoods of curves α_0 and $\beta_0 = \tau(\alpha_0)$, and in a crossover the feet (attaching regions) of the tubes are exchanged. In Figure 5.8, we show both feet of these annuli, as one foot is involved in the crossover, while the other is involved in the real handleslide.

(A4c): In this case, p_1^0 and p_2^0 are both hyperbolic singularities and there is a $\mathbb{Z}/2$ -orbit of tangency between $W^s(p_2^0)$ and $W^u(p_1^0)$ as well as a $\{1\}$ -orbit of tangency between $W^u(\tau(p_1^0))$ and $W^s(p_1^0)$. Suppose that there are $k_1 + \ell_1$ flows out of p_1^0 to index 2 critical points partitioned into two parts by the directional manifold of p_2^μ . Similarly, assume there are $k_2 + \ell_2$ flows from p_2^0 to index 2 critical points. On the right side of the diagram, there is a generalized real handleslide of type (k_1, ℓ_2) , creating or annihilating $(\ell_2 - k_1)$ flows from p_1^μ to index 2 critical points.

In this case, the bifurcation diagram consists of at least $5 + (\ell_1 - k_2) \geq 5$ curves meeting at the origin. The strata S_1 and S_3 correspond to handleslides and S_2 corresponds to a crossover

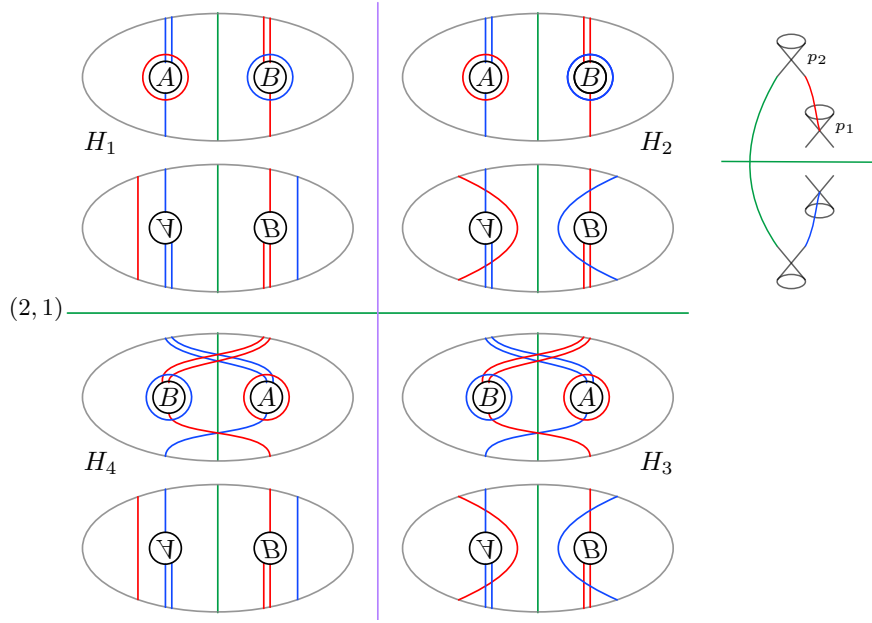


FIGURE 5.8. A commutation of a real handleslide and a crossover corresponding to the link of a bifurcation of type (A4b).

of type $(k_1, \ell_2 + (\ell_1 - k_2))$. Pick a point $x \in C_1$, and define H_1 using a separating surface $\Sigma_1 \in \Sigma^R(f_x, v_x)$. Since S_1 is a handleslide stratum, we can define $\Sigma_2 = \Sigma_1$ as the separating surface for H_2 as well. To construct the remaining splitting surfaces, we use the arc $a^{2 \rightarrow 3}$ to construct a surface Σ_3 so that H_3 is obtained from H_2 by a crossover of type $(k_1, \ell_2 + (\ell_1 - k_2))$; the type of crossover is determined by the directional manifolds associated to the arcs $a^{2 \rightarrow 3}$ and $a^{5 \rightarrow 1}$. Of the remaining strata, S_i and S_{i+1} are crossover strata of types (k_2, ℓ_2) and (k_1, ℓ_1) , and the others are real handleslides. For the diagrams $H_4, \dots, H_{i+1}$, we take Σ_3 to be the splitting surface. As above, we choose an arc $a^{i-1 \rightarrow i}$ and construct Σ_i so that it is obtained as a crossover of $\Sigma'_{i-1} \cong \Sigma_{i-1}$, producing diagrams H'_{i-1} and H_i (and inserting diffeomorphism strata). Likewise, we choose an arc $a^{i \rightarrow i+1}$ to construct diagrams H'_i and H_{i+1} . We take $H'_{i+1}, H_{i+2}, \dots, H_{5+(\ell_1-k_2)}$ to have splitting surface Σ_1 .

Type (B) bifurcations: Bifurcations of type (B) involve stabilizations. The link P contains two stabilization strata S_1 and S_3 , and there is a single stratum S_2 on the stabilized side, which is either a real handleslide or a crossover stratum.

For each bifurcation, we fix a separating surface $\Sigma \in \Sigma^R(f_\mu, v_\mu)$ for a parameter μ close to 0 on the unstabilized side of the parameter space. Let p be the index 1-2 birth-death singularity which breaks into critical points (or orbits) $q'_1 \in C_1(f_\nu)$ and $q'_2 \in C_2(f_\nu)$. We will assume that the surface on the stabilized side, $\Sigma_\nu \in \Sigma^R(f_\nu, v_\nu)$, is obtained by attaching a tube around $W^u(q'_2)$ (though we could just as well have considered the surface obtained by attaching a tube around $W^s(q'_1)$).

(B1): There are four strata; S_1 and S_3 are stabilizations while S_2 and S_4 are real handleslides. By the non-degeneracy condition (ND-2) (see [JTZ21, Definition 5.11], Remark 4.16), there is a 1-dimensional invariant submanifold $W^{uu}(p_2^0) \subset W^u(p_2^0)$ which partitions flows from p_2^0 to p_0 into two parts; we assume there are k_1 and k_2 such flows.

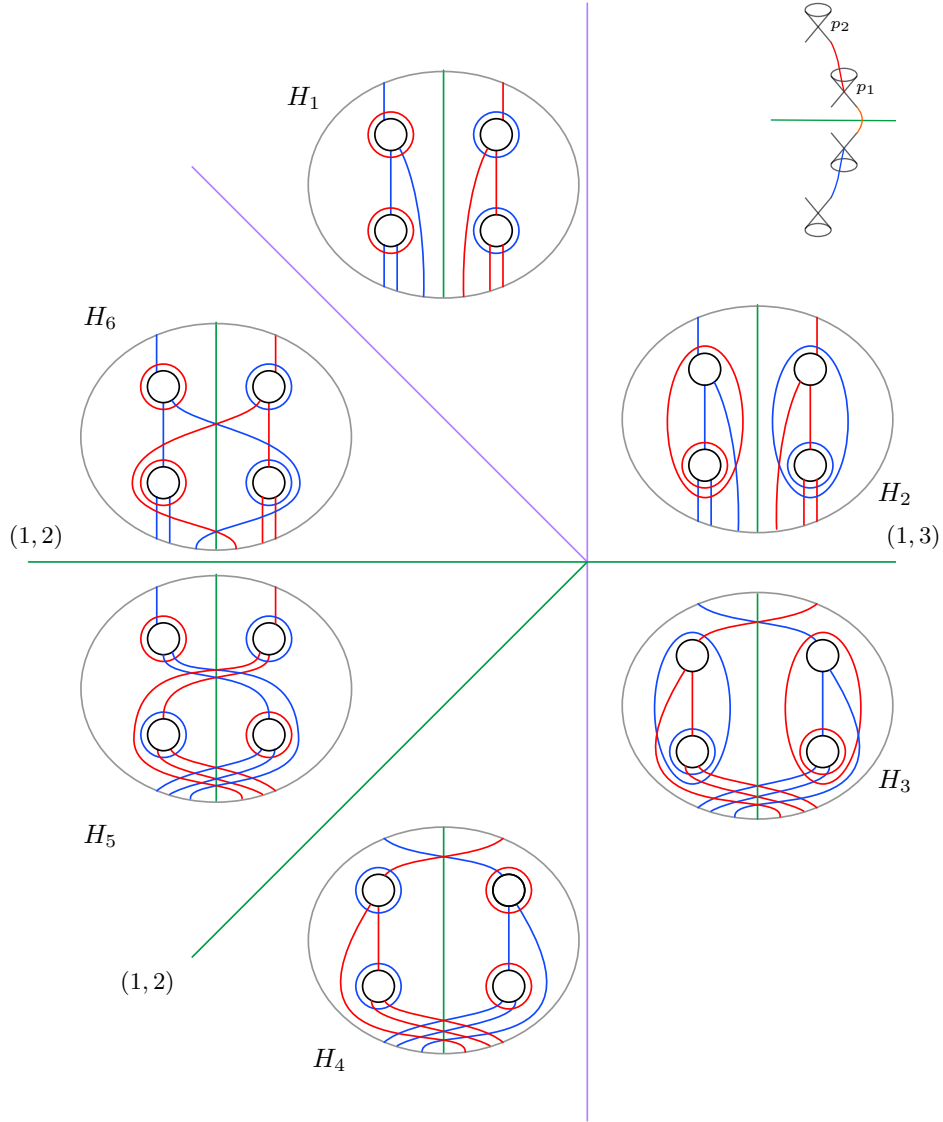


FIGURE 5.9. The link of a bifurcation of type (A4c). In this example $k_1 = k_2 = 1$, and $\ell_1 = \ell_2 = 2$.

Say there are m flows to p from index 1 critical points and ℓ flows from p to index 2 critical points (if p is a $\{1\}$ -stabilization, i.e. $\tau(p) = p$, then $m = \ell$). If p is a $\mathbb{Z}/2$ -stabilization and there are t flows between p and $\tau(p)$, then the stabilizations $H_1 \rightarrow H_2$ and $H_3 \rightarrow H_4$ are of type $(\ell + k_1 + t, m, t)$ and $(\ell + k_2 + t, m, t)$, respectively; if p_0 is a $\{1\}$ -stabilization the stabilizations are of types $(\ell + k_1)$ and $(\ell + k_2)$. See Figures 5.10 and 5.11.

(B2): In this case, there is an orbit of tangency from p to an index 1 critical point $\bar{p}^0$. For every flow from an index 1 critical point p_*^0 to p , we can perturb near p on the death side of $S_1 \cup S_3$ so that there is a flow from p_*^μ to $\bar{p}^\mu$ (likewise for the symmetric critical points). In the case that p_0 is a $\{1\}$ -orbit birth-death singularity, we can also perturb to get a flow from $\tau(\bar{p}^\mu)$

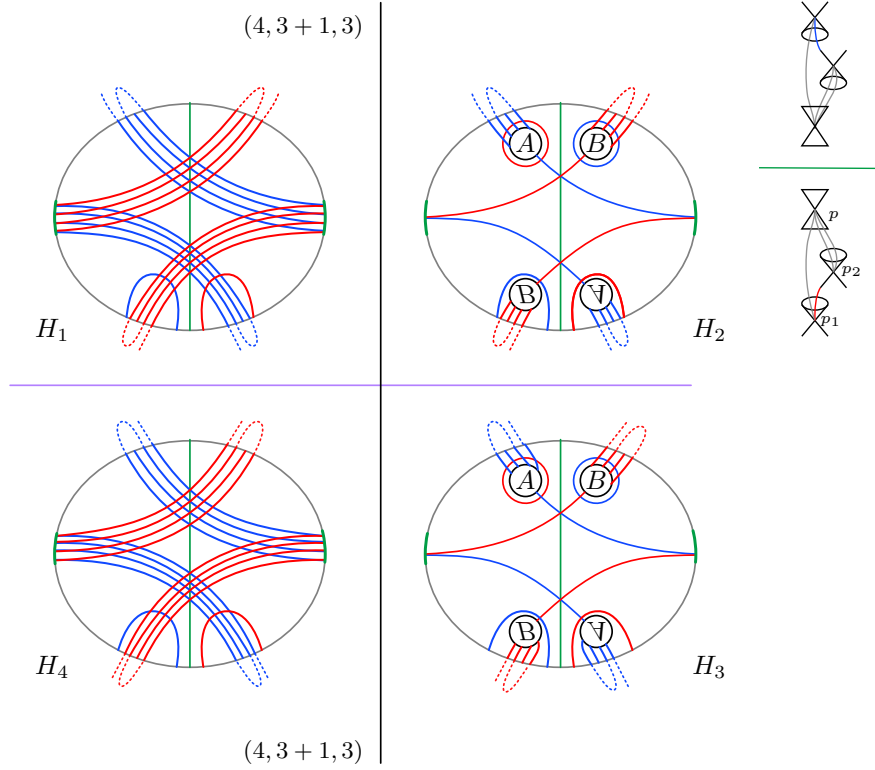


FIGURE 5.10. A commutation of a real handleslide and a $\mathbb{Z}/2$ -stabilization corresponding to the link of a bifurcation of type (B1) appearing in a $\mathbb{Z}/2$ -orbit. Here, the two green arcs on either side of the diagrams are identified and represent a portion of the fixed point set. This example has $m = \ell = 3$, $k_1 = k_2 = 1$, and $c = 3$.

to $\bar{p}^\mu$, which corresponds to a crossover stratum. There are at least three strata; again S_1 and S_3 are stabilizations and S_2 is a real equivalence.

First, let us assume that p appears in a $\mathbb{Z}/2$ -orbit. To understand the stabilizations in this case, we assume there are k flows from index 1 critical points to p and ℓ flows from p to index 2 critical points as well as t flows between p and $\tau(p)$. Further, there are $m_1 + m_2$ flows from the hyperbolic singularity $\bar{p}^0$ to index 2 critical points, partitioned into two sets by $W^{uu}(\bar{p}^0)$. The two stabilizations $H_1 \rightarrow H_2$ and $H_3 \rightarrow H_4$ are of types $(k + m_1 + t, \ell, t)$ and $(k + m_2 + t, \ell, t)$, respectively; on the stabilized side of the diagram, there is a single real handleslide stratum, whereas on the unstabilized side, there are k real handleslide strata. See Figure 5.12.

Next, assume that p appears in a $\{1\}$ -orbit. By symmetry, there are an equal number of flows (k , say) in and out of p . Say that there are $m_1 + m_2$ flows from the hyperbolic singularity $\bar{p}^0$ to index 2 critical points, partitioned into two sets by $W^{uu}(\bar{p}^0)$. Strata S_1 and S_3 are $\{1\}$ -stabilizations $H_1 \rightarrow H_2$ and $H_4 \rightarrow H_3$ of type $(k + m_1)$ and $(k + m_2)$ respectively. On the stabilized side of the diagram, there is a single real handleslide stratum. On the destabilized side of the bifurcation diagram, there are a total of k real handleslide strata separated into two collections by a single crossover of type (m_1, m_2) . See Figure 5.13.

We need to be slightly careful in constructing the splittings. As above, we fix a surface Σ_1 for H_1 and take Σ_2 to be obtained by a stabilization. We can choose $\Sigma_3 = \Sigma_2$. We cannot, however,

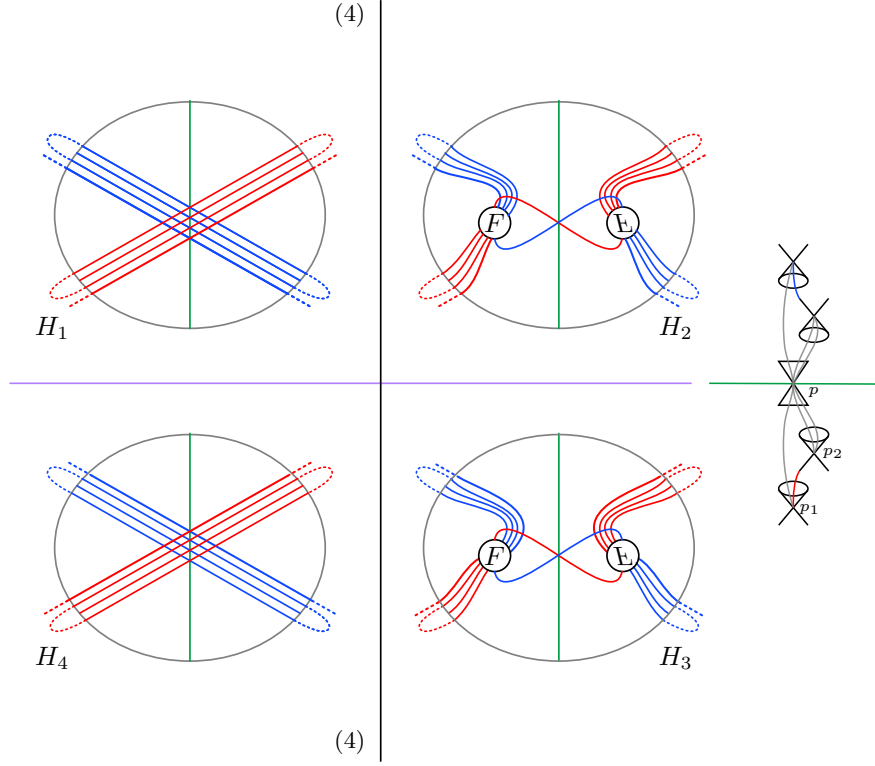


FIGURE 5.11. A commutation of a real handleslide and a $\{1\}$ -stabilization, which corresponds to the link of a bifurcation of type (B1) appearing in a trivial orbit. This example has $m = \ell = 3$ and $k_1 = k_2 = 1$.

take $\Sigma_4 = \Sigma_1$, as on the unstabilized side of the diagram, they are separated by the crossover stratum. We therefore choose Σ_4 so that it is obtained by a destabilization of Σ_3 . If S_i , for $i > 3$, is the crossover stratum, we define the remaining surfaces as follows. Let $\Sigma_5 = \dots = \Sigma_i := \Sigma_4$ and $\Sigma'_{i+1} = \Sigma_{i+2} = \dots = \Sigma_{r-1} := \Sigma_1$. To define Σ'_i and Σ_{i+1} , we choose a small arc $a^{i \rightarrow i+1}$ transverse to S_i , and choose $\Sigma_{i+1} \in \Sigma^R(f_\nu, v_\nu)$, where $\nu = \partial a^{i \rightarrow i+1} \cap C_{i+1}$ and $\Sigma'_i \in \Sigma^R(f_\eta, v_\eta)$, where $\eta = \partial a^{i \rightarrow i+1} \cap C_i$ so that H_i is obtained from H_{i+1} via a crossover. In particular, C_i and C_{i+1} each contain two vertices, and therefore each contains a diffeomorphism stratum.

(B3): There is an orbit of tangency between the strong stable manifold of an index 1-2 birth-death p and the unstable manifold of an index 1 critical point $\bar{p}$. In this case $r = 3$, and S_1 and S_3 are again stabilizations and S_2 is a handleslide.

Consider the case when p appears in a trivial orbit. Say there are k flow lines from index 1 critical points to p (and symmetric flows to index 2 critical points); then the stabilizations $H_1 \rightarrow H_2$ and $H_1 \rightarrow H_3$ are of type $(k+1)$ and (k) respectively.

If p appears in a $\mathbb{Z}/2$ -orbit, suppose that there are k flow lines from index 1 critical points to p , ℓ flow lines from p to index 2 critical points, and c flows between p and $\tau(p)$. Then the stabilization $H_1 \rightarrow H_2$ and $H_1 \rightarrow H_3$ are of type $(k+c, \ell+1, c)$ and $(k+2c, \ell, c)$ respectively.

(B4): Suppose $\{p, \tau(p)\}$ is a $\mathbb{Z}/2$ -orbit of index 1-2 birth-death singularities, that p_1^μ is an index 1 critical point, and that there is a real quasi-transversal orbit of tangency from $\tau(p_1^0)$ to p_1^0 . In this case, $r = 4$, the strata S_1 and S_3 are stabilizations while S_2 and S_4 are crossovers.

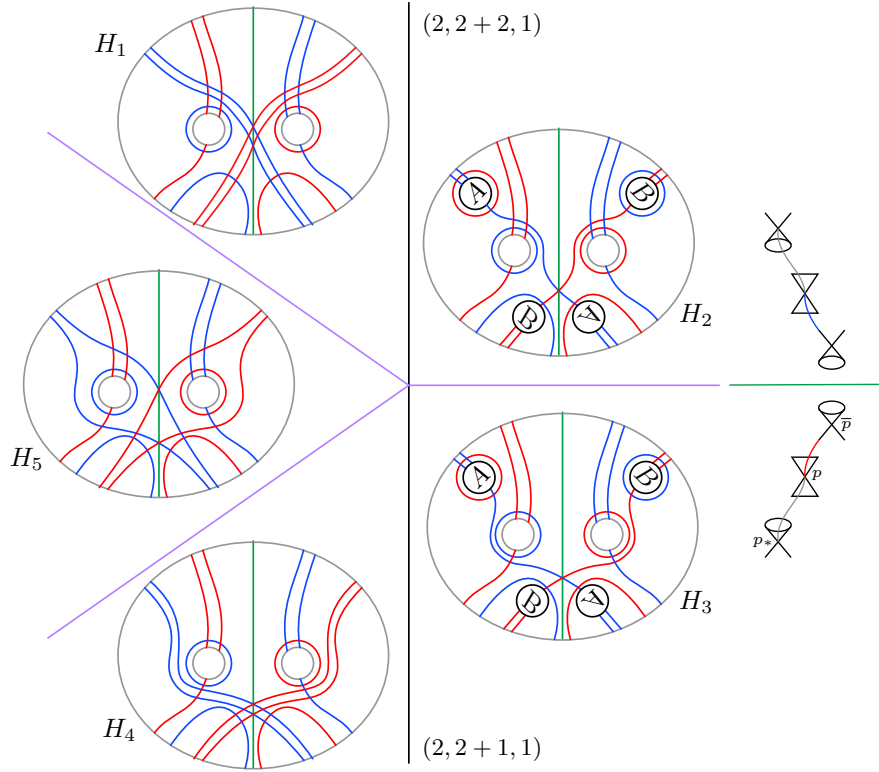


FIGURE 5.12. The link of a bifurcation of type (B2) appearing in a $\mathbb{Z}/2$ -orbit. In this example, $k = 3$, $\ell = 2$, $c = 1$, $m_1 = 2$, and $m_2 = 1$.

Again, flows out of p_1^μ are partitioned into two subsets. Suppose there are n flows from p to critical points of index 2, $r_1 + r_2$ flows from p_1^0 to p_1 (partitioned as such), and m additional flows from index 1 critical points to p ; additionally, say there are c flows from $\tau(p)$ to p . Finally, assume there are $\ell_1 + \ell_2$ flows from p_1^μ to critical points of index 2. The strata S_1 and S_3 correspond to $\mathbb{Z}/2$ -stabilizations of type $(m + r_1 + r_2 + c, n, c)$ and $(m + r_1 + r_2 + c + r_1(r_1 + \ell_1) + r_2(r_2 + \ell_2), n, c + r_1 + r_2)$. The strata S_2 and S_4 are crossovers of types $(r_1 n + \ell_1, r_2 n + \ell_2)$ and $(r_1 + \ell_1, r_2 + \ell_2)$ respectively. See Figure 5.16.

To construct the various splittings, we choose short arcs $a^{i \rightarrow i+1}$ transverse to S_i and use the parameter values $\partial a^{i \rightarrow i+1}$ to choose Σ'_i and Σ_{i+1} . This gives us eight vertices on our link, and each chamber contains a diffeomorphism stratum.

The case that p is fixed by τ is much simpler, as there can be no flows from p to p_1^0 . If there are n flows from p to index 2 critical points and $k_1 + k_2$ flows from p_1^0 to index 2 critical points, then S_1 and S_3 both correspond to $\{1\}$ -stabilizations $H_1 \rightarrow H_2$ and $H_4 \rightarrow H_3$ of type (m) while S_2 and S_4 correspond to crossovers $H_2 \rightarrow H_3$ and $H_1 \rightarrow H_4$ of type (k_1, k_2) . See Figure 5.17.

(B5): Let $\{p, \tau(p)\}$ be a pair of index 1-2 birth-death singularities with a quasi-transversal orbit of tangency from $W^{uu}(\tau(p))$ to $W^{ss}(p)$. Say there are m flows from p to critical points of index 2 and also that there are n flows from critical points of index 1 and c flows from $\tau(p)$ to p .

The bifurcation diagram has three strata S_1, S_2 , and S_3 . Strata S_1 and S_3 are $\mathbb{Z}/2$ -stabilizations of type $(m + (c + 1), n, c + 1)$ and $(m + c, n, c)$ while S_2 is a crossover of type $(n, 1)$. See Figure 5.18. Fix $\nu \in C_1$ to choose $\Sigma_1 \in \Sigma^R(f_\nu, \nu_\nu)$. We construct Σ_2 and Σ'_3 by attaching tubes to Σ_1 . We define Σ'_2 and Σ_3 as before, by picking surfaces in $\Sigma^R(f_\eta, \nu_\eta)$ for $\eta \in \partial a^{2 \rightarrow 3}$.

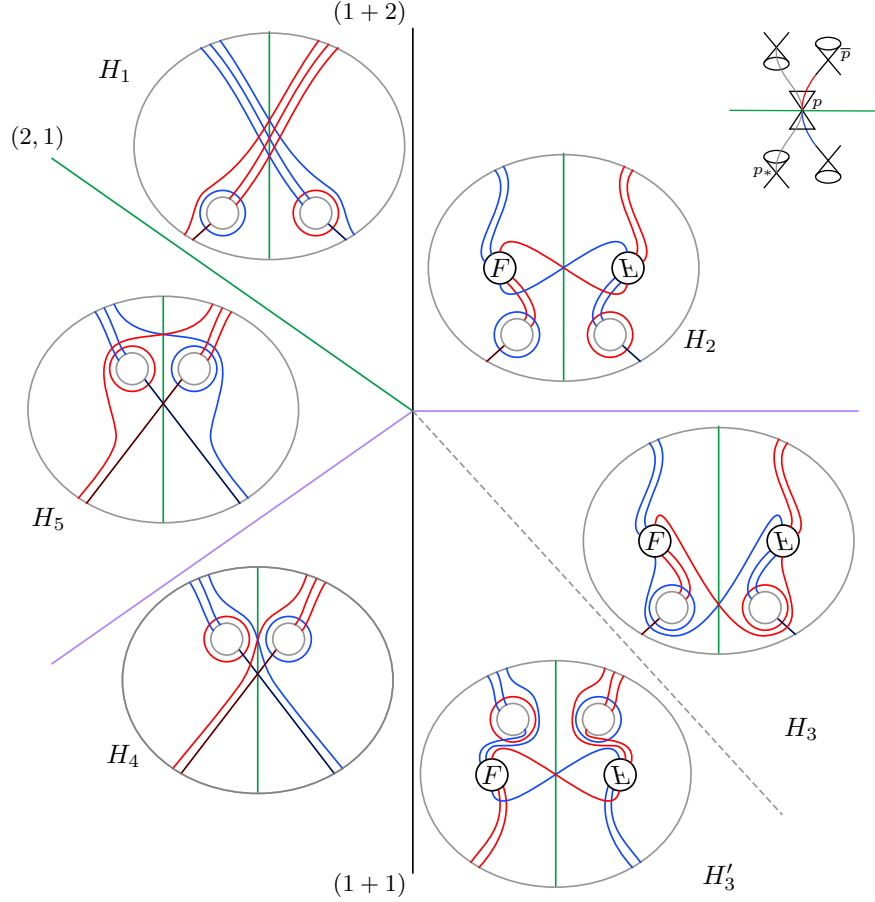


FIGURE 5.13. The link of a bifurcation of type (B2) appearing in a trivial orbit. In this example, $k = 1$, $m_1 = 2$ and $m_2 = 1$. In this example, there are no additional handleslides after the crossover occurs.

There are two diffeomorphism strata relating Σ_2 and Σ'_2 as well as Σ_3 and Σ'_3 .

Type (C) bifurcations: Type (C) bifurcations involve simultaneous stabilizations, and therefore come in three varieties, depending on the combination of orbit types.

Consider the case that (f_0, v_0) has a pair of $\mathbb{Z}/2$ -orbit index 1-2 birth-death bifurcations, $\{p_1, \tau(p_1)\}$ and $\{p_2, \tau(p_2)\}$. Generically, $f(p_1) < f(p_2)$. In this case, (f_0, v_0) is separable: indeed, we place $p_1, p_2 \in C_{01}(f)$ and $\tau(p_1), \tau(p_2) \in C_{23}(f)$ and choose $\Sigma_0 \in \Sigma^R(f_0, v_0)$ so that for $\epsilon > 0$ small enough, $\Sigma_0 \pitchfork v_\mu$ for all $|\mu| < \epsilon$. In this case, there are four strata: S_1 and S_3 correspond to $\mathbb{Z}/2$ -stabilizations at p_1 and $\tau(p_1)$, and S_2 and S_4 correspond to $\mathbb{Z}/2$ -stabilizations at p_2 and $\tau(p_2)$.

We now consider the various flow lines, beginning with flow lines between the degenerate critical points. Suppose that there are t flows from p_1 to p_2 , c_i flows from p_i to $\tau(p_i)$ for $i \in \{1, 2\}$, as well as s flows from p_1 to $\tau(p_2)$. Next, assume there are m flows from p_1 to index 2 critical points, n flows from index 1 critical points to p_1 , and likewise, let k and ℓ be the number of flows from and to p_2 . The stabilization $H_1 \rightarrow H_2$ is of type $(\ell + nt + c_2, k + ns, c_2)$,

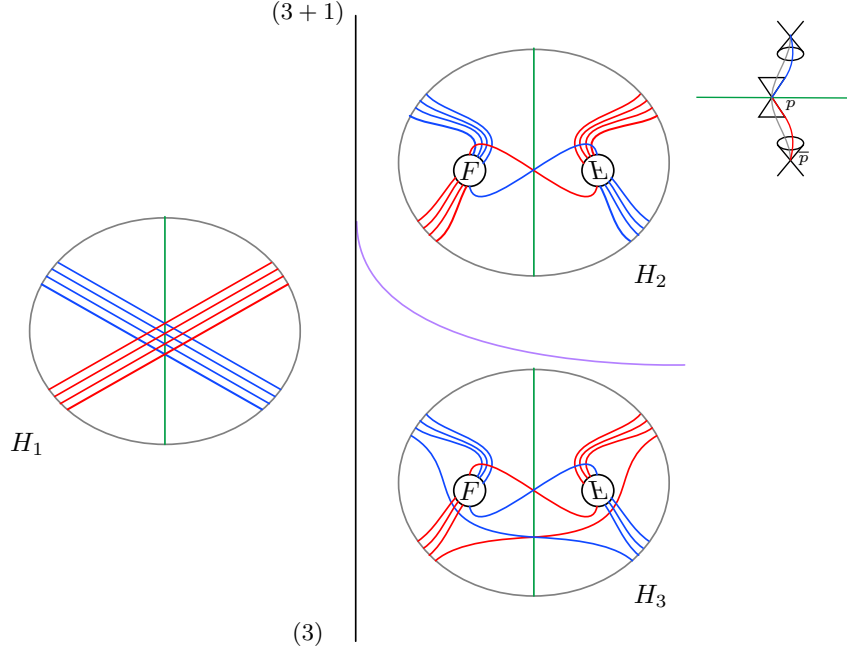


FIGURE 5.14. The link of a bifurcation of type (B3) appearing in a trivial orbit. In this example, $k = 3$.

$H_2 \rightarrow H_3$ is of type $(n, m + s + c_1, c_1)$, $H_1 \rightarrow H_4$ is of type $(n, m + (t + \ell + c_2)s, c_1 + sc_2)$, and $H_4 \rightarrow H_2$ is of type $(\ell + c_2 + t, k + s, c_2)$.

The underlying Heegaard splittings are constructed as follows. Choose μ_1 to be any point in C_1 and define $\Sigma_1 = \Sigma_0$. For $\nu \in C_2 \cup S_2 \cup C_3$, we let $q_1^2(\nu)$ be the index 2 critical point which splits off from p_1 and for $\eta \in C_3 \cup S_3 \cup C_4$, let $q_2^1(\eta)$ be the index 1 critical point which splits off from p_2 . By shrinking ϵ if necessary, we may assume that $W^u(q_1^2(\nu)) \cap W^s(q_2^1(\eta))$ is empty for all $\nu, \eta \in S_2 \cap C_3 \cap S_3$. Now, fix $\nu \in S_2$ and $\eta \in S_3$. We can obtain a surface $\Sigma_\nu \in \Sigma^R(f_\nu, v_\nu)$ from Σ_0 by attaching a pair of tubes around $W^u(q_1^2(\nu))$ and $W^s(\tau(q_1^2(\nu)))$; likewise, we construct $\Sigma_\eta \in \Sigma^R(f_\eta, v_\eta)$ from Σ_0 by attaching a pair of tubes around $W^s(q_2^1(\eta))$ and $W^u(\tau(q_2^1(\eta)))$. Choose arcs a_2 and a_3 transverse to S_2 and S_3 at ν and η , respectively, and define μ_2 to be the boundary component of a_2 in C_2 , μ_3 the boundary component of a_2 in C_3 , μ'_3 the boundary component of a_3 in C_3 , and μ_4 the boundary component of a_3 in C_4 . Define $\Sigma_2 = \Sigma_\nu$ and $\Sigma_4 = \Sigma_\eta$. Finally, we construct Σ_3 from Σ_2 by attaching a pair of tubes around $W^s(q_2^1(\mu_3))$ and $W^u(\tau(q_2^1(\mu_3)))$ and construct Σ'_3 from Σ_4 by attaching a pair of tubes around $W^u(q_1^2(\mu'_3))$ and $W^s(\tau(q_1^2(\mu'_3)))$. See Figure 5.19.

Assume now that the orbit types of p_1 and p_2 differ: concretely assume that p_1 corresponds to a $\mathbb{Z}/2$ -stabilization and p_2 corresponds to a $\{1\}$ -stabilization. Suppose that there are c flows from p_1 to p_2 , r flows from p_1 to $\tau(p_1)$, n (and k) flows from index 1 critical points to p_1 (and p_2 , respectively), m (and ℓ) flows from p_1 (and p_2 , respectively) to index 2 critical points.

There are four strata: S_1 and S_3 are $\mathbb{Z}/2$ -stabilizations of types $(n + t\ell + r + t, m, r + t)$ and $(n + r + \ell, m, r)$ while S_2 and S_4 are $\{1\}$ -stabilizations. Unlike the previous case, the gradient (f_0, v_0) is not separable, but we construct the Heegaard diagrams in essentially the same way: we choose some point $\mu_1 \in C_1$ and fix $\Sigma_1 \in \Sigma^R(f_{\mu_1}, v_{\mu_1})$; we then use Σ_1 to construct surfaces Σ_2 and Σ_4 which represent the $\mathbb{Z}/2$ - and $\{1\}$ -stabilized diagrams, respectively; finally, we use Σ_2 to construct a surface Σ_3 which is obtained from it via a $\{1\}$ -stabilization, and we use Σ_4 to

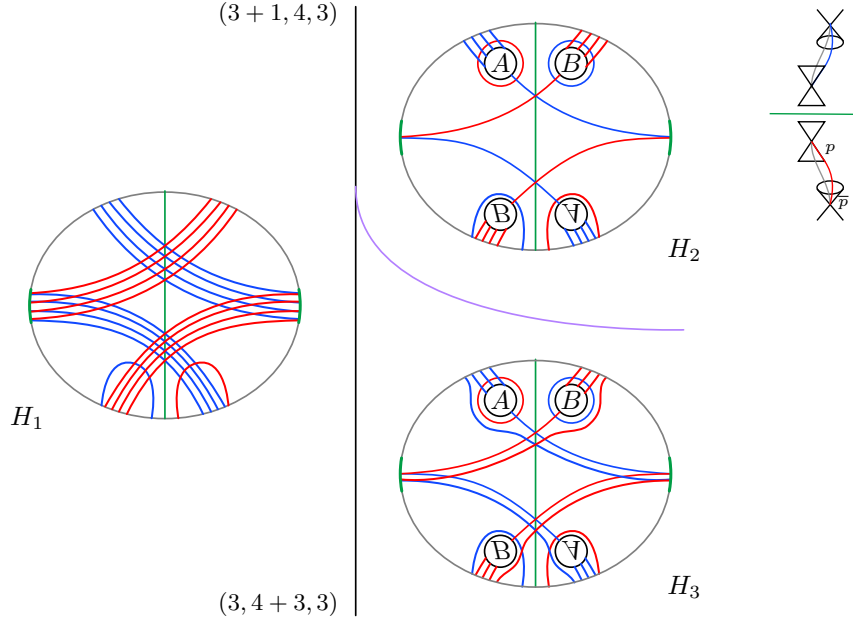


FIGURE 5.15. The link of a bifurcation of type (B3) appearing in a $\mathbb{Z}/2$ -orbit. Here, the two green arcs on either side of the diagrams are identified and represent a portion of the fixed point set. In this example, $k = 4$, $\ell = 3$, and $c = 3$.

construct Σ'_3 as a $\mathbb{Z}/2$ -stabilization. Then, we choose the (non-minimal) link exactly as above, with five vertices and five edges (four of which are transverse to the four codimension-1 strata and one which is contained in C_3). See Figure 5.20.

The case that p_1 and p_2 both appear in trivial orbits is comparatively much more straightforward; since the two singularities both appear on C , there can be no flows connecting the two degenerate critical points. Therefore, the bifurcation diagram has four strata, meeting at the origin. Suppose there are k and ℓ flows from index 1 critical points to p_1 and p_2 respectively (and symmetrically k and ℓ flows from p_1 and p_2 to index 2 critical points). Then, S_1 and S_3 correspond to $\{1\}$ -stabilizations $H_1 \rightarrow H_2$ and $H_4 \rightarrow H_3$ of type (k) and S_2 and S_4 correspond to stabilizations $H_1 \rightarrow H_4$ and $H_2 \rightarrow H_3$ of type (ℓ) . The link is obtained in the same way as the previous two cases. See Figure 5.21.

Type (D) bifurcations: Next, we consider bifurcations of type (D), i.e., singularities with (real) codimension 2.

(D1): A pair consisting of an index 1-2-1 (A_3^+) and an index 2-1-2 (A_3^-) degenerate critical point. In this case, $r = 2$ and on the stabilized side there are three index 1 critical points, p_1 , p_2 , and p_3 , and three index 2 critical $\tau(p_1)$, $\tau(p_2)$, and $\tau(p_3)$. The critical point p_1 can cancel with $\tau(p_2)$ or $\tau(p_3)$ (and symmetrically $\tau(p_1)$ can cancel with p_2 or p_3). We assume that there are c_2 and c_3 flows from p_2 and p_3 to $\tau(p_2)$ and $\tau(p_3)$ as well as k and ℓ flows from p_2 and p_3 to index 2 critical points (excluding the two flow lines to $\tau(p_2)$). Then, the two $\mathbb{Z}/2$ -stabilizations passing S_1 and S_2 are of types $(1, k + c_2, c_2)$ and $(1, \ell + c_3, c_3)$, respectively. See Figure 5.22.

(D2): An A_4 degenerate critical point. Here, $r = 3$, and in S_1 there are four critical points, p_1 , p_2 , $\tau(p_1)$, and $\tau(p_2)$, where p_1 and p_2 are index 1. There is a unique flow line from $\tau(p_2)$ to p_1 which is taken by the involution to the unique flow line from $\tau(p_1)$ to p_2 (making possible a

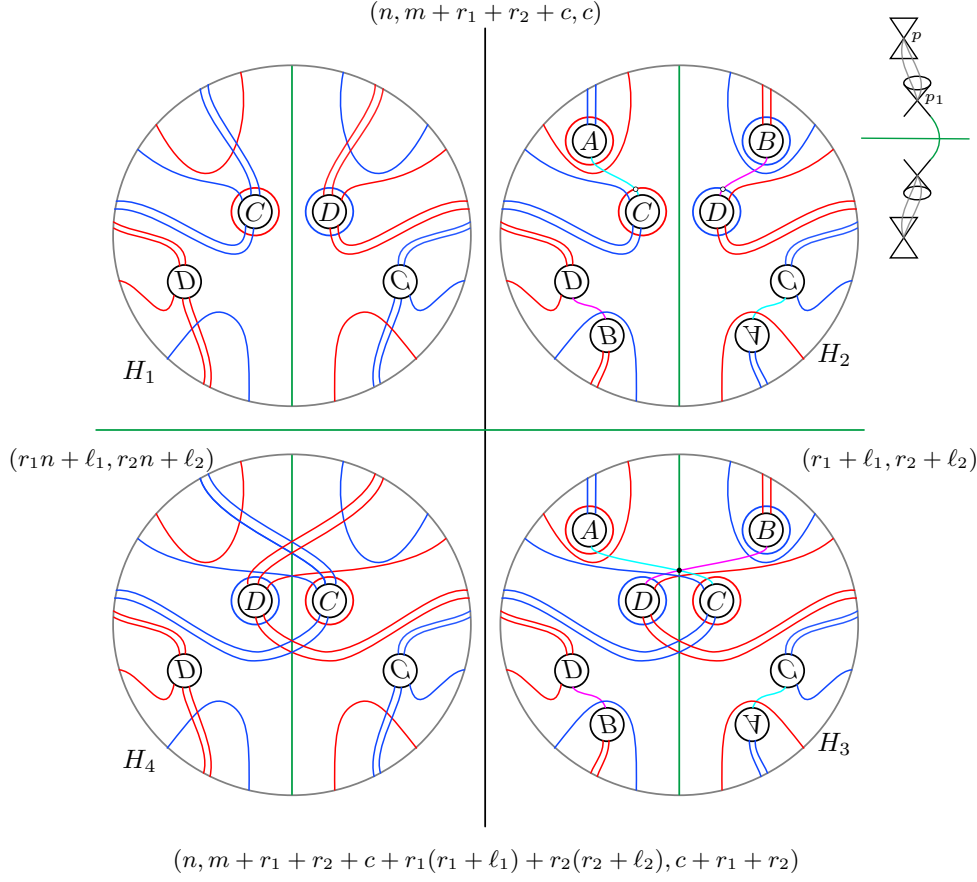


FIGURE 5.16. A commutation of a crossover and a $\mathbb{Z}/2$ -stabilization corresponding to a bifurcation of type (B4) involving a $\mathbb{Z}/2$ -stabilization. In this example, $n = 2$, $r_1 = 1$, $r_2 = 0$, $m = 2$, $c = 0$, $\ell_1 = 1$, $\ell_2 = 2$. We highlight some of the more subtle aspects of the figure. The stable manifold of the index 2 critical point which splits off from p is drawn in light blue; the unstable manifold of the symmetric index 1 critical point is drawn in pink. In H_2 , we have indicated the pair of intersection points corresponding to the r_1 flows from p_1 to p which give rise to the change in the type of crossover on the two sides of the diagram. Similarly, in H_3 we have indicated the intersection point corresponding to the $r_1 + r_2$ additional flow lines from $\tau(p)$ to p which is created when the parameter passes through the crossover stratum while moving along the stabilization stratum.

$\mathbb{Z}/2$ -destabilization) while simultaneously, there is a unique flow line from $\tau(p_2)$ to p_2 which is fixed setwise by τ (making possible a $\{1\}$ -destabilization). Say there are k and ℓ flows from index 2 critical points into p_1 and p_2 respectively (not including the flow line from $\tau(p_2)$ to p_2). Then, the stratum S_1 corresponds to a $\mathbb{Z}/2$ -destabilization of type $(k + \ell + 1, \ell + 2, 1)$, the stratum S_2 corresponds to a $\{1\}$ -stabilization of type (k) , and stratum S_3 corresponds to a $\{1\}$ -stabilization of type $(\ell + 1)$. See Figure 5.23.

Type (E) bifurcations: Type (E) bifurcations correspond to a failure of real quasi-transversality.

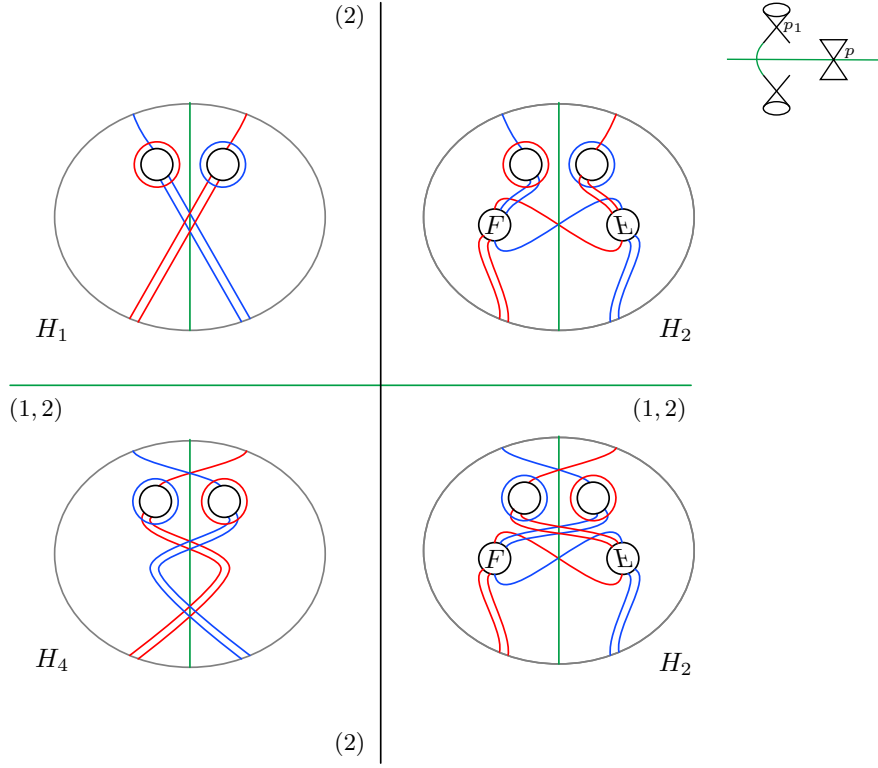


FIGURE 5.17. A commutation of a crossover and a $\{1\}$ -stabilization corresponding to a bifurcation of type (B4) involving a $\{1\}$ -stabilization.

(E1): In this case, we have index 1 critical points p_1^μ and p_2^μ and a pair of orbits of tangency from p_1^0 to $\tau(p_2^0)$ and p_2^0 to $\tau(p_1^0)$. There are two cases, depending on whether or not $p_2^\mu = p_1^\mu$.

Suppose the two orbits are the same, i.e., $p_2^\mu = p_1^\mu$, and that the two flows out of p_1^0 connect with the two flows into $\tau(p_1^0)$. Suppose there are k flows from p_1 to index 2 critical points and ℓ flows from index 1 critical points to p_1 . Then, $r = k + \ell$ and all strata correspond to $\mathbb{Z}/2$ -handleslides.

As in Proposition 5.19, let $\tilde{\Sigma}_0$ be the boundary of a thin regular neighborhood $\tilde{N}_0$ of

$$\bigcup \{W^s(p) : p \in C_0(f_0) \cup C_1(f_0) \setminus \{p_1^0\}\}.$$

We then construct the various Heegaard diagrams exactly as in [JTZ21]. Choose values ν_i in S_i and let A_i be a small tube $\partial\nu(W^s(p_1^{\nu_i})) \setminus N_0$ and let B_i be an annulus around $W^s(p_2^{\nu_i}) \cap \tilde{\Sigma}$. Ensure that A_i is small enough that it is disjoint from the stable and unstable manifolds of $p_2^{\nu_i}$. Let D_i and D'_i be the disks $\nu(W^s(p_1^{\nu_i})) \cap \Sigma$. These are the same diagrams constructed in [JTZ21], and are not symmetric. To obtain real diagrams, we push all of these along the gradient flow. By shrinking A_i if necessary, arrange that $D_i \cap \tau(D_i)$ and $D'_i \cap \tau(D'_i)$ are disjoint and define Σ_i to be $(\Sigma_0 \setminus (D_i \cup D'_i) \cup \tau(D_i \cup D'_i)) \cup A_i \cup B_i$. Compare to Figure 5.5.

Finally, to construct the link of the singularity, we choose arcs $a^{i \rightarrow i+1}$ transverse to S_i short enough that $D_i \cap W^s(p_2^\mu) = \emptyset$ for every $\mu \in a^{i \rightarrow i+1}$. Take $\partial a^{i \rightarrow i+1} = \{\mu_i, \mu'_{i+1}\}$ to be vertices of our link, and complete the link by choosing any arcs connecting μ_i to μ'_i . The resulting Heegaard diagrams are all identical away from the tubes A_i and B_i , and near those tubes, the loop of diagrams is shown in Figure 5.24.

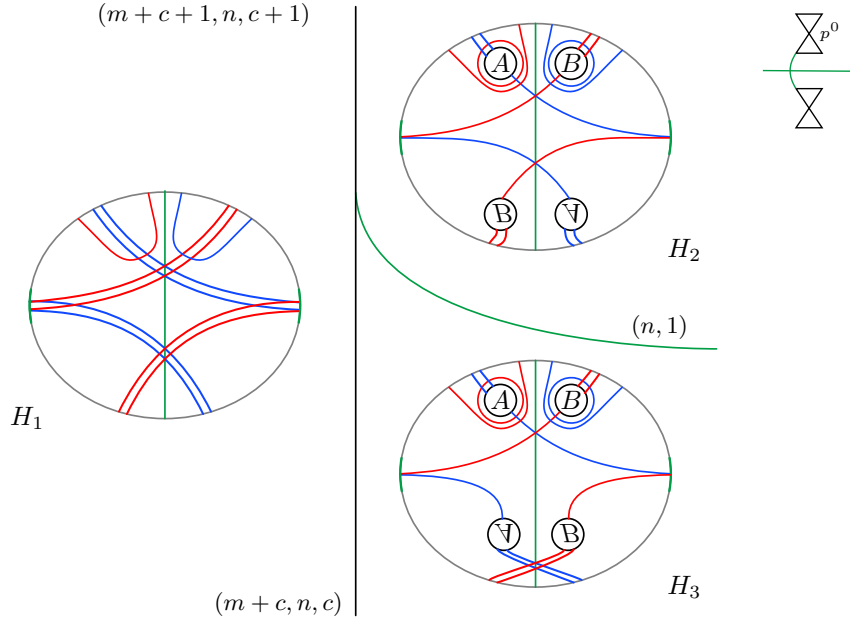


FIGURE 5.18. A bifurcation of type (B5) involving a $\{1\}$ -stabilization. The two green arcs on either side of the diagrams are identified and represent a portion of the fixed point set. In this example, $m = 1$, $n = 2$, $c = 1$.

The case that p_1^μ and p_2^μ are distinct is similar. We construct the Heegaard diagram in the same way, following the construction in [JTZ21], and pushing forward diagrams by the gradient flow to obtain invariant ones. In this case, however, as the annuli A_i traverse the annulus B , there are simultaneously annuli $B_i := \tau(A_i)$ which are looping around $A := \tau(B)$.

Suppose that there are k flows from p_1 to index 2 critical points, r flows from p_2^μ to $\tau(p_2^\mu)$, and ℓ additional flows from p_2^μ to other index 2 critical points. The bifurcation diagram consists of $(k + \ell + r)$ strata which meet at a point. When we resolve the diagram, the k flows out of p_1 will glue to $\tau(p_1)$ to become flows from $\tau(p_2^\mu)$ to index-2 critical points, corresponding to handleslides; the ℓ flows out of p_2 glue to the flow from $\tau(p_1^0)$ to p_2^0 and also give rise to handleslides; finally, the r flows from p_2^0 to $\tau(p_2^0)$ glue to become crossovers between p_1^μ and $\tau(p_1^\mu)$. See Figure 5.24 for an example.

Remark 5.26. We note that the first case is a particular case of the second; we take all A_i to be isotopic to A , and all B_i to be isotopic to B . In this case, we see that one foot of A is looping around one foot of B while symmetrically the second foot of B loops the second foot of A .

(E2): In this case, p_1^0 is a hyperbolic singularity of index 1 with a $\{1\}$ -orbit of tangency γ^0 from $\tau(p_1^0)$ to p_1^0 . Additionally, there is a flow from p_1^0 to an index 2 critical point p^* which is tangent to the directional manifold of $p_1^{(t,0)}$, where $t \in (-\epsilon, \epsilon)$ for $\epsilon > 0$ small. For $y < 0$ close to 0, the directional manifold $D(p_1^{(t,y)}, \gamma^\mu)$ partitions the flows from p_1^0 to index 2 critical points into two subsets of size $k + 1$ and ℓ , while for small $y > 0$ the directional manifold $D(p_1^{(t,y)}, \gamma^\mu)$ partitions the flows into two subsets of size k and $\ell + 1$.

The bifurcation diagram has three strata which meet in a point: S_1 and S_3 are crossovers of types $(k + 1, \ell)$ and $(k, \ell + 1)$, respectively, while S_2 is a handleslide. See Figure 5.25.

□

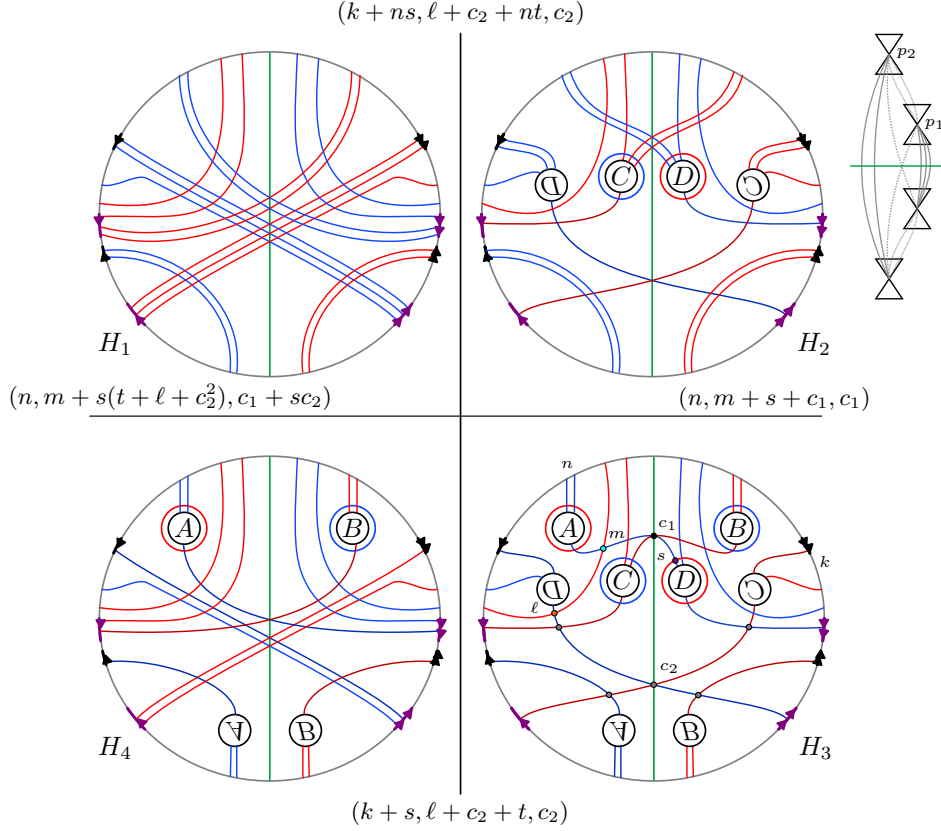


FIGURE 5.19. The link of a bifurcation of type (C) involving two generalized $\mathbb{Z}/2$ -stabilizations. In each quadrant, there are four pairs of arcs which are identified, e.g. the black arcs decorated with a single arrow on the right side of the diagram are glued together. Here, we have color-coded and labeled the various intersection points which correspond to the flow parameters determining the bifurcation type. In this example, $m = 1$, $n = 2$, $k = 2$, $\ell = 1$, $c_1 = 1$, $c_2 = 5$, $s = 1$, and $t = 0$.

6. SIMPLIFYING MOVES AND LOOPS OF REAL HEEGAARD DIAGRAMS

In this section, we decompose the moves and loops appearing in the previous sections into the simpler moves appearing in the definition of a strong real Heegaard invariant. Throughout, we will work with overcomplete diagrams, only later choosing spanning trees to obtain actual diagrams.

In the first subsection, we describe resolutions of the basic codimension-1 strata, and discuss some relations between possible resolutions. In the remaining subsections, we return to codimension-2 bifurcations. We will resolve type (A) bifurcations first as well as type (E2) bifurcations; the remaining codimension-2 bifurcations will typically be reduced to these cases. We will then turn to the resolution of real handleslides, which will ultimately be reduced to the unreal case. Finally, we simplify bifurcations of type (D2).

Let us briefly recall the combinatorial structures from [JTZ21] which are useful in simplifying loops of real Heegaard diagrams. We state the relevant lemmas without proofs, as they are identical to those in the unreal case.

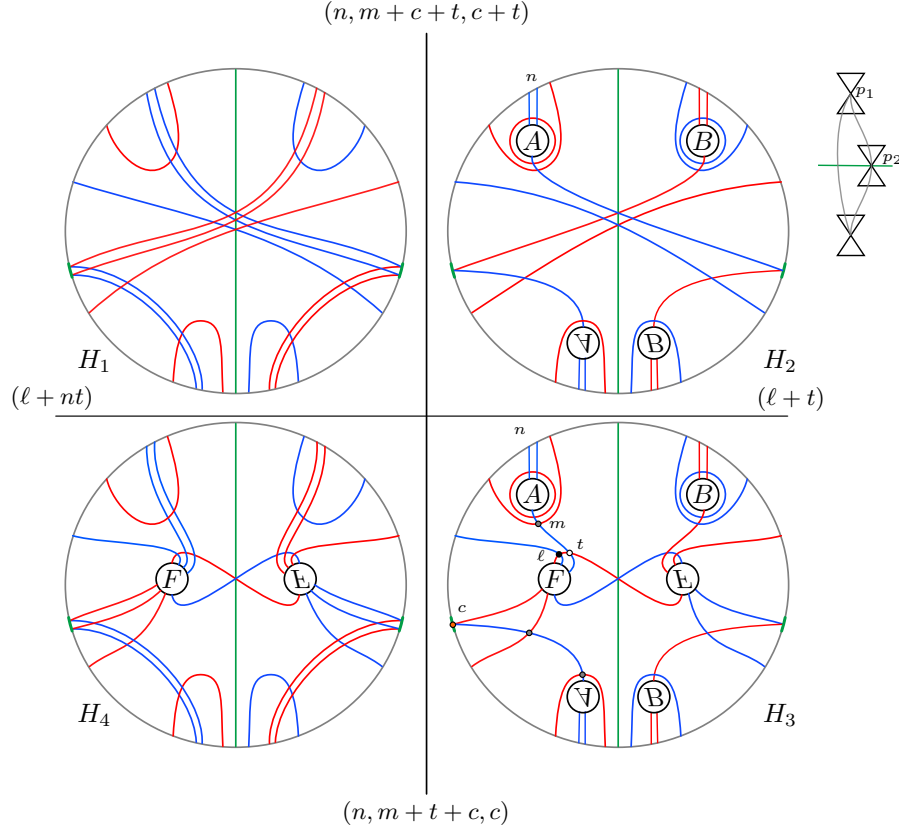


FIGURE 5.20. The link of a bifurcation of type (C) involving a generalized $\mathbb{Z}/2$ -stabilization and a generalized $\{1\}$ -stabilization. Again, there is a pair of green arcs which are identified and represent portions of the fixed point set. In this example, $m = 3$, $n = 2$, $\ell = 1$, $t = 1$, $c = 1$.

Definition 6.1. [JTZ21, Definition 7.1] A *polyhedral decomposition* of D^2 is a regular CW decomposition of D^2 (i.e., the attaching map of every cell is an embedding) such that every closed 1-cell is smoothly embedded in D^2 .

A *bordered polyhedral decomposition* of D^2 is a partition of D^2 that arises as follows: Choose a polyhedral decomposition of D^2 such that every 0-cell in the boundary S^1 has valence 3, every closed 1-cell not contained in S^1 intersects S^1 in at most one 0-cell, and every 2-cell intersects S^1 in at most one 1-cell. Then take the union of each open i -cell in S^1 with the neighboring open $(i + 1)$ -cell in $\text{Int}(D^2)$ for $i \in \{0, 1\}$ (where an open 0-cell is just a 0-cell). We call these *bordered $(i + 1)$ -cells*.

Let sk_i denote the i -skeleton. A polyhedral decomposition $\mathcal{P}$ and a bordered polyhedral decomposition $\mathcal{R}$ are *dual* if $\text{sk}_0(\mathcal{P}) \cap \text{sk}_1(\mathcal{R}) = \emptyset$ and $\text{sk}_0(\mathcal{R}) \cap \text{sk}_1(\mathcal{P}) = \emptyset$, in each 2-cell of $\mathcal{P}$ there is a unique vertex of $\mathcal{R}$, and in each (bordered) 2-cell of $\mathcal{R}$ there is a unique vertex of $\mathcal{P}$. Furthermore, for each 1-cell e of $\mathcal{P}$, we have $|e \cap \text{sk}_1(\mathcal{R})| = 1$, and for each (bordered) 1-cell e^* of $\mathcal{R}$, we have $|e^* \cap \text{sk}_1(\mathcal{P})| = 1$. For an example, see [JTZ21, Figure 31].

Definition 6.2. [JTZ21, Definition 7.2] We say that the partition $\mathfrak{S} = V_0 \sqcup V_1 \sqcup V_2$ is a *bordered stratification* of the disk D^2 if the following hold:

- (1) V_0 is a finite set of points in the interior of D^2 ,

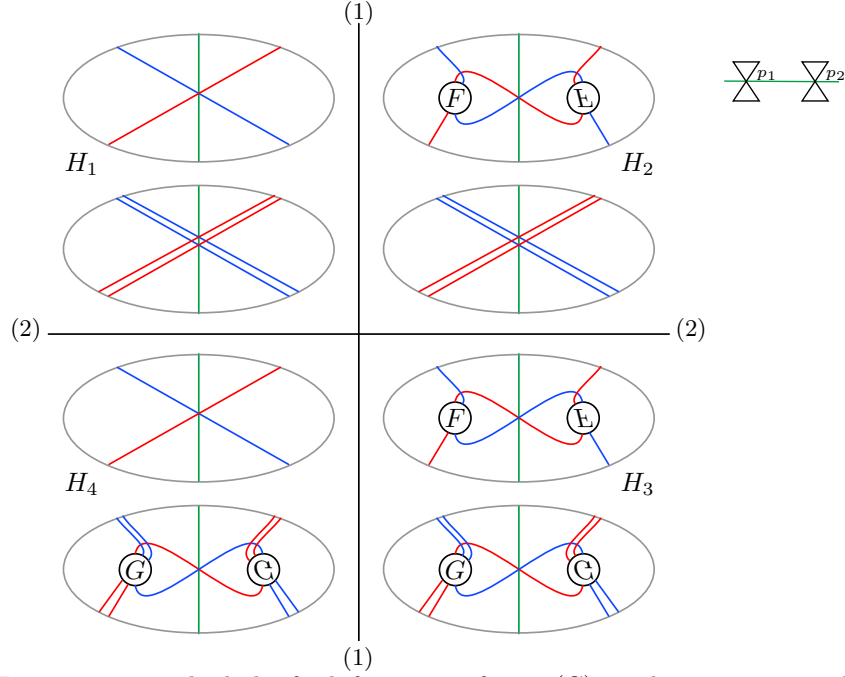


FIGURE 5.21. The link of a bifurcation of type (C) involving two generalized $\{1\}$ -stabilizations. In this example, $k = 1$ and $\ell = 2$.

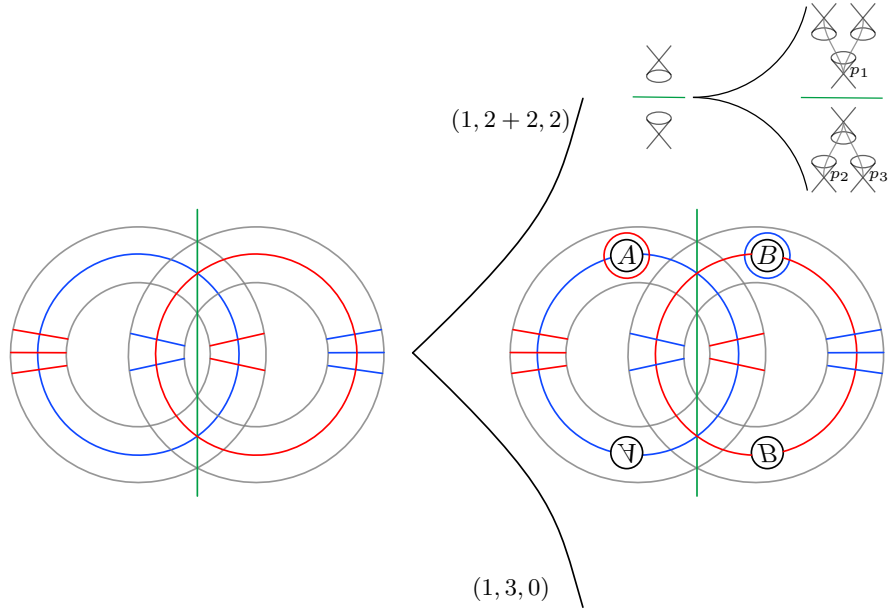


FIGURE 5.22. The link of a bifurcation of type (D1). In this example, $c_2 = 2$, $c_3 = 0$, $k = 2$, and $\ell = 3$.

(2) V_1 is a properly embedded 1-dimensional submanifold-with-boundary of $D^2 \setminus V_0$, and

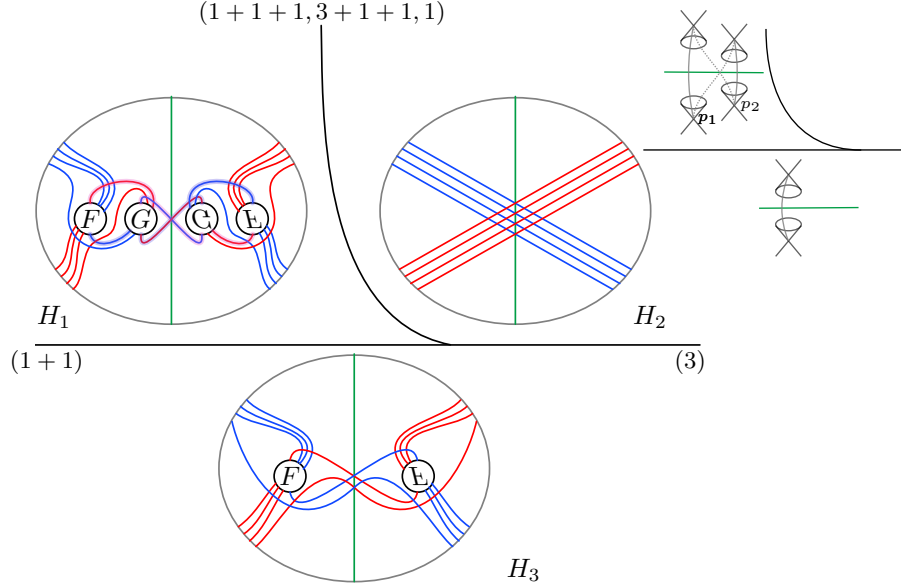


FIGURE 5.23. The link of a bifurcation of type (D2). In this example $k = 3$ and $\ell = 1$. In the first region, we have highlighted the pair of symmetric tori which are used in the $\mathbb{Z}/2$ -destabilization.

- (3) each point $x \in V_0$ has a neighborhood N_x such that the pair $(N_x, V \cap N_x)$ is homeomorphic to a cone $(D^2, I \cdot H)$ for some finite set $H \subset S^1$, where $V = V_0 \cup V_1$, and $I \cdot H$ is the union of the line segments connecting the origin with each point of H .

A bordered polyhedral decomposition $\mathcal{R}$ of D^2 is a *refinement* of $\mathfrak{S}$ if $\text{sk}_i(\mathcal{R}) \supset V_i$ for $i \in \{0, 1\}$. We say that a polyhedral decomposition of D^2 is *dual* to $\mathfrak{S}$ if it is dual to some bordered polyhedral decomposition $\mathcal{R}$ refining $\mathfrak{S}$. For an example, see [JTZ21, Figure 31].

Of course, a generic 2-parameter family of real gradients $\mathcal{F} : D^2 \rightarrow \mathcal{FV}(Y, \gamma, \tau)$ gives rise to a bordered stratification of D^2 by defining

$$V_i = \{\mu \in D^2 : \mathcal{F}(\mu) \in \mathcal{FV}_{2-i}(Y, \gamma, \tau)\}$$

for $i = 0, 1, 2$.

Definition 6.3. [JTZ21, Definition 7.3] A polyhedral decomposition $\mathcal{P}$ of D^2 is *adapted to the family $\mathcal{F}$* if

- (1) $\mathcal{P}$ is dual to $\mathfrak{S}(\mathcal{F})$,
- (2) each edge intersecting V_1 is so short that Proposition 5.19 applies to it,
- (3) if $\bar{\mu} \in V_0$ and σ is the 2-cell of $\mathcal{P}$ containing $\bar{\mu}$, then $\partial\sigma$ is a link of $\bar{\mu}$ as in Theorem 16,
- (4) every 2-cell σ of $\mathcal{P}$ that intersects V_1 but is disjoint from V_0 is a quadrilateral, and $\sigma \cap V_1$ is an arc connecting opposite sides of σ ,
- (5) any two closed 2-cells of $\mathcal{P}$ containing two different points of V_0 are disjoint, and any two closed 1-cells of $\mathcal{P}$ that intersect $V_1 \setminus S^1$ are either disjoint, or they both belong to a 2-cell containing a point of V_0 .

Lemma 6.4. [JTZ21, Lemma 7.4] Let $\mathcal{F} : D^2 \rightarrow \mathcal{FV}(Y, \gamma, \tau)$ be a generic 2-parameter family. Then there exists a polyhedral decomposition $\mathcal{P}$ of D^2 adapted to $\mathcal{F}$. Furthermore, given a

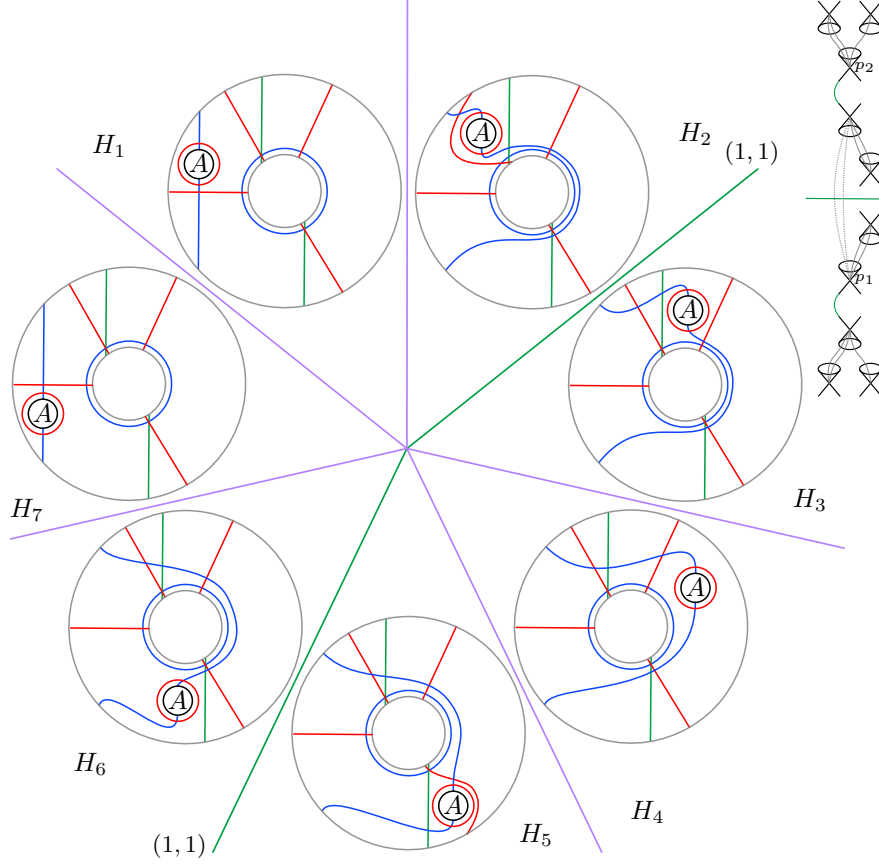


FIGURE 5.24. The link of a bifurcation of type (E1). Here, we have drawn the handleswap around the annulus A . Each time the foot crosses C , the annuli A and B intersect, though for clarity we have only drawn A . In this example, $k = 2$, $\ell = 3$, and $r = 2$.

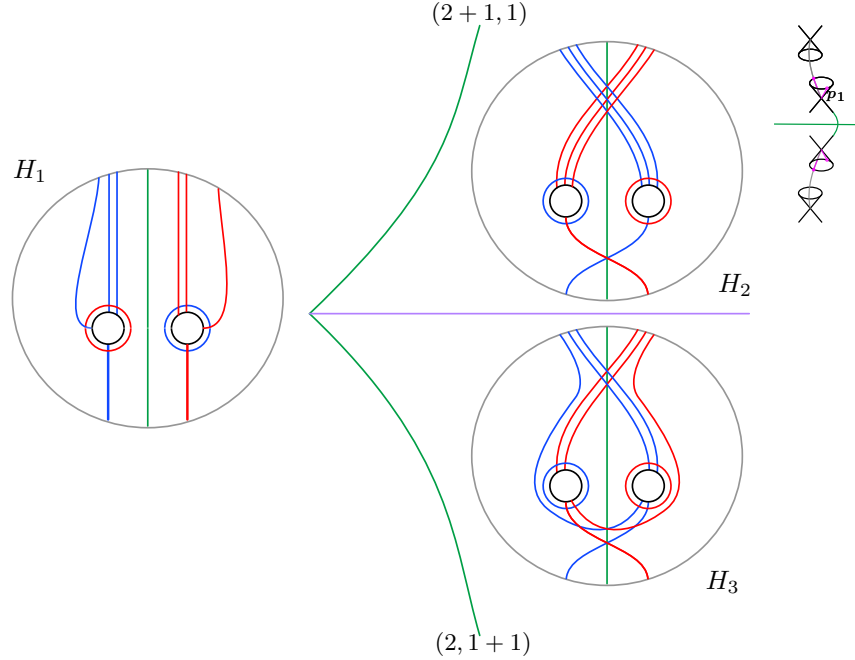
triangulation of S^1 such that each 1-cell contains at most one bifurcation point of $\mathcal{F}$ and satisfies Condition (2) of Definition 6.3, then we can choose $\mathcal{P}$ such that it extends this triangulation.

Definition 6.5. [JTZ21, Definition 7.5] Let $\mathcal{F}: D^2 \rightarrow \mathcal{FV}(Y, \gamma, \tau)$ be a generic 2-parameter family and $\mathcal{P}$ an adapted polyhedral decomposition. A choice of Heegaard surfaces

$$\{ \Sigma_\mu \in \Sigma^R(\mathcal{F}(\mu)) : \mu \in \text{sk}_0(\mathcal{P}) \}$$

is *coherent* with $\mathcal{P}$ if, for every edge e of $\mathcal{P}$ with $\partial e = \mu - \mu'$, the isotopy diagrams $[H(\mathcal{F}(\mu), \Sigma_\mu)]$ and $[H(\mathcal{F}(\mu'), \Sigma_{\mu'})]$ are related as indicated by the label of e . A *surface-enhanced polyhedral decomposition* of D^2 adapted to $\mathcal{F}$ is a polyhedral decomposition of D^2 adapted to $\mathcal{F}$, together with a coherent choice of Heegaard surfaces.

Lemma 6.6. [JTZ21, Lemma 7.6] Let $\mathcal{F}: D^2 \rightarrow \mathcal{FV}(Y, \gamma, \tau)$ be a generic 2-parameter family, and suppose that $\mathcal{P}$ is a polyhedral decomposition of D^2 adapted to $\mathcal{F}$. If we are given Heegaard surfaces $\Sigma_\mu \in \Sigma^R(\mathcal{F}(\mu))$ for $\mu \in \text{sk}_0(\mathcal{P}) \cap S^1$ such that, for every edge e of $\mathcal{P}$ in S^1 with $\partial e = \mu - \mu'$, the isotopy diagrams $[H(\mathcal{F}(\mu), \Sigma_\mu)]$ and $[H(\mathcal{F}(\mu'), \Sigma_{\mu'})]$ are related as indicated by the label of e , then this can be extended to a choice of Heegaard surfaces coherent with $\mathcal{P}$.


 FIGURE 5.25. The link of a bifurcation of type (E2). In this example, $k = \ell = 2$.

6.1. Codimension-1 resolutions. First, we break down generalized stabilizations (of both types) and crossovers.

The case of generalized $\{1\}$ -stabilizations is the most straightforward, and follows just as in [JTZ21, Section 7.1]. Suppose H' differs from H by a generalized $\{1\}$ -stabilization of type (k) . Such a stabilization can be replaced by a single simple $\{1\}$ -stabilization and k real handleslides. The diagram H' is obtained from H by removing a disk D and replacing it with a punctured torus T , which contains a new pair of attaching curves α, β . Fix an orientation on α (which determines an orientation on $\beta = \tau(\alpha)$) and let $\beta_1, \dots, \beta_k$ be the beta curves which intersect α . By sliding each β_i over β in the direction opposite the orientation on α (and hence the symmetric slides of the α_i over α), we obtain a sequence of diagrams

$$H' = H_0 \rightarrow H_1 \rightarrow \dots \rightarrow H_k.$$

In H_k , α and β are disjoint from all other attaching curves, and hence is obtained from H by a simple $\{1\}$ -stabilization.

This construction does depend on the orientation of α . However, as in [JTZ21, Lemma 7.8], there is a 2-parameter family which interpolates between these two choices. See Figure 6.2. Traversing a horizontal path along the top of the diagram corresponds to the resolution given by one choice of orientation and traversing a horizontal path along the bottom of the diagram corresponds to the other. In their situation, there is also a choice of whether to perform slides of the alpha curves first or the beta curves first; this is irrelevant in our situation, as the two slides happen simultaneously.

Remark 6.7. Importantly, every codimension-2 bifurcation value in this 2-parameter family corresponds to one of the simple loops appearing in the definition of a strong real Heegaard invariant.

In interpolating between resolutions of codimension-1 strata with different resolutions, we will frequently make use of the following subgraphs of $\mathcal{G}^R$.

Definition 6.8. An $\mathcal{O}$ -stabilization slide is a subgraph of $\mathcal{G}^R$ of the form

$$\begin{array}{ccc} H_1 & \xrightarrow{e} & H_2 \\ & \searrow f & \downarrow g \\ & & H_3 \end{array}$$

such that

- (1) $H_i = (\Sigma_i, [\alpha_i], [\beta_i])$ are overcomplete isotopy diagrams for $i \in \{1, 2, 3\}$ so that $\Sigma_2 = \Sigma_3$;
- (2) the edges e and f are $\mathcal{O}$ -stabilizations, while g is a real-equivalence;
- (3) there is a disk $D \subset \Sigma_1$ and a punctured torus $T \subset \Sigma_2 = \Sigma_3$ so that $H_1|_D$, $H_2|_T$, and $H_3|_T$ are conjugate to the pictures in Figure 6.1;
- (4) $H_1|_{\Sigma_1 \setminus D} = H_2|_{\Sigma_2 \setminus T} = H_3|_{\Sigma_3 \setminus T}$.

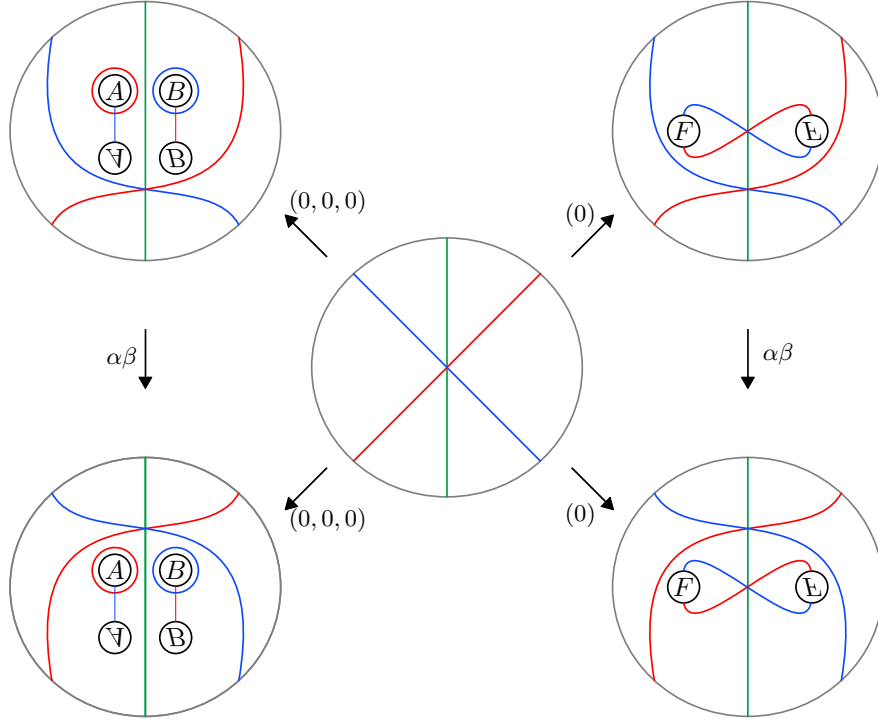


FIGURE 6.1. The two kinds of stabilization slides.

Now, suppose H' is obtained from H by a crossover of type (k, ℓ) . Suppose this crossover involves a neighborhood A of a curve α and a neighborhood B of the beta curve $\beta = \tau(\alpha)$. Let $\alpha_1, \dots, \alpha_k$ and $\alpha_{k+1}, \dots, \alpha_{k+\ell}$ be the alpha curves which intersect β and let $\beta_1, \dots, \beta_k$ and $\beta_{k+1}, \dots, \beta_{k+\ell}$ be the beta curves which intersect α . An example of such a resolution can be seen in Figure 6.3, by traversing the strata along the top of the figure. We will refer to this figure throughout the following discussion. First, we perform a simple $\{1\}$ -stabilization $H = H_0 \rightarrow H_1$, introducing a new pair of curves α_0 and β_0 which are exchanged by the involution (this is the first stratum in Figure 6.3; there are two ways this can be done, as the roles of alpha and beta in the new handle can be swapped (in the language of [GM], we can change the framing of C by either $+1$ or -1). Next, we perform a diffeomorphism which drags A and B over the new

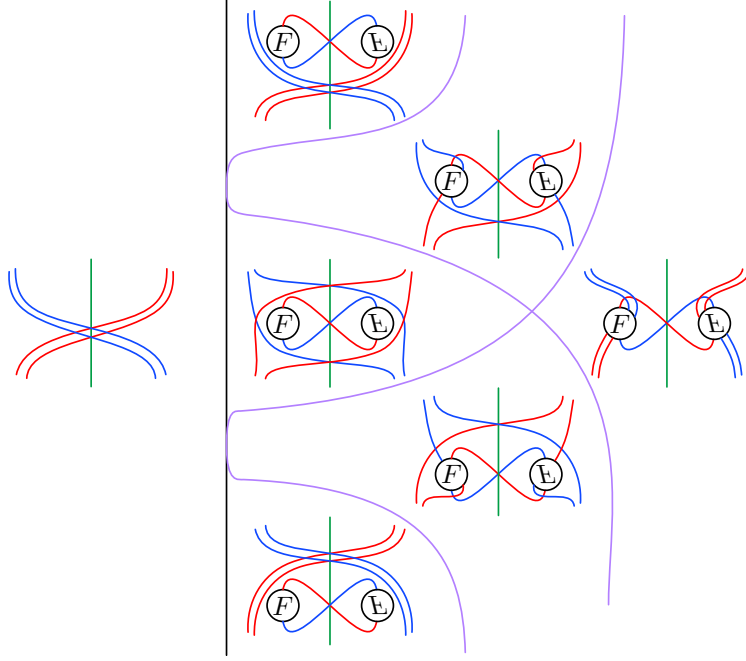


FIGURE 6.2. Switching the orientation involved in a $\{1\}$ -stabilization of type (k) . The top and bottom of the figure represent the two ways of resolving the stabilization.

handle. See the transition between the second and third chambers on the top row of Figure 6.3. Call the resulting diagram H'_1 . Next, we perform real handleslides over α_0 and β_0 ; depending on the framing of the new handle, we either perform k or ℓ handleslides. Following the top of Figure 6.3, we will for now assume we have performed ℓ handleslides:

$$H'_1 \rightarrow H_2 \rightarrow \dots \rightarrow H_{1+\ell}.$$

We then handleslide α_0 over α (and symmetrically β_0 over β), giving diagrams $H_{1+\ell} \rightarrow H_{2+\ell}$. See the transition between Frames 4 and 5 of Figure 6.3. Then, we perform k handleslides

$$H_{2+\ell} \rightarrow \dots \rightarrow H_{2+k+\ell},$$

over α_0 (and β_0) as in Frames 5 and 6 (or ℓ handleslides, in the case we have chosen the opposite framing of the $\{1\}$ -stabilization). Finally, we perform a simple $\{1\}$ -destabilization, giving the last move $H_{2+k+\ell} \rightarrow H_{3+k+\ell}$. Again, this resolution is not unique, but we can interpolate between resolutions, as shown in Figure 6.3.

Finally, we consider generalized $\mathbb{Z}/2$ -stabilizations. It is slightly easier to describe the destabilization. Suppose H' is obtained from H by a $\mathbb{Z}/2$ -stabilization of type (k, ℓ, c) . Recall that there is a punctured torus T in H' which is removed (along with $\tau(T)$) and replaced with a pair of disks D and $\tau(D)$ in H (the two tori may intersect, as might the disks). The torus T contains the two new attaching curves α and β , which we orient. By traversing β starting from α following the orientation, we obtain a sequence of alpha arcs $\alpha_1, \dots, \alpha_{\ell+c}$ (possibly with multiplicity). The curve β intersects exactly c of these α -arcs on C . Likewise, the orientation of α gives an ordering of the β -arcs which intersect it, $\beta_1, \dots, \beta_k$.

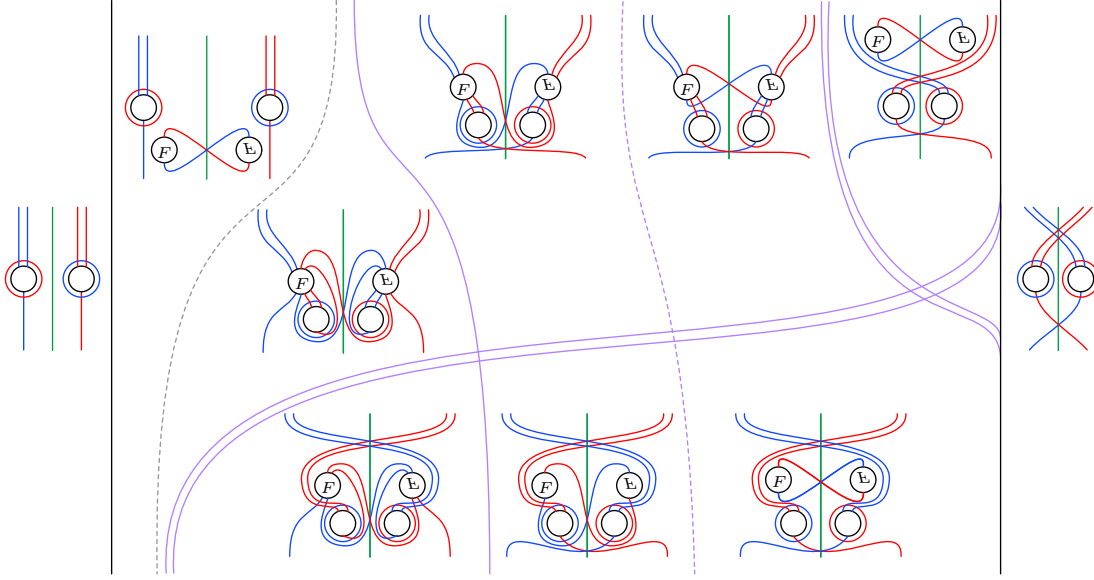


FIGURE 6.3. Two ways of resolving a crossover of type (k, ℓ) and an interpolation between these resolutions. In both cases, there are $k + 1 + \ell$ real handleslides. Here, and later, the dashed purple curve represents the handleslide involving the curves created in the $\{1\}$ -stabilization.

To perform the destabilization, we drag β along α . We obtain a sequence

$$H' = H_0 \rightarrow H_1 \rightarrow \dots \rightarrow H_{\ell+c},$$

where the diagram $H_i \rightarrow H_{i+1}$ corresponds to a handleslide of α_i over α if α_i intersects β away from the fixed set and corresponds to a crossover of type $(1, k)$ otherwise. Finally, we perform k handleslides of the $\beta_1, \dots, \beta_k$ over β :

$$H_{\ell+c} \rightarrow H_{1+\ell+c} \rightarrow \dots \rightarrow H_{k+\ell+c}.$$

At this point, α and β intersect no other curve in the diagram and can be removed by a simple destabilization. For an example of this simplification, see Figure 6.4; the sequence of moves is given by traversing a horizontal path at the top of the figure.

As in the case of $\{1\}$ -stabilizations, there is a dependence on the orientations of the alpha curves, though we can interpolate between the two; see Figure 6.4.

We now turn to the task of resolving the codimension-2 bifurcations appearing in Section 5.3 into 2-parameter families whose bifurcation values correspond only to the simple loops of diagrams which appear in the definition of strong real Heegaard invariants. We provide paradigmatic examples for each bifurcation type in Appendix A. (Since there are many large figures of this kind, we put them in an appendix to improve readability of the remaining sections.)

6.2. Bifurcations of type (A). The bifurcations of type (A1) and (A2) involve real handleslides, and do not need to be simplified, just as in the unreal case. On the other hand, bifurcations of types (A3) and (A4) require some work, as they involve real handleslides and crossovers. We must resolve these first, as we have chosen to resolve generalized $\mathbb{Z}/2$ -stabilizations in terms of crossovers.

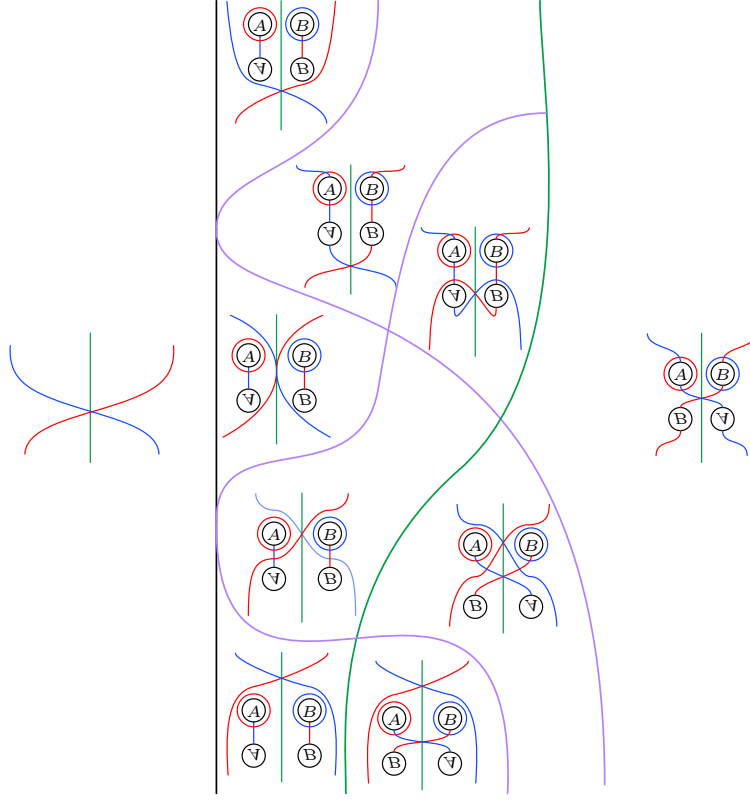


FIGURE 6.4. Switching the orientation involved in a $\mathbb{Z}/2$ -stabilization. The top and bottom of the figure represent two ways of resolving the stabilization.

Bifurcations of type (A4b) are the simplest. We simply resolve the (k, ℓ) -crossover into a simple $\{1\}$ -stabilization, followed by a sequence of $(k + 1 + \ell)$ real handleslides, and end with a $\{1\}$ -destabilization. Since the handleslide and crossover involve different critical points, the resolved diagram consists of a single $\{1\}$ -stabilization-handleslide commutation and $(k + \ell + 1)$ handleslide commutations. See Figure A.1.

(A3) is very similar: the two crossovers involve different critical points, so we may resolve them both into a $\{1\}$ -stabilization followed by handleslides and a $\{1\}$ -destabilization. The resolved diagram involves a $(\{1\}, \{1\})$ -stabilization commutation as well as many handleslide and $\{1\}$ -stabilization-handleslide commutations. See Figure A.2.

To resolve a bifurcation of type (A4a), we resolve the two crossover strata. The stratum S_4 is a crossover of type (k, ℓ) whereas S_2 is of type $(k + 1, \ell)$, hence we must introduce a link of type (A2), i.e. a handleslide pentagon as in [JTZ21, Figure 22]. An example is shown in Figure A.3; the handleslide pentagon is the result of the loop around the central bifurcation point where the five handle-slide strata meet.

Next, we consider bifurcations of type (A4c), which are the most complicated. We encourage the reader to compare the following discussion to Figure A.4. Recall that in this setting, there are two alpha curves α and α' which may be resolved either into a pair of crossovers between α and $\beta = \tau(\alpha)$ and between α' and $\beta' = \tau(\alpha')$, or, alternatively, into a real handleslide of α' over α and a crossover between the new alpha curve and its symmetric beta. Let $S_2, S_i,$

and S_{i+1} be crossover strata of types (k_3, ℓ_3) , (k_2, ℓ_2) , and (k_1, ℓ_1) respectively. Recall that $(k_3, \ell_3) = (k_1, \ell_2 + (\ell_1 - k_2))$.

A resolution of a crossover of type (k, ℓ) involves a diffeomorphism stratum, a simple $\{1\}$ -stabilization, ℓ real handleslides, a real handleslide involving the new curves introduced in the stabilization, k more real handleslides, and then finally a simple $\{1\}$ -destabilization. As in the previous section, strata which correspond to handleslides which involve the new curves from the stabilization will be drawn as dashed purple lines rather than solid. We complete the resolution as follows. Glue together the three diffeomorphism strata as in Figure A.4. There are six stabilization strata which we connect up: the stabilization strata from the (k_2, ℓ_2) -crossover and the (k_3, ℓ_3) -crossover are connected, the (k_2, ℓ_2) -crossover destabilization and (k_1, ℓ_1) -crossover stabilization are connected, and the (k_1, ℓ_1) -crossover destabilization and (k_3, ℓ_3) -crossover destabilization are connected.

Since $k_1 = k_3$, we can connect the k_3 handleslide strata from the resolution of the (k_3, ℓ_3) -crossover to the k_1 handleslide strata from the resolution of the (k_1, ℓ_1) -crossover. Similarly, the ℓ_2 handleslide strata from the (k_2, ℓ_2) -crossover can be connected to the ℓ_2 corresponding strata in the (k_3, ℓ_3) -crossover (again, recall that $\ell_3 = \ell_2 + (\ell_1 - k_2)$).

The remaining strata are more complicated. In the simplest case, when $\ell_1 = k_2$, we can connect the ℓ_1 strata from the (k_1, ℓ_1) -crossover to the k_2 strata from the (k_2, ℓ_2) -crossover. This leaves only the strata involving the curves introduced in the stabilization (i.e., the dashed purple strata). We glue these three strata together on the handleslide strata S_1 ; these five strata meet in a point, forming a handleslide pentagon. See Figure A.4 for an illustration in this case. Notice that we have used two different crossover resolutions in this example to simplify matters. We may interpolate between the two kinds of crossover resolutions as in Figure 6.3.

In general, it may be the case that $\ell_1 > k_2$. Such an example is shown in Figure A.5. In this situation, we cannot match up all the handleslide strata in the three crossover resolutions. To describe the resolution, let α_1 be the alpha curve associated to the birth-death p_1 and α_2 be the alpha curve associated to the birth-death p_2 (i.e., these are the curves corresponding to the dashed purple strata). There are $(\ell_1 - k_2)$ beta arcs emanating from α_1 which do not intersect α_2 . These are partitioned into two groups, of size m_1 and m_2 (there are m_1 arcs to the left of α_2 and m_2 to the right).

These additional arcs create two issues when trying to resolve the bifurcation. First, we must add in additional stabilization slides along the stabilization strata to interpolate between the various crossovers. This is due to the fact that the m_2 β -arcs which do not intersect α_2 must be slid over $\beta_2 = \tau(\alpha_2)$ before the (k_2, ℓ_2) crossover can be performed. On the right side of the diagram, there is a unique component of $\Sigma \setminus \alpha \setminus \beta$ where the stabilization may take place. On the left half of the diagram there are three regions of the diagram in which we can stabilize, created by the handleslides preceding the (k_2, ℓ_2) crossover. Therefore, below the topmost stabilization strata, we must add in stabilization slides to interpolate between the two stabilization locations. There is a similar phenomenon for the other two stabilization strata.

Second, in each crossover resolution, there is a distinguished handleslide which involves the curves α_0 and β_0 introduced by the stabilization (the dashed purple curves). At these strata, α_0 slides over either α_1 or α_2 (depending on which crossover we are near). In the interior of the diagram, there are parameters at which β_0 cannot slide over β_2 , because it is obstructed by the m_2 beta arcs which also slide over β_2 at various points of the resolution. At these points, we must introduce handleslide pentagons between β_0 , β_2 , and the m_2 beta arcs (i.e., we must add in singularities of type (A2)). For an example in the general case, see Figure A.5.

6.3. Bifurcations of type (E2). Next, we resolve bifurcations of type (E2), as they are relatively straightforward and will come up in later resolutions. We resolve the two crossovers. Both resolutions begin with a $\{1\}$ -stabilization and end with a $\{1\}$ -destabilization. In between

the (de)-stabilizations, the first crossover involves ℓ real handleslides, a distinguished handleslide involving the new alpha and beta curves coming from the stabilization, and then k more real handleslides. The second crossover involves $\ell + 1$ real handleslides, a distinguished handleslide involving the new alpha and beta curves coming from the stabilization, and then $k - 1$ more real handleslides. The first ℓ slides are identical, as are the last $k - 1$. The remaining real handleslides can be resolved into an (A2) singularity, forming a pentagon. See Figure A.6.

6.4. Bifurcations of type (B). We now turn to bifurcations involving stabilizations. Since we have chosen to resolve generalized $\mathbb{Z}/2$ -stabilizations into simple $\mathbb{Z}/2$ -stabilizations, handleslides, and crossovers, bifurcations of type (A) will appear in our resolutions. To avoid further complicating the figures, we will not depict the resolutions of crossover strata. Additionally, we will at times reduce to diagrams which contain bifurcations of types (A1)–(A4), which we have already shown how to resolve.

The simplest type (B) bifurcations are those of types (B3) and (B5). Recall that there are two cases, depending on the orbit type of the stabilization. In the trivial case, we simply resolve the $\{1\}$ -stabilization stratum and obtain a resolution with no singularities. See Figure A.7.

In the case of $\mathbb{Z}/2$ -orbits, we must be more careful. The S_1 stratum is a stabilization of type $(k, \ell + t, t)$ while S_3 is of type $(k - 1, \ell, t)$, so when we resolve the two generalized $\mathbb{Z}/2$ -stabilizations, there are an unequal number of handleslide strata. More specifically, in the resolution of the S_3 stratum, for each crossover, we must perform an additional handleslide that is not required in the S_1 resolution. To account for this, we simply merge the extra handleslide with the corresponding crossover, creating an (A4c) singularity. See Figure A.8 for an example.

Case (B5) is very similar. Stratum S_1 is a generalized $\mathbb{Z}/2$ -stabilization of type $(m + c + 1, n, c + 1)$ and S_3 is a generalized $\mathbb{Z}/2$ -stabilization of type $(m + c, n, c)$. We simply resolve S_1 , and since the resolution of S_3 is a sub-resolution of S_1 , there is no additional work to do. See Figure A.9.

Bifurcations of type (B1) are also relatively straightforward. In the case of a $\{1\}$ -stabilization, the resolution is exactly like the unreal case. We resolve the stabilization into a simple $\{1\}$ -stabilization and $k = k_1 + k_2$ real handleslides, though we must introduce k pentagon slides (singularities of type (A2)). For an example, see Figure A.10.

The case of generalized $\mathbb{Z}/2$ -stabilizations is very similar. We break the type (k, ℓ, r) stabilization into a single simple $\mathbb{Z}/2$ -stabilization and then a sequence of $k = k_1 + k_2$ handleslides, then a sequence of crossovers and handleslides. To resolve strata S_1 and S_3 , we again introduce k pentagon slides, just as in the previous case. The remaining handleslides and crossovers are identical for both strata, so no additional care is needed. For an example, see Figure A.11.

Next, consider bifurcations of type (B2). The case involving a generalized $\mathbb{Z}/2$ -stabilization is nearly identical to the resolution in the unreal case. Recall that we have a birth-death singularity p^0 with k flows in, ℓ flows out, and c flows between p^0 and $\tau(p^0)$; additionally, there are $m_1 + m_2$ flows from $\bar{p}^0$ to index 2 critical points. The stabilization strata S_1 and S_3 are of types $(k + m_1 + c, \ell, c)$ and $(k + m_2 + c, \ell, c)$ respectively; the m_1 real handleslides from resolving S_1 cannot always be matched up with the m_2 real handleslides from resolving S_3 . Therefore, we remove a small disk D where these $m_1 + m_2$ strata terminate. See Figure A.12. The loop around this disk corresponds exactly to a real handleswap (i.e., a bifurcation of type (E1)).

The case of a $\{1\}$ -stabilization is similar. The relevant parameters are the k flows from index 1 critical points to the birth-death singularity p^0 and the $m_1 + m_2$ flows from the hyperbolic critical point $\bar{p}^\mu$. Stratum S_1 is a stabilization of type $(k + m_1)$ and $(k + m_2)$. As in the previous case, we resolve S_1 and S_2 into a simple stabilization followed by a sequence of real handleslides, but the m_1 handleslides from resolving S_1 may not match up with the m_2 handleslides from resolving S_2 , so we remove a disk D . Unlike in the previous case, there is also a crossover stratum

which terminates on ∂D as well. Nevertheless, a loop around ∂D is again a real handleswap, which is permitted to involve crossover strata as well as handleslides. See Figure A.13.

Let us consider the case of (B4) bifurcations. The $\{1\}$ -stabilization case is straightforward. Recall that there is a birth-death singularity p^0 with m flows in and out as well as a tangency between the 1-dimensional stable manifolds of p_1^0 and $\tau(p_1^0)$; we assume there are $k_1 + k_2$ flows from p_1^0 to index 2 critical points partitioned into two sets. Strata S_1 and S_2 both correspond to $\{1\}$ -stabilizations of type (m) , and therefore can be resolved simultaneously, and likewise S_3 and S_4 are both crossover strata of type (k_1, k_2) , so can also be resolved simultaneously. See Figure A.14 for an example.

The case of generalized $\mathbb{Z}/2$ -stabilizations is more subtle. The relevant parameters are as follows: there are n flows out of p^0 to index 2 critical points; there are $r_1 + r_2$ flows from p_1^0 to p^0 partitioned into two subsets by the directional manifold of p_1^0 as well as m additional flow lines from additional index 1 critical points; there are also t flow lines from $\tau(p^0)$ to p^0 ; there are $\ell_1 + \ell_2$ flow lines out of p_1^0 to index 2 critical points. Recall that strata S_1 and S_3 are generalized $\mathbb{Z}/2$ -stabilizations of types $(m + r_1 + r_2 + t, n, t)$ and $(m + r_1 + r_2 + t + r_1(r_1 + \ell_1) + r_2(r_2 + \ell_2), n, t + r_1 + r_2)$ while strata S_2 and S_4 are crossovers of types $(r_1 n + \ell_1, r_2 n + \ell_2)$ and $(r_1 + \ell_1, r_2 + \ell_2)$.

We will begin by resolving the stabilization strata, and as we will see, this will be sufficient to reduce to previous cases. The resolutions for S_1 and S_3 are identical except that the resolution for S_3 involves additional crossovers (and adjacent handleslides). More precisely, let α_0 be the alpha curve associated to the index-1 critical point born from p^0 and let α be the alpha curve for p_1 which is involved in the crossover. Each time α slides over α_0 on the upper half of the diagram, the corresponding sequence on the lower half of the diagram after the crossover involves a sequence of handleslides, a crossover of α_0 and $\tau(\alpha_0)$, a second series of real handleslides, and then α slides over α_0 . These two paths are precisely the loop around a singularity of type (A4c). We simply repeat for each time α slides over α_0 (which happens $r_1 + r_2$ times in total). See Figure A.15 for an example of the resolved diagram.

6.5. Bifurcations of type (C). Next, we resolve bifurcations of type (C), which involve simultaneous $\mathcal{O}$ -stabilizations (where $\mathcal{O} \in \{\mathbb{Z}/2, \{1\}\}$). The simplest case is that of a pair of $\{1\}$ -stabilizations. Since there can be no flows between two birth-death singularities appearing in trivial orbits, we simply resolve the two $\{1\}$ -stabilizations into a pair of simple $\{1\}$ -stabilizations and real handleslides. See Figure A.16.

Next, we consider a generalized $\mathbb{Z}/2$ -stabilization and a generalized $\{1\}$ -stabilization. The generalized $\{1\}$ -stabilization is resolved into a simple $\{1\}$ -stabilization and a sequence of handleslides and the generalized $\mathbb{Z}/2$ -stabilization is resolved into a simple $\mathbb{Z}/2$ -stabilization and a combination of handleslides and crossovers. The key parameter is the number of flows t between the two kinds of birth-death singularities. For each such flow line, we must remove a small disk in the parameter space; the horizontal flow lines correspond to handleslides over the new $\{1\}$ -handle, while the vertical flow lines correspond to crossovers of the new $\mathbb{Z}/2$ -handles and handleslides of the same alpha and beta curves. The loop around this disk corresponds exactly to a real handleswap (E1); the feet of the $\mathbb{Z}/2$ -handles are sliding around the alpha and beta curves introduced in the $\{1\}$ -stabilization. See Figure A.17 for the resolution.

The case of two generalized $\mathbb{Z}/2$ -stabilizations is the most complicated. We deal with the $\{1\}$ -orbit flow lines between the degenerate critical points just as in the unreal case. For each flow line of this kind, there are strata which cannot be completed, so we remove a small disk from the parameter space; a loop around the removed disk is a handleswap.

The new phenomenon is due to the s pairs of flow lines from p_1 to $\tau(p_2)$. In these cases, the curve α_2 associated to the index 1 critical point born from p_2 slides over the curve α_1 associated to the index 1 critical point born from p_1 . On the half of the parameter space where α_2 has not slid over α_1 , there are many handleslides over α_1 involved in resolving the stabilization coming

from p_1 . On the half of the parameter space where α_2 has slid over α_1 , these handleslides do not occur, as they are slid over α_2 instead. Similarly, crossovers involving α_1 on the left half of the diagram may be traded for crossovers involving α_2 on the right-hand side.

To address these issues, we will glue the extra handleslide and crossover strata on the left side of the diagram to the handleslide strata corresponding to the slide of α_2 over α_1 . For the handleslides, this amounts to adding additional (A2) bifurcations and for the crossovers, this amounts to adding additional (A4c) bifurcations. See Figure A.18 for an example.

6.6. Bifurcations of type (D). Bifurcations of type (D) involve codimension-2 singularities. Bifurcation (D1) is a pair consisting of an A_3^+ singularity and an A_3^- singularity. We resolve the singularity just as in the unreal case: we introduce a simple $\mathbb{Z}/2$ -stabilization together with a sequence of real handleslides and crossovers; the sequences corresponding to the original two strata may not match up, in which case we can resolve the singularity by adding in a real handleswap and a handleslide triangle. See Figure A.19; in the figure, we have only drawn an annular neighborhood of the beta curve corresponding to the singularity, and not its symmetric alpha curve.

Next, we consider a singularity of type (D2). Let p^0 be the singularity of type $\{1\}$ - A_4 and suppose that there are k flows in and out of p^0 . The loop around the singularity involves two generalized $\{1\}$ -stabilizations and then a single generalized $\mathbb{Z}/2$ -destabilization. We resolve the first generalized $\{1\}$ -stabilization stratum into a simple $\{1\}$ -stabilization and k real handleslides; starting from the destabilized side, the generalized $\mathbb{Z}/2$ -stabilization is resolved into a simple $\mathbb{Z}/2$ -stabilization followed by k handleslides followed by a crossover (which we resolve momentarily). We can connect the k real handleslides from the generalized $\{1\}$ -stabilization with the k slides from the generalized $\mathbb{Z}/2$ -stabilization. Next, we resolve the crossover into a simple $\{1\}$ -stabilization followed by two real handleslides and a simple $\{1\}$ -destabilization; we connect the stabilization and destabilization strata. Finally, we resolve the second generalized $\{1\}$ -stabilization into a simple $\{1\}$ -stabilization and a real handleslide. It remains to extend the resolution over the central disk of Figure A.20. The six remaining strata cannot be glued together to form the simple loops appearing in the definition of a strong real Heegaard invariant. Therefore, we simply glue the remaining strata at a point, a loop around which is exactly a simple trade loop. See Figure A.21.

6.7. Bifurcations of type (E1). Finally, we resolve singularities of type (E1), real handleswaps. Let us briefly recall the set-up and introduce some terminology.

Definition 6.9. A *real handleswap of type (k, ℓ, r)* is a loop of overcomplete diagrams $\mathcal{H}_0, \dots, \mathcal{H}_{k+\ell+r}$, as follows. There is a surface Σ such that $\mathcal{H}_i = (\Sigma, \alpha_i, \beta_i)$ for each i and there is a pair of pants P so that

$$\alpha_i \cap (\Sigma \setminus P) = \alpha_j \cap (\Sigma \setminus P) \text{ and } \beta_i \cap (\Sigma \setminus P) = \beta_j \cap (\Sigma \setminus P)$$

for each i and j . Inside P , there is a full alpha circle α_0 . The boundary of P consists of three circles: A which is parallel to α_0 , B , and Z . The boundary component B is parallel to a beta curve β_0 , which may or may not be contained in P . The collection $(\alpha \setminus \alpha_0) \cap P$ is a collection of arcs connecting B to Z , $(\beta \setminus \beta_0) \cap P$ is a collection of arcs connecting A to Z , and $C \cap P$ is a collection of arcs connecting A to Z . Precisely k of the real Heegaard moves $\mathcal{H}_i \rightarrow \mathcal{H}_{i+1}$ are given by a real handleslide of an alpha curve over α_0 followed by a diffeomorphism dragging B along α_0 , ℓ of the moves are handleslides of beta curves over β_0 (followed by a diffeomorphism dragging B along α_0), and the remaining are crossovers of β_0 with $\tau(\beta_0)$.

A real handleswap of type (k, ℓ, r) of isotopy diagrams is a loop $H_0, \dots, H_{k+\ell+r}$ such that there are representatives $\mathcal{H}_i$ for H_i such that $\mathcal{H}_0, \dots, \mathcal{H}_{k+\ell+r}$ is a real (k, ℓ, r) -handleswap.

The first step in the simplification process is to reduce to the case of $(k, \ell, 0)$ crossovers. Suppose S_i is a crossover stratum. Each crossover stratum corresponds to an intersection point p between C and α_0 . Introduce a circular stratum corresponding to a $\{1\}$ -stabilization of type (1) around the singularity which removes this intersection point. We resolve this into a simple $\{1\}$ -stabilization stratum S near p and then perform a real handleslide of α_0 and $\tau(\alpha_0)$ over the new handle. The new stratum S meets S_i , introducing a (B2) bifurcation involving the $\{1\}$ -stabilization. On the stabilized side of S , we have a handleslide stratum, rather than a crossover; in particular, we see a real handleswap loop of type $(k, \ell + 1, r - 1)$. See Figure A.22. We repeat this process for each crossover stratum, after which we will have obtained a real handleswap loop of type $(k, \ell + r, 0)$.

Having eliminated all intersections between α_0 and $\tau(\alpha_0)$, we can reduce to the case of ordered real handleswaps following the discussion after [JTZ21, Definition 7.9] by introducing commutation moves between diffeomorphism and handleslide strata (see especially [JTZ21, Figures 41 and 42]).

At this point, we are actually in a position to apply the simplification process of [JTZ21, Section 7.3] equivariantly, as we have almost reduced real handleswaps to disjoint pairs of unreal handleswaps. There is one subtlety, namely that it could be the case that α_0 and $\tau(\beta_0)$ intersect (this was the first case considered in the description of (E1) in Section 4.3), in which case the local pictures shown in Figure A.23 are actually connected; we identify annular neighborhoods of the two alpha curves as well as annular neighborhoods of the two beta curves.

More precisely, Juhász-Thurston-Zemke reduce their analysis to the case of $(1;1)$ -handleswaps; this simply means that there is a single beta curve β which intersects α_0 and a single alpha curve α which intersects β_0 . In their setting, the final step is to consider all alpha arcs that intersect α and slide them over α_0 , and likewise slide beta arcs intersecting α over β_0 . However, this may be impossible in our setting, as it could be the case that $\beta = \tau(\alpha)$ and these curves intersect. In this case, we cannot apply real handleslides to reduce to the case of simple handleslides. By applying the Juhász-Thurston-Zemke algorithm slightly more carefully we can actually preclude this possibility, as we now explain.

The first step in their simplification process is to reduce to the case that there is a beta curve which intersects α_0 exactly once and an alpha curve which intersects β_0 exactly once (Juhász-Thurston-Zemke call this reducing to $(k, 1; \ell; 1)$ -handleswaps). They do this by performing a $(k, 1)$ -stabilization involving α_0 and a $(\ell, 1)$ -stabilization involving β_0 and then resolving the stabilization strata (see [JTZ21, Figure 54]). We may apply this step equivariantly as well. The crucial point for us is that in performing the $\mathbb{Z}/2$ -stabilization, we introduce a curve β' which intersects α_0 once and does not intersect $\tau(\beta')$, and it is this curve that appears when we reduce to $(1, 1)$ -handleswaps. See Figure A.23 for an example. In short, we need to apply step 1 of their simplification process even if we began in the case of a pair of $(1, 1)$ -handleslides, as we need to ensure the handleswaps are disjoint.

At this point, we can apply the rest of their algorithm equivariantly with no issue to reduce to the case of a simple real handleswap.

7. STRONG REAL HEEGAARD FLOER INVARIANTS HAVE NO MONODROMY

We now have the tools to prove Proposition 2.29 and Theorem 9, which we restate here.

Proposition 2.29. *Any two isotopy diagrams in $\mathcal{G}_{(Y, \gamma, \tau)}^R$ can be connected by an oriented path.*

Proof. This is the same as [JTZ21, Proof of Proposition 2.36] and follows from our analysis of 1-parameter families of real gradients. See Corollary 4.23, Proposition 5.19, and Lemma 5.11. $\square$

Theorem 9. *Let $\mathcal{S}^R$ be a set of diffeomorphism types of real sutured manifolds containing $[(Y, \gamma, \tau)]$. Suppose that $F : \mathcal{G}^R(\mathcal{S}^R) \rightarrow \mathcal{C}$ is a strong real Heegaard invariant. Given isotopy*

diagrams $H, H' \in |\mathcal{G}_{(Y, \gamma, \tau)}^R|$ and any two oriented paths η and ν in $\mathcal{G}_{(Y, \gamma, \tau)}^R$ connecting H to H' , we have

$$F(\eta) = F(\nu).$$

Proof. This is nearly identical to the proof in the unreal case [JTZ21, Theorem 2.38], so we only sketch the proof, emphasizing the necessary modifications. By the Functoriality Axiom, it suffices to prove that, for any loop η of the form

$$H_0 \xrightarrow{e_1} H_1 \xrightarrow{e_2} \dots \xrightarrow{e_{n-1}} H_{n-1} \xrightarrow{e_n} H_0,$$

we have $F(\eta) = \text{id}_{F(H_0)}$. We then construct a generic 2-parameter family $\mathcal{F} : D^2 \rightarrow \mathcal{FV}(Y, \gamma, \tau)$ and a surface-enhanced polyhedral decomposition $\mathcal{P}$ of D^2 so that along ∂D^2 we have the loop η just as in [JTZ21]. We then apply the resolution process of Section 6 to obtain a decomposition $\mathcal{P}'$ with the property that each 2-cell has boundary corresponding to a simple loop appearing in the axioms of a strong real Heegaard invariant: a loop of real equivalences, a loop of equivariant diffeomorphisms, a distinguished rectangle, a simple real handleswap, a stabilization slide (which is just a degenerate distinguished rectangle), or a simple trade. The composition of such a loop of diffeomorphisms is the identity by Lemma 5.13. The vertices of $\mathcal{P}'$ may be overcomplete, but we can follow [JTZ21] to obtain a decomposition $\mathcal{P}''$ decorated by actual real Heegaard diagrams without altering the boundary. Each 2-cell of $\mathcal{P}''$ corresponds uniquely to a cell (of some dimension) of $\mathcal{P}'$, and we have to check that F commutes along each of these. The analysis in [JTZ21] carries through in our setting, so it suffices to address the case that σ is a 2-cell of $\mathcal{P}''$ corresponding to a 2-cell σ_0 in $\mathcal{P}'$ so that $\partial\sigma_0$ is a simple trade loop. But in this case, the spanning tree plays no essential role. Let $K_1, \dots, K_5$ be the real isotopy diagrams decorating the vertices of $\partial\sigma_0$. Start with a spanning tree T^1 for $\Gamma(K_1)$; as K_2 and K_3 are obtained by stabilizations, we naturally obtain spanning trees T^2 and T^3 for $\Gamma(K_2)$ and $\Gamma(K_3)$; a spanning tree T^4 for $\Gamma(K_4)$ is obtained by sliding the new edges over one another and one for $\Gamma(K_5)$ is obtained by pushing T^4 forward by the diffeomorphism relating K_4 and K_5 . We then take $D_i = H(K_i, T^i)$ and $D_1, \dots, D_5$ clearly still form a simple trade loop, and hence F commutes. $\square$

8. NATURALITY OF REAL HEEGAARD FLOER HOMOLOGY

In this section, we establish naturality of real Heegaard Floer homology. First, we review the construction of [GM] and our proof of invariance up to isomorphism. There we proved real handleslide invariance by appealing to Perutz [Per08]. For ease of computation, we will reformulate this statement in terms of holomorphic polygon counting maps. Then we will verify the remaining axioms of a strong real Heegaard Floer theory: the most interesting verification is that of the simple trade loop. To verify the computation, we will briefly set up a cylindrical reformulation of real Heegaard Floer homology.

8.1. Real Lagrangian Floer homology. We begin with some background on our theory. Let (M, ω) be a symplectic manifold equipped with an anti-symplectic involution $R : M \rightarrow M$; i.e. a smooth involution satisfying $R^*(\omega) = -\omega$. Let L be a Lagrangian submanifold of M . Let $\mathcal{J}$ be the space of time-dependent almost complex structures on M and let $\mathcal{J}_R \subset \mathcal{J}$ be the subset satisfying the condition

$$J_t \circ R_* = -R_* \circ J_{1-t}.$$

Let us write M^R for the fixed point set of R ; M^R is a smooth submanifold of M and is, in fact, a Lagrangian submanifold. Therefore, for a path of real-invariant almost complex structures

$J_s \in \mathcal{J}_R$, we define

$$CFR_{J_s}^\circ(L) := CF_{J_s}^\circ(L, M^R).$$

Note that as usual, we can construct continuation maps

$$(6) \quad \Psi_{J_s \rightarrow J'_s} : CFR_{J_s}^\circ(L) \rightarrow CFR_{J'_s}^\circ(L)$$

which show these complexes are independent of our choice of almost complex structure. Furthermore, the notation is intended to indicate that the real Floer chain complex associated to the Lagrangian L and involution R is simply the usual Lagrangian Floer chain complex of L and M^R ; in particular, we impose no symmetry relation on the holomorphic strips which contribute to the differential. However, as J is assumed to be real-invariant, there is a correspondence between holomorphic strips between L and M^R and real invariant strips between L and $R(L)$, i.e. holomorphic maps satisfying:

$$u : [0, 1] \times \mathbb{R} \rightarrow M, \quad \begin{cases} u(\{1\} \times \mathbb{R}) \subset L \\ u(\{0\} \times \mathbb{R}) \subset R(L) \\ u(1 - s, t) = R \circ u(s, t). \end{cases}$$

Furthermore, the generators of $CF_J^\circ(L, M^R)$ (i.e. elements of $L \cap M^R$) are exactly the intersection points of $L \cap R(L)$ which are fixed by R . In short, we will frequently make use of the fact that we may either view $CF_J^\circ(H)$ as the usual Lagrangian Floer chain complex of L and M^R , or the complex generated by $(L \cap R(L))^R$ with differential which counts rigid real invariant holomorphic strips.

8.2. Real holomorphic polygons. As is typical in Floer theory, we obtain maps between our Floer complexes by counting holomorphic triangles. Suppose we have two Lagrangian submanifolds L_1 and L_2 of (M, ω) . Just as there is a correspondence between J -holomorphic strips between M^R and L_1 (as well as between M^R and L_2) with real invariant J -holomorphic strips between $R(L_1)$ and L_1 (as well as between $R(L_2)$ and L_2), there is also a correspondence between J -holomorphic triangles between M^R , L_1 , and L_2 and invariant J -holomorphic rectangles between L_1 , L_2 , $R(L_2)$, and $R(L_1)$.

More generally, let $\mathbb{P}_k$ be a k -gon in the plane; we do not fix a complex structure on $\mathbb{P}_k$. Label the edges of $\mathbb{P}_k$ by $e_1, \dots, e_k$. Let $L_1, \dots, L_k$ be a collection of Lagrangian subspaces of (M, ω) . A *Whitney n -gon* is a map $\phi : \mathbb{P}_n \rightarrow M$ such that $\phi(e_i) \subset L_i$. Let $\mathcal{C}(\mathbb{P}_k)$ be the space of conformal structures on $\mathbb{P}_k$. Given $\mathbf{x}_i \in L_i \cap L_{i+1}$ for all $i \in \mathbb{Z}/k$ and a class $\psi \in \pi_2(\mathbf{x}_1, \dots, \mathbf{x}_k)$, we define $\mathcal{M}(\psi)$ to be the space of holomorphic k -gons $\mathbb{P}_k \rightarrow M$ in the class ψ . The expected dimension of $\mathcal{M}(\psi)$ is given by

$$\mu(\psi) + \dim \mathcal{C}(\mathbb{P}_k) = \mu(\psi) + (k - 3).$$

In our setting, we consider Lagrangian subspaces $L_1, \dots, L_{k-1}, M^R \subset M$. Given a class $\psi \in \pi_2(\mathbf{x}_1, \dots, \mathbf{x}_k)$ and $J \in \mathcal{J}_R$, we define $\mathcal{M}_R^J(\psi)$ to be the space of J -holomorphic k -gons $\mathbb{P}_k \rightarrow M$ in the class ψ . The expected dimension of $\mathcal{M}_R(\psi)$ is given by

$$\mu_R(\psi) + \dim \mathcal{C}(\mathbb{P}_k) = \mu_R(\psi) + (k - 3).$$

The symmetry condition on J sets up our correspondence between Whitney k -gons between Lagrangians $L_1, \dots, L_{k-1}, M^R$ and real invariant $(2k - 2)$ -gons between

$$R(L_{k-1}), \dots, R(L_1), L_1, \dots, L_{k-1}.$$

We remark that the space of real conformal structures on $\mathbb{P}_{2k-2}$ is $\mathcal{C}^R(\mathbb{P}_{2k-2}) = \mathcal{C}(\mathbb{P}_k)$.

We now specialize to the case of triangles (that is, real-invariant rectangles). Let $\diamond$ be the unit disk in the complex plane with four marked points. The space of conformal structures of the rectangle $\mathcal{M}(\diamond)$ is one dimensional, parametrized by the cross ratio. In [OS04b], they

consider the moduli space $\mathcal{M}(\psi)$ of holomorphic maps of the rectangle in the homotopy class ψ into $\text{Sym}^m(\Sigma)$ without fixing the complex structure on $\diamond$. In our setting, we will fix the almost complex structure on $\diamond$ which is preserved by complex conjugation (which, of course, agrees with the fact that there is a unique conformal structure on the triangle). In the usual setting, holomorphic rectangle maps are not chain maps; when the cross-ratio goes to 0 or infinity, rectangles break into a pair of triangles meeting at a vertex. However, in our setting, this kind of degeneration cannot occur. Because of the symmetry, at the ends of 1-parameter families of real rectangles, only strips can break off. It follows that the real holomorphic rectangle counting maps

$$F_{L_1, L_2} : CF^\circ(L_2, L_1) \otimes CFR^\circ(L_1, R(L_1)) \rightarrow CFR^\circ(L_2, R(L_2)),$$

are actually chain maps in the real setting. We note here that while F_{L_1, L_2} does have four inputs, the input coming from $R(L_1) \cap R(L_2)$ is determined by the one coming from $L_1 \cap L_2$.

As in the case of bigons, there is a simple relationship between the real index of a rectangle and its ordinary index. The space of conformal structures of the rectangle is one dimensional. However, the space of conformal structures on the rectangle which respect the real structure is zero dimensional, as the symmetry forces the cross-ratio to be one. More generally, the space of conformal structures of a $(2k-2)$ -gon is $(2k-5)$ -dimensional, while the space of real conformal structures is $(k-3)$ -dimensional (since the quotient is a k -gon).

Lemma 8.1. *Let $\psi \in \pi_2^R(\theta, \mathbf{x}, R(\theta), \mathbf{y})$ be a homotopy class of invariant rectangles and let $\tilde{\psi}$ be its image in $\pi_2(\theta, \mathbf{x}, R(\theta), \mathbf{y})$. Then*

$$(7) \quad \text{ind}(\tilde{\psi}) = 2 \text{ind}_R(\psi) + \frac{\sigma(L_0, \mathbf{x}) - \sigma(L_1, \mathbf{y})}{2}.$$

Proof. This is similar to [GM, Proposition 2.3]. The index of a holomorphic rectangle u in the class ψ is given by trivializing u^*TM over the disk and considering the loop λ of Lagrangian subspaces in $\mathbb{C}^n$ determined by L_0 , L_1 , $R(L_0)$, and $R(L_1)$ which in turn determines an integer, as the fundamental group of the Lagrangian Grassmannian $\pi_1(U(n)/O(n)) \cong \mathbb{Z}$. We write $\mu(\lambda) = \text{ind}(\tilde{\psi})$ and $\mu_R(\lambda) = \text{ind}_R(\psi)$.

Consider the quantity

$$\delta(\lambda) = 2\mu_R(\lambda) - \mu(\lambda).$$

This quantity only depends on the Lagrangian subspaces at the vertices, as concatenating with the generator of $\pi_1(U(n)/O(n))$ increases $\mu(\lambda)$ by two and increases $\mu_R(\lambda)$ by one.

Both ind and ind_R are additive under splicing in bigons at $\mathbf{x}$ and $\mathbf{y}$ and pairs of bigons at θ and $R(\theta)$. Any configuration of Lagrangians at the vertices can be obtained by splicing in bigons, so it suffices to check the formula holds in a single case. Consider the rectangle $\psi : \mathbb{D} \rightarrow \text{Sym}^2(\Sigma)$ shown in Figure 8.1. This rectangle has $\text{ind}(\psi) = 0$ (and the corresponding moduli space has expected dimension 1). Indeed, for each component and for each slit length, there is a map from $\mathbb{P}_4$ (of varying of conformal class) to the domain. In the real case, the conformal structure on $\diamond$ is not permitted to vary, and the cuts must be made equivariantly. Therefore, there is a unique slit length for which there is a holomorphic representative for ψ . Hence, $\text{ind}_R(\psi) = 0$ (and so the moduli space of real rectangles has expected dimension 0). In this case, Equation (7) holds. $\square$

Remark 8.2. We note that a simple modification of this proof shows that for a $2n$ -gon

$$\psi \in \pi_2^R(\theta_{n-1}, \dots, \theta_1, \mathbf{x}, R(\theta_1), \dots, R(\theta_{n-1}), \mathbf{y}),$$

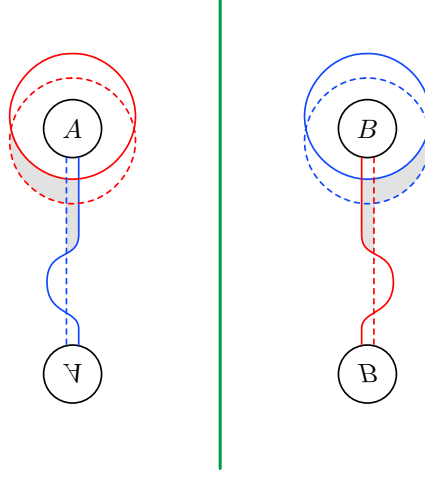


FIGURE 8.1. A class of rectangles admitting a unique (real) holomorphic representative.

we have

$$(8) \quad \text{ind}(\tilde{\psi}) = 2 \text{ind}_R(\psi) + \frac{\sigma(L_0, \mathbf{x}) - \sigma(L_1, \mathbf{y})}{2}$$

as well.

8.3. Real Heegaard Floer homology. Given a real sutured manifold (Y, γ, τ) , the associated real Heegaard Floer complex is defined as a special case of real Lagrangian Floer homology. We choose an embedded real sutured Heegaard diagram $\mathcal{H} = (\Sigma, \alpha, \beta, \tau)$ representing (Y, γ, τ) , and consider $M = \text{Sym}^m(\Sigma)$ equipped with the involution $R : M \rightarrow M$ induced by the restriction of τ to Σ ; since τ restricts to an orientation-reversing involution of Σ , the map R is anti-symplectic. Utilizing the work of Perutz, we may pick an anti-invariant symplectic form ω on M with respect to which $\mathbb{T}_\alpha = \prod_{i=1}^m \alpha_i$ is a Lagrangian submanifold of M . We then define

$$CFR^\circ(\mathcal{H}) := CFR^\circ(\mathbb{T}_\alpha, M^R).$$

In [GM], we proved that $CFR^\circ(H)$ is a weak real Heegaard Floer invariant; namely, that it is an invariant up to isomorphism of (Y, γ, τ) . This amounts to verifying that real isotopies, handleslides, and stabilizations induce chain homotopy equivalences. In [GM], we defined maps associated to real isotopies and handleslides in terms of continuation maps. However, for the sake of computations, it is more convenient to define these maps in terms of counts of holomorphic rectangles.

First, we recall a basic fact in the theory.

Lemma 8.3. *Let (Σ, β, β') be an admissible (unreal) Heegaard diagram such that $\beta \sim \beta'$. Then, there is a unique Spin^c -structure $\mathfrak{s}_0$ with $c_1(\mathfrak{s}_0) = 0$ as well as a highest-graded element $\Theta_{\beta, \beta'} \in CF^\circ(\Sigma, \beta, \beta', \mathfrak{s}_0)$, well defined up to boundary.*

Definition 8.4. Let $(\Sigma, \alpha', \alpha, \beta, \beta', \tau)$ be an admissible quadruple diagram with $\beta \sim \beta'$. We will write $\Psi_{\beta \rightarrow \beta'}^{\alpha \rightarrow \alpha'}$ for the map

$$F_{\alpha', \alpha, \beta, \beta'}(\Theta_{\alpha', \alpha} \otimes - \otimes \Theta_{\beta, \beta'}) : CFR^\circ(\Sigma, \alpha, \beta, \tau) \rightarrow CFR^\circ(\Sigma, \alpha', \beta', \tau).$$

Note that $HF^\circ(\Sigma, \alpha', \alpha)$ and $HF^\circ(\Sigma, \beta, \beta')$ are isomorphic via τ , and in particular, $\tau_*(\Theta_{\beta, \beta'}) = \Theta_{\alpha', \alpha}$.

Before proceeding, we state the following lemma regarding admissibility.

Lemma 8.5. *[JTZ21, Lemma 9.5] For every $i \in \{1, \dots, k\}$, let*

$$(\Sigma, \tau(\eta_n), \tau(\eta_{n-1}^i), \dots, \tau(\eta_0^i), \eta_0^i, \dots, \eta_{n-1}^i, \eta_n, \tau)$$

be a real sutured multi-diagram such that

$$(\Sigma, \tau(\eta_{n-1}^i), \dots, \tau(\eta_0^i), \eta_0^i, \dots, \eta_{n-1}^i, \tau)$$

is admissible. Then, there is a translate η'_n of η_n that is Hamiltonian isotopic to η_n such that

$$(\Sigma, \tau(\eta'_n), \tau(\eta_{n-1}^i), \dots, \tau(\eta_0^i), \eta_0^i, \dots, \eta_{n-1}^i, \eta'_n, \tau)$$

is admissible for every $i \in \{1, \dots, k\}$.

Proof. This follows by combining [GW10] and [GM, Lemma 3.34]. $\square$

Suppose that $\mathcal{H} = (\Sigma, \alpha, \beta, \tau)$ and $\mathcal{H}' = (\Sigma, \alpha', \beta', \tau)$ are two real sutured Heegaard diagrams for (Y, γ, τ) which differ by a real Hamiltonian isotopy. Then, there is a continuation map

$$\Gamma_{\beta \rightarrow \beta'}^{\alpha \rightarrow \alpha'} : CFR^\circ(\mathcal{H}) \rightarrow CFR^\circ(\mathcal{H}'),$$

which is a chain homotopy equivalence [GM, Proposition 5.3]. On the other hand, $CF^\circ(\mathbb{T}_\alpha, \mathbb{T}_{\alpha'})$ has a canonical top-graded class $\Theta_{\alpha', \alpha}$, so there is another map

$$\Psi_{\beta \rightarrow \beta'}^{\alpha \rightarrow \alpha'} : CFR^\circ(\mathcal{H}) \rightarrow CFR^\circ(\mathcal{H}'),$$

given by counting rigid real-invariant holomorphic rectangles with one vertex on $\Theta_{\alpha', \alpha}$ and another on $R(\Theta_{\alpha', \alpha})$. The two maps agree up to homotopy.

Lemma 8.6. *Let H and H' be admissible diagrams which differ by a real Hamiltonian isotopy. Then,*

$$\Gamma_{\beta \rightarrow \beta'}^{\alpha \rightarrow \alpha'} \sim \Psi_{\beta \rightarrow \beta'}^{\alpha \rightarrow \alpha'}.$$

In particular, $\Psi_{\beta \rightarrow \beta'}^{\alpha \rightarrow \alpha'}$ is a chain homotopy equivalence and these maps compose under concatenation of isotopies.

Proof. Since we may instead view $CFR^\circ(\mathcal{H})$ and $CFR^\circ(\mathcal{H}')$ as $CF^\circ(\mathbb{T}_\alpha, M^R)$ and $CF^\circ(\mathbb{T}_{\alpha'}, M^R)$ respectively, and the maps $\Gamma_{\beta \rightarrow \beta'}^{\alpha \rightarrow \alpha'}$ and $\Psi_{\beta \rightarrow \beta'}^{\alpha \rightarrow \alpha'}$ as maps between them, this follows from the proof in the usual case, which equates the continuation and triangle counting maps; see [OS06, Theorem 2.3] and [JTZ21, Lemma 9.7]. $\square$

Next, we consider the situation in which $\mathcal{H} = (\Sigma, \alpha, \beta, \tau)$ and $\mathcal{H}' = (\Sigma, \alpha', \beta', \tau)$ are two real sutured Heegaard diagrams for (Y, γ, τ) which differ by a real handleslide. Then, by appealing to [Per08], we once again have a continuation map

$$\Gamma_{\beta \rightarrow \beta'}^{\alpha \rightarrow \alpha'} : CFR^\circ(\mathcal{H}) \rightarrow CFR^\circ(\mathcal{H}'),$$

which is a chain homotopy equivalence [GM, Proposition 5.4]. Again, $CF^\circ(\mathbb{T}_\alpha, \mathbb{T}_{\alpha'})$ has a canonical top-graded class $\Theta_{\alpha', \alpha}$, so there is another rectangle counting map

$$\Psi_{\beta \rightarrow \beta'}^{\alpha \rightarrow \alpha'} : CFR^\circ(\mathcal{H}) \rightarrow CFR^\circ(\mathcal{H}').$$

These two maps are homotopic.

Lemma 8.7. *Let $\mathcal{H}$ and $\mathcal{H}'$ be admissible diagrams which differ by a real handleslide. Then,*

$$\Gamma_{\beta \rightarrow \beta'}^{\alpha \rightarrow \alpha'} \sim \Psi_{\beta \rightarrow \beta'}^{\alpha \rightarrow \alpha'}.$$

In particular, $\Psi_{\beta \rightarrow \beta'}^{\alpha \rightarrow \alpha'}$ is a chain homotopy equivalence.

Proof. This again follows from the unreal case, since the statement only involves $\mathbb{T}_\alpha$, not $\mathbb{T}_\beta$ nor M^R . Compare with [Per08, Remark after Proof of Corollary 1.3] or [Lip06, Proposition 11.4]. The continuation map counts strips with dynamic boundary conditions; by degenerating the source, one can see that the continuation map is homotopic to the composition of a monogon map and a triangle counting map. The monogon map picks out the top generator of $CF^\circ(\mathbb{T}_\alpha, \mathbb{T}_{\alpha'})$, and therefore composing with the triangle counting map is precisely the map associated to the handleslide. $\square$

Having related the continuation maps to holomorphic rectangle maps in the real setting, we show that if $\mathcal{H} = (\Sigma, \alpha, \beta, \tau)$ and $\mathcal{H}' = (\Sigma, \alpha', \beta, \tau)$ are real equivalent, then $\Psi_{\beta \rightarrow \beta'}^{\alpha \rightarrow \alpha'}$ is a chain homotopy equivalence.

Proposition 8.8.

- (1) *Suppose that $(\Sigma, \alpha', \alpha, \beta, \beta')$ is an admissible quadruple and we have $\beta \sim \beta'$. Then the map*

$$\Psi_{\beta \rightarrow \beta'}^{\alpha \rightarrow \alpha'} : CFR^\circ(\Sigma, \alpha, \beta, \tau) \rightarrow CFR^\circ(\Sigma, \alpha', \beta', \tau)$$

is a chain homotopy equivalence.

- (2) *These isomorphisms are compatible in the sense that if the quadruple diagrams $(\Sigma, \alpha', \alpha, \beta, \beta', \tau)$, $(\Sigma, \alpha'', \alpha', \beta', \beta'', \tau)$, and $(\Sigma, \alpha'', \alpha, \beta, \beta'', \tau)$ are admissible, then*

$$\Psi_{\beta' \rightarrow \beta''}^{\alpha' \rightarrow \alpha''} \circ \Psi_{\beta \rightarrow \beta'}^{\alpha \rightarrow \alpha'} \sim \Psi_{\beta \rightarrow \beta''}^{\alpha \rightarrow \alpha''}.$$

Proof. This is an analogue of [JTZ21, Proposition 9.10], and the proof is identical, since we may simply replace the role of $\mathbb{T}_\beta$ with M^R . $\square$

Hence, when $(\Sigma, \alpha', \alpha, \beta, \beta', \tau)$ is admissible, we have canonical identifications between $CFR^\circ(\Sigma, \alpha, \beta, \tau)$ and $CFR^\circ(\Sigma, \alpha', \beta', \tau)$. In the case that $(\Sigma, \alpha', \alpha, \beta, \beta', \tau)$ is not admissible, we make use of the following lemma.

Lemma 8.9. *Suppose $(\Sigma, \alpha, \beta, \tau)$ and $(\Sigma, \alpha', \beta', \tau)$ are admissible real diagrams and that $\alpha \sim \alpha'$ and therefore $\beta \sim \beta'$. Suppose that $\bar{\alpha}$ and $\bar{\beta}$ are attaching curves isotopic to α and β respectively so that the quadruples $(\Sigma, \bar{\alpha}, \alpha, \beta, \bar{\beta}, \tau)$ and $(\Sigma, \alpha', \bar{\alpha}, \bar{\beta}, \beta', \tau)$ are admissible. Then the map*

$$\Psi_{\bar{\beta} \rightarrow \beta'}^{\bar{\alpha} \rightarrow \alpha'} \circ \Psi_{\beta \rightarrow \bar{\beta}}^{\alpha \rightarrow \bar{\alpha}}$$

is a homotopy equivalence. Furthermore, the composite map $\Psi_{\bar{\beta} \rightarrow \beta'}^{\bar{\alpha} \rightarrow \alpha'} \circ \Psi_{\beta \rightarrow \bar{\beta}}^{\alpha \rightarrow \bar{\alpha}}$ is independent of $\bar{\alpha}$ and $\bar{\beta}$.

Proof. The proof of [JTZ21, Proposition 9.13] carries over to our context with no change. $\square$

Therefore, for any quadruple $(\Sigma, \alpha', \alpha, \beta, \beta')$ such that $\alpha \sim \alpha'$ and $\beta \sim \beta'$, there is a canonical homotopy equivalence

$$\Phi_{\beta \rightarrow \beta'}^{\alpha \rightarrow \alpha'} := \Psi_{\bar{\beta} \rightarrow \beta'}^{\bar{\alpha} \rightarrow \alpha'} \circ \Psi_{\beta \rightarrow \bar{\beta}}^{\alpha \rightarrow \bar{\alpha}},$$

which is well defined by Lemma 8.9. In particular, there is a well-defined map $\Psi_{\alpha \rightarrow \alpha'}^{\beta \rightarrow \beta'}$, which ought to be the identity map. This is the case.

Lemma 8.10. *Suppose that the diagram $(\Sigma, \alpha, \beta, \tau)$ is admissible and $\beta \sim \beta'$. Then*

$$(9) \quad \Phi_{\beta \rightarrow \beta}^{\alpha \rightarrow \alpha} \sim \text{Id}_{CFR^\circ(\Sigma, \alpha, \beta)}.$$

In particular, if $\beta \sim \beta'$, then $\Psi_{\beta' \rightarrow \beta}^{\alpha' \rightarrow \alpha} \sim (\Psi_{\beta \rightarrow \beta'}^{\alpha \rightarrow \alpha'})^{-1}$.

Proof. Let α' be a Hamiltonian translate of α so that $(\Sigma, \alpha', \alpha, \beta, \beta')$ is admissible. Then, by Proposition 8.8, we have

$$\Phi_{\beta \rightarrow \beta}^{\alpha \rightarrow \alpha} \sim \Psi_{\beta' \rightarrow \beta}^{\alpha' \rightarrow \alpha} \circ \Psi_{\beta \rightarrow \beta'}^{\alpha \rightarrow \alpha'}.$$

But by Lemma 8.6, we have that

$$\Psi_{\beta' \rightarrow \beta}^{\alpha' \rightarrow \alpha} \circ \Psi_{\beta \rightarrow \beta'}^{\alpha \rightarrow \alpha'} \sim \Gamma_{\beta' \rightarrow \beta}^{\alpha' \rightarrow \alpha} \circ \Gamma_{\beta \rightarrow \beta'}^{\alpha \rightarrow \alpha'} \sim \text{Id}_{CFR^\circ(\Sigma, \alpha, \beta)},$$

by the naturality of the continuation maps under juxtaposition. $\square$

Let $H = (\Sigma, A, B, \tau)$ be an isotopy diagram and let $Y_{(\Sigma, A, B, \tau)}$ be the set of admissible diagrams $(\Sigma, \alpha, \beta, \tau)$ such that $[\alpha] = A$ and $[\beta] = B$. This is non-empty by Lemma 8.5. It follows from Proposition 8.8 and Lemma 8.10 that the groups $CFR^\circ(\Sigma, \alpha, \beta, \tau)$ for $(\Sigma, \alpha, \beta, \tau) \in Y_{(\Sigma, A, B, \tau)}$, together with the isomorphisms $\Phi_{\beta \rightarrow \beta'}^{\alpha \rightarrow \alpha'}$ form a transitive system of (curved) chain complexes, which will be denoted $CFR^\circ(H)$.

8.4. Real Heegaard Floer homology as a weak real Heegaard invariant. We now show that the maps induced by real equivalences, $\mathcal{O}$ -stabilizations, and diffeomorphisms between isotopy diagrams commute with the maps appearing in this transitive system. We address real equivalences first, since they are the most straightforward.

Lemma 8.11. *Let $(\Sigma, \alpha_1, \beta_1, \tau)$, $(\Sigma, \alpha'_1, \beta'_1, \tau)$, $(\Sigma, \alpha_2, \beta_2, \tau)$, and $(\Sigma, \alpha'_2, \beta'_2, \tau)$ be admissible diagrams such that $\alpha_1 \sim \alpha_2$ and $\alpha'_1 \sim \alpha'_2$. Then, the diagram*

$$\begin{array}{ccc} CFR^\circ(\Sigma, \alpha_1, \beta_1, \tau) & \xrightarrow{\Psi_{\beta_1 \rightarrow \beta'_1}^{\alpha_1 \rightarrow \alpha'_1}} & CFR^\circ(\Sigma, \alpha'_1, \beta'_1, \tau) \\ \downarrow \Psi_{\beta_1 \rightarrow \beta_2}^{\alpha_1 \rightarrow \alpha_2} & & \downarrow \Psi_{\beta'_1 \rightarrow \beta'_2}^{\alpha'_1 \rightarrow \alpha'_2} \\ CFR^\circ(\Sigma, \alpha_2, \beta_2, \tau) & \xrightarrow{\Psi_{\beta_2 \rightarrow \beta'_2}^{\alpha_2 \rightarrow \alpha'_2}} & CFR^\circ(\Sigma, \alpha'_2, \beta'_2, \tau) \end{array}$$

commutes up to homotopy.

Proof. This follows immediately from Proposition 8.8. $\square$

Definition 8.12. Let $(\Sigma, \alpha, \beta, \tau)$ be an admissible diagram and $d: \Sigma \rightarrow \Sigma'$ a diffeomorphism. We write $\alpha' = d(\alpha)$, $\beta' = d(\beta)$, and $\tau' = d \circ \tau \circ d^{-1}$. Then d induces an isomorphism

$$d_*: CFR^\circ(\Sigma, \alpha, \beta, \tau) \rightarrow CFR^\circ(\Sigma', \alpha', \beta', \tau'),$$

as follows. We fix a generic path of symmetric almost complex structures J_s on $M = \text{Sym}^{|\alpha|}(\Sigma)$. Pushing J_s forward along d produces an almost complex structure J'_s on $M' = \text{Sym}^{|\alpha|}(\Sigma')$. Of course, d induces an isomorphism

$$d_{J_s, J'_s}: CFR_{J_s}^\circ(\Sigma, \alpha, \beta, \tau) \rightarrow CFR_{J'_s}^\circ(\Sigma', \alpha', \beta', \tau').$$

Furthermore, since the map d_{J_s, J'_s} commutes with the continuation maps $\Psi_{J_s \rightarrow J'_s}$, these diffeomorphism maps descend to a map d_* of transitive systems.

Lemma 8.13. *Suppose that $(\Sigma, \alpha', \alpha, \beta, \beta')$ is an admissible quadruple. Then, the diagram*

$$\begin{array}{ccc} CFR^\circ(\Sigma, \alpha, \beta, \tau) & \xrightarrow{\Psi_{\beta \rightarrow \beta'}^{\alpha \rightarrow \alpha'}} & CFR^\circ(\Sigma, \alpha', \beta', \tau) \\ \downarrow d_* & & \downarrow d_* \\ CFR^\circ(d(\Sigma), d(\alpha), d(\beta), d_*(\tau)) & \xrightarrow{\Psi_{d(\beta) \rightarrow d(\beta')}^{d(\alpha) \rightarrow d(\alpha')}} & CFR^\circ(d(\Sigma), d(\alpha'), d(\beta'), d_*(\tau)) \end{array}$$

is commutative up to homotopy. Here, $d_*(\tau) = d \circ \tau \circ d^{-1}$.

Proof. This follows exactly as in [JTZ21, Lemma 9.24]. $\square$

Importantly, for diffeomorphisms isotopic to the identity, the induced maps are homotopic to maps induced by real equivalences.

Proposition 8.14. *Let $(\Sigma, \alpha, \beta, \tau)$ be an admissible diagram. Suppose that $d: \Sigma \rightarrow \Sigma$ is an equivariant diffeomorphism equivariantly isotopic to Id_Σ , and let $\alpha' = d(\alpha)$, $\beta' = d(\beta)$, and $\tau' = d \circ \tau \circ d^{-1}$. Then*

$$(10) \quad d_* = \Phi_{\beta \rightarrow \beta'}^{\alpha \rightarrow \alpha'}: CFR^\circ(\Sigma, \alpha, \beta, \tau) \rightarrow CFR^\circ(\Sigma, \alpha', \beta', \tau').$$

Proof. This follows just as [JTZ21, Proposition 9.27]. The point is to decompose d into a composition of diffeomorphisms $H_0 \xrightarrow{d_1} \dots \xrightarrow{d_n} H_0$ where H_{i+1} differs from H_i by an equivariant Hamiltonian isotopy so that $|\alpha \cap d_i(\alpha)| = 2$ for each i . They argue that both d_* and $\Phi_{\beta \rightarrow \beta'}^{\alpha \rightarrow \alpha'}$ are homotopic to continuation maps. They then construct an explicit homotopy between the continuation maps by constructing a 1-dimensional moduli space whose ends give the desired relation. $\square$

Finally, we turn to stabilizations. Suppose that $\mathcal{H}' = (\Sigma', \alpha', \beta', \tau')$ is obtained from $\mathcal{H} = (\Sigma, \alpha, \beta, \tau)$ by a $\{1\}$ -stabilization. Then, for a suitable almost complex structure, the map

$$\sigma_{\mathcal{H} \rightarrow \mathcal{H}'}^{\{1\}}: CFR^\circ(\mathcal{H}) \rightarrow CFR^\circ(\mathcal{H}')$$

is given by $\mathbf{x} \mapsto \mathbf{x} \times \{c\}$, where c is the unique new intersection point introduced in the $\{1\}$ -stabilization. Similarly, if $\mathcal{H}'$ is obtained from $\mathcal{H}$ by a $\mathbb{Z}/2$ -stabilization, then, for a suitable almost complex structure, the map

$$\sigma_{\mathcal{H} \rightarrow \mathcal{H}'}^{\mathbb{Z}/2}: CFR^\circ(\mathcal{H}) \rightarrow CFR^\circ(\mathcal{H}')$$

is given by $\mathbf{x} \mapsto \{a\} \times \mathbf{x} \times \{\tau(a)\}$, where a and $\tau(a)$ are the two new intersection points introduced in the $\mathbb{Z}/2$ -stabilization. To prove that these induce maps of transitive systems, we appeal to the following result.

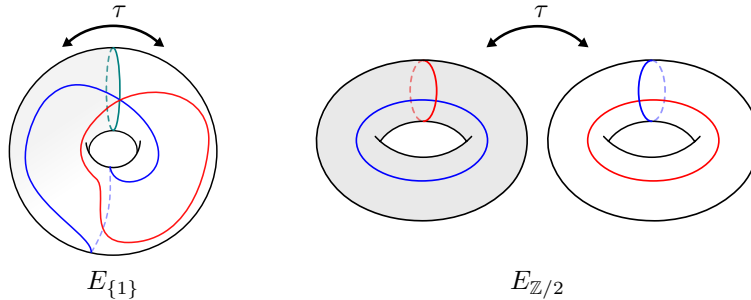


FIGURE 8.2. The diagrams $E_{\mathbb{Z}/2}$ and $E_{\{1\}}$ appearing in Proposition 8.15.

Proposition 8.15. *Let $\mathcal{H} = (\Sigma, \alpha, \beta, \tau, z)$ be a real Heegaard diagram and $E^{\{1\}}$ be a genus 1, real Heegaard diagram $(\Sigma_0, \alpha_0, \beta_0, \tau_0, z_0)$ for (S^3, rot) shown in the first frame of Figure 8.2. Let $\mathcal{H} \# E_{\{1\}}$ be the connected sum of $\mathcal{H}$ and $E_{\{1\}}$ at the points z and z_0 , extending the two involutions in the obvious way. Let $\psi \in \pi_2^R(\tau(\mathbf{w}), \mathbf{x}, \mathbf{y}, \mathbf{w})$ be a class of real-invariant rectangles in $\mathcal{H}$ and let $\psi_0 \in \pi_2^R(\tau(\mathbf{w}_0), \mathbf{x}_0, \mathbf{y}_0, \mathbf{w}_0)$ be a class of real invariant rectangles in $E_{\{1\}}$ with $n_{z_0}(\psi_0) = 0$. Then, for an appropriate almost complex structure, there is a diffeomorphism*

$$\mathcal{M}_R(\psi \# \psi_0) \cong \mathcal{M}_R(\psi) \times \mathcal{M}_R(\psi_0).$$

Further, define $E_{\mathbb{Z}/2} = (\Sigma_0 \sqcup \Sigma_1, \alpha_0 \cup \alpha_1, \beta_0 \cup \beta_1, z_0 \cup z_1, \tau_{\mathbb{Z}/2})$ where Σ_0 and Σ_1 are two copies of a genus 1 surface and $\tau_{\mathbb{Z}/2}$ is the involution exchanging them, swapping the alphas and betas. See the second frame of Figure 8.2. Also, let z_L and z_R be two distinct points of Σ near z that are exchanged by τ . Then, let $\mathcal{H} \#_{\mathbb{Z}/2} E_{\mathbb{Z}/2}$ be the diagram obtained by taking an equivariant connected sum, identifying z_L with z_0 and z_R with z_1 , extending the involution in the obvious way. Let $\psi \in \pi_2^R(\tau(\mathbf{w}), \mathbf{x}, \mathbf{y}, \mathbf{w})$ be a class of real-invariant rectangles in $\mathcal{H}$ and let $\psi_0 \in \pi_2^R(\tau(\mathbf{w}_0), \mathbf{x}_0, \mathbf{y}_0, \mathbf{w}_0)$ be a class of real invariant rectangles in $E_{\mathbb{Z}/2}$ with $n_{z_0}(\psi_0) = 0 = n_{z_1}(\psi_0)$. Then, for an appropriate almost complex structure, there is a diffeomorphism

$$\mathcal{M}_R(\psi \# \psi_0) \cong \mathcal{M}_R(\psi) \times \mathcal{M}_R(\psi_0).$$

Proof. This follows just as in [OS04a] and [OS06, Theorem 9.4]. In the first case, we choose holomorphic representatives u and u_0 , which produce a map to $\text{Sym}^{k+1}(\Sigma \vee \Sigma_0)$. We splice in spheres to construct a map $u \# u_0$ to $\text{Sym}^{k+1}(\Sigma \# \Sigma_0)$ which is approximately holomorphic. Then, we insert a long connected-sum tube; as the length goes to infinity, we can apply the inverse function theorem to find a nearby pseudoholomorphic rectangle. The second case is nearly identical, though we insert two long connected-sum tubes. $\square$

As a consequence, the maps for stabilizations descend to the transitive systems.

Lemma 8.16. *Let $\mathcal{H}_1 = (\Sigma, \alpha_1, \beta_1, \tau_1)$ and $\mathcal{H}_2 = (\Sigma, \alpha_2, \beta_2, \tau_2)$ be two admissible Heegaard diagrams such that $\alpha_1 \sim \alpha_2$. If $\mathcal{H}'_1 = (\Sigma', \alpha'_1, \beta'_1)$ and $\mathcal{H}'_2 = (\Sigma', \alpha'_2, \beta'_2)$ are $\mathcal{O}$ -stabilizations of $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively, then the diagram*

$$\begin{array}{ccc} \text{CFR}^\circ(\Sigma, \alpha_1, \beta_1, \tau) & \xrightarrow{\Phi_{\beta_1 \rightarrow \beta_2}^{\alpha_1 \rightarrow \alpha_2}} & \text{CFR}^\circ(\Sigma, \alpha_2, \beta_2, \tau) \\ \downarrow \sigma_{\mathcal{H}_1 \rightarrow \mathcal{H}'_1}^\circ & \Phi_{\beta'_1 \rightarrow \beta'_2}^{\alpha'_1 \rightarrow \alpha'_2} & \downarrow \sigma_{\mathcal{H}_2 \rightarrow \mathcal{H}'_2}^\circ \\ \text{CFR}^\circ(\Sigma, \alpha'_1, \beta'_1, \tau) & \xrightarrow{\quad} & \text{CFR}^\circ(\Sigma, \alpha'_2, \beta'_2, \tau) \end{array}$$

commutes up to homotopy. In particular, there are well-defined maps $\sigma_{H \rightarrow H'}^\circ : \text{CFR}^\circ(H) \rightarrow \text{CFR}^\circ(H')$ associated to stabilizations.

Proof. This follows from Proposition 8.15. $\square$

It follows from the previous lemmas that CFR° is a weak real Heegaard invariant. We now verify the remaining axioms to upgrade it to a strong invariant.

8.5. Real Heegaard Floer homology as a strong real Heegaard invariant. We now verify the axioms from Definition 2.26.

- (1) **Functoriality:** The real equivalence maps are functorial according to Proposition 8.8 and Lemma 8.10. Functoriality of diffeomorphisms is a consequence of the definition. Similarly, the destabilization map $\sigma_{H' \rightarrow H}^\circ$ is defined to be the homotopy inverse of $\sigma_{H \rightarrow H'}^\circ$.

- (2) **Commutativity:** For the four kinds of distinguished rectangles, we must have homotopy commutativity. Squares of type (1) commute according to Lemma 8.16. Squares of type (2) commute by Lemma 8.13. Squares of type (3) commute on the nose for the appropriate choice of almost complex structure. For instance, suppose that H_1 and H_2 (as well as H_3 and H_4) differ by a $\{1\}$ -stabilization and that H_1 and H_3 (as well as H_2 and H_4) differ by a $\mathbb{Z}/2$ -stabilization, then we have that

$$\begin{aligned} \sigma_{H_3 \rightarrow H_4}^{\{1\}} \circ \sigma_{H_1 \rightarrow H_3}^{\mathbb{Z}/2}(\mathbf{x}) &= \sigma_{H_3 \rightarrow H_4}^{\{1\}}(\{a\} \times \mathbf{x} \times \{\tau_4(a)\}) \\ &= \{a\} \times \mathbf{x} \times \{\tau_4(a)\} \times \{c\} \\ &= \{a\} \times \mathbf{x} \times \{c\} \times \{\tau_4(a)\} \\ &= \sigma_{H_2 \rightarrow H_4}^{\mathbb{Z}/2}(\mathbf{x} \times \{c\}) \\ &= \sigma_{H_2 \rightarrow H_4}^{\mathbb{Z}/2} \circ \sigma_{H_1 \rightarrow H_2}^{\{1\}}(\mathbf{x}). \end{aligned}$$

The other two cases are similar.

Finally, for squares of type (4), we have that H_2 is an $\mathcal{O}$ -stabilization of H_1 and H_4 is an $\mathcal{O}$ -stabilization of H_3 and $f(H_1) = H_3$ and $g(H_2) = H_4$. Further, we assume that f and g satisfy $f(\mathcal{O} \times D) = \mathcal{O} \times D'$, $g(\mathcal{O} \times T) = \mathcal{O} \times T'$, and $f|_{\Sigma_1 \setminus (\mathcal{O} \times D)} = g|_{\Sigma_2 \setminus (\mathcal{O} \times T)}$. In the case of $\{1\}$ -stabilizations, we have that $\alpha_2 \cap \beta_2 \cap T = \{c\}$ and $\alpha_4 \cap \beta_4 \cap T' = \{c'\}$; note also that $g(c) = c'$. Hence, for the appropriate choice of almost complex structure, we have

$$\begin{aligned} g_* \circ \sigma_{H_1 \rightarrow H_2}^{\{1\}}(\mathbf{x}) &= g(\mathbf{x} \times \{c\}) \\ &= g(\mathbf{x}) \times \{g(c)\} \\ &= f(\mathbf{x}) \times \{c'\} \\ &= \sigma_{H_3 \rightarrow H_4}^{\{1\}} \circ f_*(\mathbf{x}). \end{aligned}$$

The case of $\mathbb{Z}/2$ -stabilizations is identical.

- (3) **Continuity:** Continuity is precisely Proposition 8.14.

The remaining two axioms (real handleswap and trade invariance) are more subtle, and require explicit counts of holomorphic rectangles. In order to make these computations more tractable, one typically makes use of Lipshitz's cylindrical reformulation of Heegaard Floer homology [Lip06]. We will describe the analogue of his approach in the real setting.

8.6. Cylindrical real Floer homology. Our treatment here follows [LO], so we will be terse, only emphasizing the differences between the bordered setup and ours.

Let $W = \Sigma \times [0, 1] \times \mathbb{R}$ and define $\pi_\Sigma : W \rightarrow \Sigma$ and $\pi_{\mathbb{D}} : W \rightarrow [0, 1] \times \mathbb{R}$ to be the natural projections. An involution τ of Σ induces an involution τ of W defined by $\tau(x, s, t) = (\tau(x), 1 - s, t)$. This involution clearly exchanges the cylinders $\alpha \times \{1\} \times \mathbb{R}$ and $\beta \times \{0\} \times \mathbb{R}$. Fix a point z_i in each region R_i of $\Sigma \setminus (\alpha \cup \beta)$ which satisfies

$$\begin{cases} \tau(z_i) = z_i & \text{if } \tau(R_i) = R_i, \\ \tau(z_i) = z_j & \text{if } \tau(R_i) = R_j. \end{cases}$$

We will write P for this collection of points and D_i for a small disk in R_i centered at z_i .

Fix a τ -anti-invariant area form dA on Σ and a complex structure j_Σ which is tamed by dA and anti-holomorphic, i.e.

$$j_\Sigma \circ \tau_* = -\tau_* \circ j_\Sigma.$$

Let $\omega = ds \wedge dt + dA$ be a split symplectic form on W . Note that τ is an anti-symplectic involution: $\tau^*\omega = -\omega$. In the cylindrical setting, one considers the set $\mathcal{J}$ of almost complex structures on W which satisfy

- (J1) J is tamed by ω ;
- (J2) In a cylindrical neighborhood of $P \times [0, 1] \times \mathbb{R}$, $J = j_\Sigma \times j_\mathbb{D}$ is split.
- (J3) J is translation invariant in the $\mathbb{R}$ -factor.
- (J4) $J(\partial_t) = \partial_s$
- (J5) J preserves $T_p\Sigma \times (s, t)$ for all $(s, t) \in [0, 1] \times \mathbb{R}$.

As in [Lip06, HHSZ], we will sometimes work with a more general class of almost complex structures, which satisfy

- (J5') $T_p\Sigma \oplus \{0\}$ is a complex line near $(\alpha \cup \beta) \times [0, 1] \times \mathbb{R}$ and $\Sigma \times \{0, \frac{1}{2}, 1\} \times \mathbb{R}$.

Let $\mathcal{J}_R$ be the subset of $\mathcal{J}$ consisting of almost complex structures J satisfying

$$J \circ \tau_* = -\tau_* \circ J.$$

We refer to these as *symmetric almost complex structures*.

Let (S, j) be a Riemann surface with boundary which contains $m = |\alpha| = |\beta|$ negative punctures $\mathbf{p} = \{p_1, \dots, p_m\}$ and m positive punctures $\mathbf{q} = \{q_1, \dots, q_m\}$. For a symmetric almost complex structure J satisfying (J1)-(J5) (or (J5')), we are interested in J -holomorphic maps $u : S \rightarrow W$ which satisfy:

- (M0) The source S is smooth.
- (M1) $u(\partial(S)) \subset (\alpha \times \{1\} \times \mathbb{R}) \cup (\beta \times \{0\} \times \mathbb{R})$.
- (M2) There are no components of S on which $\pi_\mathbb{D} \circ u$ is constant.
- (M3) For each i , $u^{-1}(\alpha_i \times \{1\} \times \mathbb{R})$ and $u^{-1}(\beta_i \times \{0\} \times \mathbb{R})$ consist of exactly one component of $\partial S \setminus (\mathbf{p} \cup \mathbf{q})$.
- (M4) $\lim_{w \rightarrow p_i} \pi_\mathbb{R} \circ u(w) = -\infty$ and $\lim_{w \rightarrow q_i} \pi_\mathbb{R} \circ u(w) = +\infty$.
- (M5) The energy of u is finite.
- (M6) u is an embedding.

If u is in the class $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$, we will write $\mathcal{M}^{cyl}(\phi)$ for the moduli space of J -holomorphic curves which satisfy (M0)-(M6). Let $\mathcal{M}_R^{cyl}(\phi)$ be the subspace of *real holomorphic curves* $u : S \rightarrow W$ for which there is an anti-holomorphic involution σ of S which satisfies $u \circ \sigma = \tau \circ u$. We will also at times consider the moduli space $\mathcal{M}^{cyl}(\phi; S)$ of curves with a fixed source S and the associated subspace $\mathcal{M}_R^{cyl}(\phi; S)$.

We say that $J \in \mathcal{J}^R$ is *generic* if $\mathcal{M}_R^{cyl}(\phi; S)$ is transversely cut out. Let $\mathcal{B}^R$ be the space of triples (j, J, u) where j is a symmetric complex structure on S , and J is a symmetric almost complex structure on W and u is a real smooth map from S to W . Let $\mathcal{E}^R \rightarrow \mathcal{B}^R$ be the bundle of symmetric $(0, 1)$ -forms on S valued in $u^*(TW)$, i.e. those α satisfying

$$\alpha \circ d\sigma = d\tau \circ \alpha.$$

We write $\mathcal{B}_{j,J}^R$ for the subspace of $\mathcal{B}^R$ with fixed j and J and similarly for $\mathcal{E}_{j,J,u}^R$. The linearization of the $\bar{\partial}$ -operator gives a map $T\mathcal{B}^R \rightarrow \mathcal{E}^R$. It follows from [LO, Lemma 3.3] that for any real u and symmetric j and J , the map $D_u \bar{\partial} : T_u \mathcal{B}_{j,J}^R \rightarrow \mathcal{E}_{j,J,u}^R$ is well-defined and Fredholm. Let $\mathcal{M}^R$ be the subspace of $\mathcal{B}^R$ consisting of those triples (j, J, u) such that $\bar{\partial}_{j,J}(u) = 0$.

An *annoying curve* is a map $u : S \rightarrow W$ such that there is a nonempty open subset of S on which $\pi_\mathbb{D} \circ u$ is constant.

Proposition 8.17. *With respect to a symmetric almost complex structure satisfying (J1)-(J5), the space $\mathcal{M}^R$ is a smooth manifold away from annoying curves.*

Proof. This is standard; see [MS94, Proposition 3.4.1], [Lip06, Lemma 3.3], or [GM, Proposition 2.2]. The key point is that according to [Lip06, Proposition 3.3] u is somewhere injective, provided that u is not annoying. Then, we can run the usual transversality argument, obtaining symmetric objects by averaging when needed. See [GM, Proposition 2.2] or [LO, Proposition 3.4] for details. $\square$

In particular, the moduli spaces $\mathcal{M}_R^{cyl}(\phi) \subset \mathcal{M}^R$ are smooth manifolds of dimension $\text{ind}_R(\phi)$ for generic symmetric almost complex structures. In particular, we can define a cylindrical version of real Heegaard Floer homology.

Just as in the unreal setting, there is a tautological correspondence between $j \times j_{\mathbb{D}}$ -holomorphic curves in W and $\text{Sym}^m(j)$ -pseudo-holomorphic strips in $M = \text{Sym}^m(\Sigma)$.

Lemma 8.18. *Let τ be an involution on Σ , let j be a symmetric complex structure on Σ , and let $J := \text{Sym}(j)$ be the induced almost complex structure on $\text{Sym}^m(\Sigma)$. There is a one-to-one correspondence between J -holomorphic maps*

$$u : \mathbb{D} \rightarrow \text{Sym}^m(\Sigma)$$

satisfying $\tau \circ u = u \circ \rho$ and diagrams of the form

$$\begin{array}{ccccc} & & S & \xrightarrow{\hat{u}} & \Sigma \\ & \nearrow \sigma & \downarrow \hat{u} & & \nearrow \tau \\ S & \xrightarrow{\quad} & \Sigma & & \\ \downarrow & & \downarrow & & \\ \mathbb{D} & \nearrow \rho & \mathbb{D} & & \end{array}$$

where S is a Riemann surface with real structure, $\hat{u}$ is a j -holomorphic map, and $S \rightarrow \mathbb{D}$ is a holomorphic d -fold branched cover. The latter data is equivalent to a $j \times j_{\mathbb{D}}$ -holomorphic map $S \rightarrow \Sigma \times [0, 1] \times \mathbb{R}$.

Proof. This follows just as in [OS04b, Lemma 3.6] while keeping track of the real structures. $\square$

Following [Lip06], we will show that the moduli spaces in the cylindrical setting agree with those in the Lagrangian Floer setting. By Lemma 8.18, we can identify the two moduli spaces when $j \times j_{\mathbb{D}}$ achieves transversality, but this cannot always be achieved. To circumvent this issue, Lipshitz considers a broader class of almost complex structures than those considered by Ozsváth and Szabó. Fix an open neighborhood V_1 of $z_i \times \text{Sym}^{m-1}(\Sigma)$ and an open neighborhood V_2 of the diagonal, Δ . Let $\pi : \Sigma^m \rightarrow \text{Sym}^m(\Sigma)$ be the natural projection and let $\omega_0 = (dA)^m$. Fix a complex structure j_{Σ} on Σ .

Definition 8.19. [Lip06, Definition 13.1] We say that an almost complex structure $\tilde{J}$ on $\text{Sym}^m(\Sigma)$ is a *quasi-nearly-symmetric almost complex structure* if:

- (1) $\tilde{J}$ is tamed by $\pi_*(\omega_0)$ on $\text{Sym}^m(\Sigma) \setminus V_2$.
- (2) $\tilde{J}$ agrees with $\text{Sym}^m(j_{\Sigma})$ on V_1 .
- (3) There is some complex structure j on Σ such that $\text{Sym}^m(j)$ agrees with $\tilde{J}$ on V_2 .

We will refer to such almost complex structures as *QNS almost complex structures*, so that when we consider *symmetric QNS almost complex structures* we can avoid referring to

“symmetric quasi-nearly-symmetric almost complex structures”. If J is a symmetric almost complex structure on W satisfying (J1)-(J5), then J determines a path J_s of symmetric complex structures on Σ ; furthermore $\text{Sym}^m(J_s)$ is a path of symmetric QNS almost complex structures on $\text{Sym}^m(\Sigma)$. Conversely, if $\tilde{J}_t$ is a path of symmetric QNS almost complex structures on $\text{Sym}^m(\Sigma)$, we write j_t for the complex structure on Σ which agrees with $\tilde{J}_t$ near the diagonal.

Lemma 8.20. *The class of paths of symmetric almost complex structures of the form $\text{Sym}^m(J_s)$ is sufficient to achieve transversality for real-invariant strips $u : (\mathbb{D}, \partial\mathbb{D}) \rightarrow (\text{Sym}^m(\Sigma), \mathbb{T}_\alpha \cup M^R)$.*

Proof. This again follows along the same lines as [GM, Proposition 2.2] and [LO, Proposition 3.4]; we appeal to the fact that transversality can be achieved in the unreal case and then average to obtain symmetric objects when necessary. $\square$

Therefore, we have a tautological correspondence in the real setting.

Proposition 8.21. *Equipped with symmetric almost complex structures J_s and $\text{Sym}^m(J_s)$, the moduli spaces $\mathcal{M}_R^{cyl}(\phi)$ and $\mathcal{M}_R(\phi)$ agree.*

Proof. This follows just as in the unreal case, making use of [Lip06, Proposition 13.2]. $\square$

Remark 8.22. Having established the equivalence of the two theories, we will henceforth write $\mathcal{M}_R(\phi)$ rather than $\mathcal{M}_R^{cyl}(\phi)$.

8.7. Real cylindrical boundary degenerations. It will ultimately be more useful to work with the larger class of almost complex structures satisfying (J1)-(J4) and (J5'). This larger class has the advantage that the $\bar{\partial}$ -operator achieves transversality for all curves appearing in codimension-1 degenerations. For this reason it will be useful to have an explicit count of boundary degenerations in the real cylindrical setting.

Definition 8.23. (cf. [HHSZ, Definition 7.6]) A *cylindrical beta boundary degeneration* is a smooth map

$$u : (S, \partial S) \rightarrow (\Sigma \times [0, \infty) \times \mathbb{R}, \beta \times \{0\} \times \mathbb{R}),$$

satisfying the following:

- (N-1) u is (j, J) -holomorphic.
- (N-2) u is proper.
- (N-3) For each $t \in \mathbb{R}$ and $i \in \{1, \dots, n\}$, $u^{-1}(\beta_i \times \{t\})$ consists of a single point.
- (N-4) u has finite energy.
- (N-5) $\pi_{\mathbb{H}} \circ u$ is non-constant on each component of S , where $\pi_{\mathbb{H}}$ denotes the projection onto $[0, \infty) \times \mathbb{R}$.

Definition 8.24. A $\mathbb{Z}/2$ -cylindrical boundary degeneration is a smooth map

$$u \sqcup (\tau \circ u) : (S \sqcup -S, \partial(S \sqcup -S)) \rightarrow (\Sigma \times ([0, \infty) \sqcup (-\infty, 1]) \times \mathbb{R}, (\alpha \cup \beta) \times \{0, 1\} \times \mathbb{R}),$$

such that u is a cylindrical beta boundary degeneration.

A $\{1\}$ -cylindrical boundary degeneration is a smooth map

$$u : (S, \partial S) \rightarrow (\Sigma \times (\mathbb{R} \times \mathbb{R}, C \times \{1/2\} \times \mathbb{R}),$$

such that there exists an anti-holomorphic involution σ of S satisfying $u \circ \sigma = \tau \circ u$.

If B is a relative 2-cycle in (Σ, β) , we will write $\mathcal{N}_J^{\mathbb{Z}/2}(\tau(B) \sqcup B, \mathbf{x})$ for the moduli space of $\mathbb{Z}/2$ -cylindrical boundary degenerations, and $\tilde{\mathcal{N}}_J^{\mathbb{Z}/2}(\tau(B) \sqcup B, \mathbf{x})$ for the quotient $\mathcal{N}_J^{\mathbb{Z}/2}(\tau(B) \sqcup B, \mathbf{x}) / \text{Aut}(\mathbb{H})$, where $\text{Aut}(\mathbb{H})$ is the group of conformal automorphisms of the half-plane which acts pointwise on the $[0, \infty) \times \mathbb{R}$ factor. In the $\{1\}$ -orbit case, we write $\mathcal{N}_J^{\{1\}}(B, \mathbf{x})$ for the

moduli space of $\{1\}$ -cylindrical boundary degenerations, and $\tilde{\mathcal{N}}_J^{\{1\}}(B, \mathbf{x})$ for the quotient $\mathcal{N}_J^{\{1\}}(B, \mathbf{x}) / \text{Aut}(\mathbb{H})$.

Definition 8.25. We say an almost complex structure J on $\Sigma \times [0, \infty) \times \mathbb{R}$ is *admissible* if it satisfies the following:

- (J-1') J is tamed by ω .
- (J-2') J is split on $(D_1 \cup \dots \cup D_n) \times [0, \infty) \times \mathbb{R}$.
- (J-3') J is invariant under the action of $\text{Aut}(\mathbb{H})$.
- (J-4') $J\partial/\partial s = \partial/\partial t$.
- (J-5') $T_p\Sigma \times \{0\}$ is a complex line in a neighborhood of $\beta \times [0, \infty) \times \mathbb{R}$ and $C \times [0, \infty) \times \mathbb{R}$, and along $\Sigma \times \{0\} \times \mathbb{R}$.

We note that any admissible almost complex structure on $\Sigma \times [0, \infty) \times \mathbb{R}$ clearly induces one on $\Sigma \times ([0, \infty) \sqcup (-\infty, 0]) \times \mathbb{R}$. Additionally, Hendricks-Hom-Stoffregen-Zemke show that any almost complex structure on $\Sigma \times [0, 1] \times \mathbb{R}$ gives rise to one on $\Sigma \times [0, \infty) \times \mathbb{R}$.

Proposition 8.26. (cf. [HHSZ, Proposition 7.9]) *Let $(\Sigma, \alpha, \beta, \mathbf{w}, \tau)$ be a real Heegaard diagram, and let J be a generic symmetric split almost complex structure on $\Sigma \times ([0, \infty) \cup (-\infty, 1]) \times \mathbb{R}$. If $B \cup \tau(B)$ is a real Maslov-index-2 class of real boundary degenerations at $\mathbf{x}$, then the moduli space $\tilde{\mathcal{N}}_J^R(B \cup \tau(B), \mathbf{x})$ is transversely cut out and*

$$\#\tilde{\mathcal{N}}_J^{\mathbb{Z}/2}(\tau(B) \cup B, \mathbf{x}) \equiv 1 \pmod{2},$$

independent of $|\mathbf{w}|$. Similarly, if the class $[\Sigma]$ has real index 2, then

$$\#\tilde{\mathcal{N}}_J^{\{1\}}([\Sigma], \mathbf{x}) \equiv 1 \pmod{2}.$$

In particular, the real cylindrical differential satisfies

$$\partial^2 = \begin{cases} 0 & |\mathbf{w}| = 1 \\ \sum_{w_i \in \mathbf{w}} U_{w_i}^2 & |\mathbf{w}| > 1. \end{cases}$$

Proof. The transversality result and the curve count follow by the same proof as [HHSZ, Proposition 7.9]; indeed, our $\mathbb{Z}/2$ -cylindrical degenerations are simply symmetric pairs of the boundary degenerations appearing in [HHSZ, Proposition 7.9], limiting to curves in $\Sigma \times \{0\} \times \mathbb{R}$ and $\Sigma \times \{1\} \times \mathbb{R}$. Our $\{1\}$ -boundary degenerations are given by copies of Σ mapping to $\Sigma \times \{1/2\} \times \mathbb{R}$.

In the case $|\mathbf{w}| > 1$, there is a $\mathbb{Z}/2$ -boundary degeneration for each of the $|\mathbf{w}|$ basepoints, just as in [OS08, Theorem 5.5], represented by $B \cup \tau(B)$ for $B \in \Sigma \setminus \beta$. In this case, there are no analogous $\{1\}$ -boundary degenerations with which these contributions can cancel, hence they give rise to the curvature term appearing in [GM].

In the case that $|\mathbf{w}| = 1$, the class $[\Sigma]$ has real index 1, and so the class $[\Sigma \cup \tau(\Sigma)]$ has real index 2. Therefore the moduli spaces $\#\tilde{\mathcal{N}}_J^{\mathbb{Z}/2}(\Sigma \cup \tau(\Sigma), \mathbf{x})$ and $\#\tilde{\mathcal{N}}_J^{\{1\}}(\Sigma \cup \tau(\Sigma), \mathbf{x})$ always contribute to $\partial^2(\mathbf{x})$ and cancel according to [HHSZ, Proposition 7.9]. $\square$

8.8. Compactness, transversality, and gluing results. In the cylindrical setting, holomorphic rectangle maps are replaced by counts of holomorphic curves in $\Sigma \times \diamond$, where $\diamond$ is the subset of $\mathbb{C}$ shown in Figure 8.3, with four cylindrical ends modeled on $[0, 1] \times [0, \infty)$.

In this setting we consider almost complex structures on $\Sigma \times \diamond$ satisfying:

- (J'1') J is tamed by the split symplectic form on $\Sigma \times \diamond$.
- (J'2') There is a finite collection of points $P \subset \Sigma \setminus (\alpha' \cup \alpha \cup \beta \cup \beta')$, with at least one point in each component of $\Sigma \setminus (\alpha' \cup \alpha \cup \beta \cup \beta')$, such that the almost complex structure is split on a product neighborhood of $P \times \diamond$; i.e., $J = j_\Sigma \times j_\diamond$.

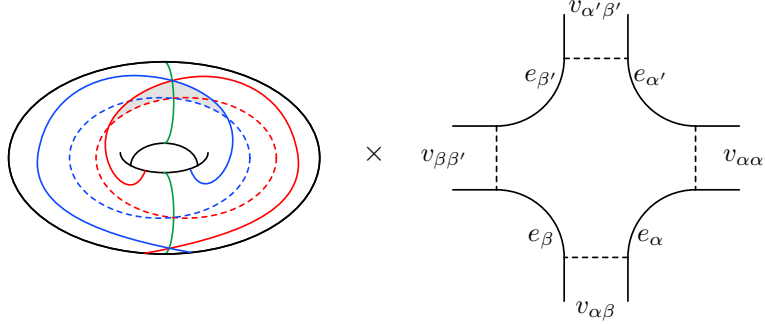


FIGURE 8.3. The four-manifold $\Sigma \times \diamond$ used to define the holomorphic quadrilateral maps.

- (J'3') Near the cylindrical ends of $\diamond$, the almost complex structure J agrees with cylindrical almost complex structures on $\Sigma \times [0, 1] \times \mathbb{R}$ satisfying condition (J5') above.
- (J'4') The 2-planes $T_d(\{p\} \times \diamond)$ are complex lines of J for all $(p, d) \in \Sigma \times \diamond$.
- (J'5') The 2-planes $T_p(\Sigma \times \{d\})$ are complex lines of J for (p, d) near $(\alpha' \cup \alpha \cup \beta \cup \beta') \times \diamond$ and for $(p, d) \in \Sigma \times U$, where $U \subset \diamond$ is an open subset containing $\partial \diamond \setminus \{v_{\alpha\alpha'}, v_{\alpha\beta}, v_{\beta\beta'}, v_{\alpha'\beta'}\}$.

We will write $\mathcal{J}_R(\diamond)$ for the set of symmetric almost complex structures on $\Sigma \times \diamond$ satisfying (J'1')-(J'5').

Lemma 8.27. *Let $J \in \mathcal{J}_R(\diamond)$. If $u : S \rightarrow \Sigma \times \diamond$ is a real J -holomorphic curve and $\pi_\Sigma \circ u$ is nonconstant on a component S_0 of S , then $\pi_\Sigma \circ u|_{S_0}$ is an open map. Moreover, there are coordinates near any critical point of $\pi_\Sigma \circ u|_{S_0}$ where $\pi_\Sigma \circ u$ has the form $z \mapsto z^k$ for some $k > 0$.*

Proof. This follows from [Lip06, Proposition 3.1]. $\square$

A (possibly nodal) holomorphic rectangle in a real quadruple diagram $\mathcal{Q} = (\Sigma, \alpha', \alpha, \beta, \beta')$ with $d = |\alpha| = |\alpha'| = |\beta| = |\beta'|$ is a map $u : S \rightarrow \Sigma \times \diamond$ that satisfies the following:

- (M1) (S, j) is a (possibly nodal) Riemann surface with boundary and $4d$ punctures on ∂S .
- (M2) u is locally nonconstant and (j, J) -holomorphic.
- (M3) $u(\partial S) \subset (\alpha' \times e_{\alpha'}) \cup (\alpha \times e_\alpha) \cup (\beta \times e_\beta) \cup (\beta' \times e_{\beta'})$.
- (M4) u has finite energy.
- (M5) For each $i \in \{1, \dots, d\}$ and $\eta \in \{\alpha', \alpha, \beta, \beta'\}$, the preimage $u^{-1}(\eta_i \times e_\eta)$ consists of exactly one component of the punctured boundary of S .
- (M6) As one approaches the punctures along ∂S , the map u converges to a collection of Reeb chords (i.e., intersection points on the Heegaard quadruple) in the cylindrical ends of $\Sigma \times \diamond$.

For the holomorphic rectangle maps, we also require:

- (M7) $\pi_\diamond \circ u$ is nonconstant on each component of S .
- (M8) S is smooth (i.e., not nodal), and the map $u : S \rightarrow \Sigma \times \diamond$ is an embedding.

If ψ is a homology class of rectangles on a real Heegaard quadruple $\mathcal{T}$, we write $\mathcal{M}_R(\psi)$ for the moduli space of real holomorphic maps $u : S \rightarrow \Sigma \times \diamond$ that satisfy (M1)–(M6). If S is a decorated surface (possibly with nodes), then we write $\mathcal{M}_R(\psi, S)$ for the subset of $\mathcal{M}_R(\psi)$ consisting of holomorphic curves with underlying source S . We will see that if ψ is a real Maslov-index-0 homology class of rectangles, then, for a generic choice of symmetric almost complex structure J in $\mathcal{J}_R(\diamond)$, any real J -holomorphic map $u : S \rightarrow \Sigma \times \diamond$ representing ψ , and satisfying (M1)–(M6) also satisfies (M7) and (M8).

Definition 8.28. Let $\mathcal{Q} = (\Sigma, \alpha', \alpha, \beta, \beta')$ be a quadruple diagram. A *broken holomorphic rectangle* on $\mathcal{Q}$ representing a homology class ψ of rectangles is a collection of (possibly nodal) (j, J) -holomorphic real surfaces $(u_1, v_1, \dots, v_n, w_1, \dots, w_m)$ such that

- (BR1) u_1 maps into $\Sigma \times \diamond$ and satisfies axioms (M1) and (M3)-(M6) for rectangles.
- (BR2) $v_1, \dots, v_n$ map into $\Sigma \times [0, 1] \times \mathbb{R}$ and satisfy axioms (M1) and (M3)-(M6) for bigons and represent homology classes of disks in the diagrams $(\Sigma, \alpha', \alpha)$, (Σ, α, β) , (Σ, β, β') , and $(\Sigma, \alpha', \beta')$.
- (BR3) $w_1, \dots, w_m$ map into $\Sigma \times \diamond$ and $\Sigma \times [0, 1] \times \mathbb{R}$. For every $i \in \{1, \dots, m\}$, the curve $w_i \sqcup R(w_i)$ has $2d$ boundary components, each with a single puncture, and maps onto one set of attaching curves (boundary degenerations).
- (BR4) The total homology class of the curves $u_1, v_1, \dots, v_n, w_1, \dots, w_m$ represents ψ .

Proposition 8.29. Let $J \in \mathcal{J}_R(\diamond)$. If u_i is a sequence of real J -holomorphic rectangles satisfying axioms (M1)-(M6) in $\mathcal{M}_R(\psi)$ for a homology class ψ , then there is a subsequence that converges to a broken holomorphic rectangle.

Proof. This follows from [Lip06, Section 7]. The real structure prevents rectangles from splitting into triangles. $\square$

Given symmetric almost complex structures J and J_0 on $\Sigma \times \diamond$ and $\Sigma_0 \times \diamond$, form their connected sum by removing small disks D and D_0 which are fixed setwise by the involution. Let $J(T)$ be the almost complex structure on $(\Sigma \# \Sigma_0) \times \diamond$ obtained by inserting a connected sum tube $S^1 \times [0, T]$ between Σ and Σ_0 equipped with the involution $(z, t) \rightarrow (\bar{z}, t)$. The following compactness result will be useful.

Proposition 8.30. If u_{T_i} is a sequence of holomorphic rectangles on $(\Sigma \# \Sigma_0) \times \diamond$ for a sequence of real-invariant almost complex structures $J(T_i)$ with $T_i \rightarrow \infty$, then a subsequence can be extracted that converges to a triple (U, V, U_0) , where U is a broken real holomorphic rectangle in $\Sigma \times \diamond$ and U_0 is a broken real holomorphic rectangle in $\Sigma_0 \times \diamond$. Moreover, V is a collection of real holomorphic curves mapping into the neck regions $S^1 \times \mathbb{R} \times \diamond$ or $S^1 \times \mathbb{R} \times [0, 1] \times \mathbb{R}$ which are asymptotic to possibly multiply covered Reeb chord orbits of the form $S^1 \times \{d\}$ in $S^1 \times \diamond$ or $S^1 \times \mathbb{R} \times [0, 1]$ for $d \in \diamond$ or $d \in [0, 1] \times \mathbb{R}$, respectively.

Since this statement does not depend on transversality and the real moduli spaces are subspaces of the unreal ones, we can deduce it from the corresponding result in the unreal theory (which is a consequence of [Lip06, Appendix A]). Compare with [JTZ21, Proposition 9.40].

Fix a point p in Σ in the complement of the attaching curves. Let $u : S \rightarrow \Sigma \times \diamond$ be a real holomorphic curve. By Lemma 8.27, u intersects $p \times \diamond$ in finitely many points, with positive intersections. Let $(x_1, \dots, x_{n_p(u)}) =: (\pi_\Sigma \circ u)^{-1}(p) \in \text{Sym}^{n_p(u)}(S)$ and define

$$\rho^p(u) = (\pi_\diamond \circ u(x_1), \dots, \pi_\diamond \circ u(x_{n_p(u)})) \in \text{Sym}^{n_p(u)}(\diamond).$$

Definition 8.31. Let ψ be a homology class of real rectangles with $n_p(\psi) = k$. Given $X \subset \text{Sym}^k(\diamond)$, define

$$\mathcal{M}_R(\psi, X) := \{u \in \mathcal{M}_R(\psi) : \rho^p(u) \in X\}.$$

As above, for a particular real source S , we write $\mathcal{M}_R(\psi, X, S)$ for the subspace of holomorphic curves in the homology class of ψ with source S that match X at p .

Lemma 8.32. Let $\mathcal{Q}$ be a quadruple diagram and fix a point p in the complement of the attaching curves. Say $\mathbf{d} \in (\text{Sym}^k(\diamond))^R$ is not in the diagonal. Let $u : S \rightarrow \Sigma \times \diamond$ be a real J -holomorphic curve in $\mathcal{M}_R$ for a symmetric almost complex structure $J \in \mathcal{J}_R$ so that $\rho^p(u) = \mathbf{d}$. Then each component of u is somewhere injective.

Proof. This follows from [JTZ21, Lemma 9.45] and [Lip06, Lemma 3.3]. $\square$

For a homology class of holomorphic rectangles ψ , the Maslov index $\mu(\psi)$ is the expected dimension of $\mathcal{M}(\psi)$. If we fix a holomorphic rectangle $u : S \rightarrow \Sigma \times \diamond$ in the class of ψ , the expected dimension of the moduli space $\mathcal{M}(\psi, S)$ is given by the Fredholm index of the linearization of the $\bar{\partial}$ -operator at u , which we write as $\text{ind}(\psi, S)$, and satisfies

$$(11) \quad \text{ind}(\psi, S) = \mu(\psi) - 2\text{sing}(u),$$

where $\text{sing}(u)$ is the order of singularity of u ; double points in the interior of S contribute $+1$ and double points along the boundary contribute $+1/2$.

If ψ represents a real holomorphic rectangle, then $\mu_R(\psi)$ is the expected dimension of $\mathcal{M}_R(\psi)$ and if $u : S \rightarrow \Sigma \times \diamond$ is a real holomorphic representative, then $\text{ind}_R(\psi, S)$ is the expected dimension of $\mathcal{M}_R(\psi, S)$. By [GM, Proposition 2.3], we have the relation:

$$\text{ind}(\psi) - 2\text{ind}_R(\psi) = \mu(\psi) - 2\mu_R(\psi).$$

Combining this observation with (11), it follows that

$$(12) \quad \text{ind}_R(\psi, S) = \mu_R(\psi) - \text{sing}(u).$$

We will also make use of the following transversality result.

Proposition 8.33. *Let $\mathcal{Q}$ be a quadruple diagram and p a point in the complement of the attaching curves. Suppose $X \subset (\text{Sym}^k(\diamond))^R$ for some k is a non-empty submanifold which does not intersect the fat diagonal. Suppose that for each $x \in X$, the tuple x has no coordinate in the open set $U \subset \diamond$ from (J'5') which contains $\partial\Diamond \setminus \{v_{\alpha\alpha'}, v_{\alpha\beta}, v_{\beta\beta'}, v_{\alpha'\beta'}\}$. For a generic symmetric almost complex structure J , the moduli space $\mathcal{M}_R(\psi, S, X)$ is a smooth manifold of dimension*

$$\text{ind}_R(\psi, S) - \text{codim}(X).$$

When $X = (\text{Sym}^k(\diamond))^R$, the same result holds near any curve u which has no component T on which $\pi_\diamond \circ u|_T$ is constant and has image in U and such that all components of u are somewhere injective.

Proof. This is a modification of [JTZ21, Proposition 9.47] and follows by their argument. $\square$

Finally, we state gluing results for real holomorphic rectangles. We will be interested in the setting in which ψ is a homology class of real rectangles on $\mathcal{Q}$ and ψ_0 is a class of real rectangles on $\mathcal{Q}_0$. We consider the class $\psi \# \psi_0$ in $\mathcal{Q} \# \mathcal{Q}_0$.

Proposition 8.34. *Let u and u_0 be real holomorphic rectangles in $\Sigma \times \diamond$ and $\Sigma_0 \times \diamond$ of real index 0 and k , respectively, representing homology classes ψ and ψ_0 . Suppose*

$$\rho^p(u) = \rho^p(u_0) \in (\text{Sym}^k(\diamond))^R \setminus \text{Diag}^k(\diamond)$$

and that $\mathcal{M}_R(\psi)$ and $\mathcal{M}_R(\psi_0)$ are transversely cut out near u and u_0 . Then, there is a homeomorphism h between a neighborhood of (u, u_0) in the compactified moduli space

$$\overline{\bigcup_T \mathcal{M}_{R, J(T)}(\psi \# \psi_0)}$$

and $[0, 1)$ so that $h(u, u_0) = \{0\}$.

Proof. This follows immediately from [Lip06, Appendix] together with the dimension counts from Proposition 8.33. $\square$

8.9. Simple trade loops. In this section, we prove that real Heegaard Floer homology satisfies the trade axiom from Definition 2.26 Item 5. Most of the work will consist in describing the equivalence g that appears in the trade pentagon. This equivalence splits into three handleslides, shown in Figures 8.4 and 8.6.

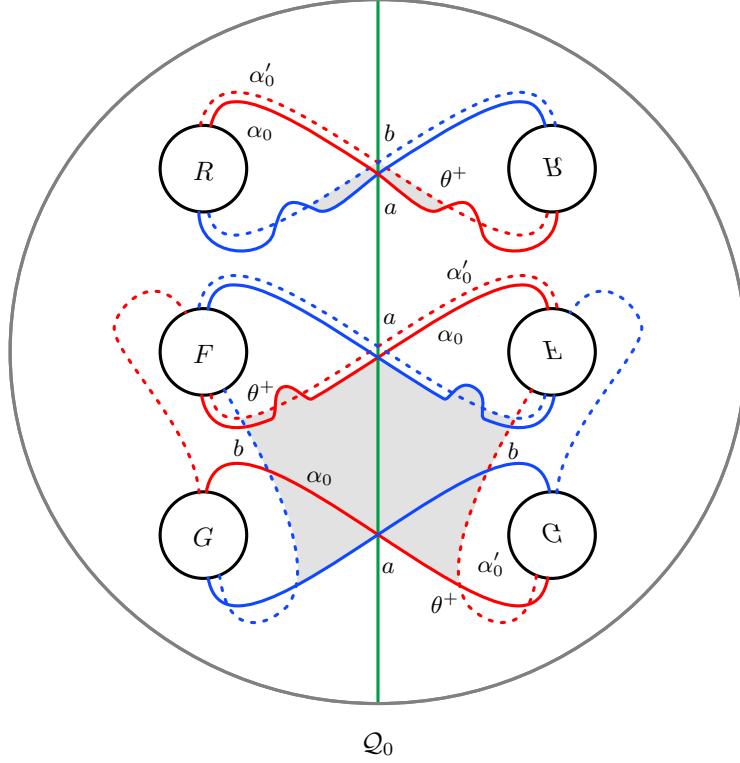


FIGURE 8.4. A quadruple diagram used for computing the real handleslides in a trade loop.

Consider the real quadruple diagram $\mathcal{Q}_{a,b} = (\Sigma_0, \alpha_b, \alpha_a, \beta_a, \beta_b, \tau_0)$ shown in the first frame of Figure 8.4. If $\mathcal{Q} = (\Sigma, \alpha', \alpha, \beta, \beta', \tau)$ is any other real quadruple diagram, we consider the quadruple

$$\mathcal{Q} \# \mathcal{Q}_{a,b} = (\Sigma \# \Sigma_0, \alpha' \cup \alpha_b, \alpha \cup \alpha_a, \beta \cup \beta_a, \beta' \cup \beta_b, \tau \# \tau_0),$$

where the connected sum is taken at points $p \in \Sigma$ and $p_0 \in \Sigma_0$ fixed by τ and τ_0 , respectively. Let $\mathbb{T}_{\alpha_a} \cap \mathbb{T}_{\beta_a} = \mathbf{a}$ and $\mathbb{T}_{\alpha_b} \cap \mathbb{T}_{\beta_b} = \mathbf{b}$ and

$$\mathbb{T}_{\alpha_b} \cap \mathbb{T}_{\alpha_a} = \{\theta_1^{\epsilon_1} \theta_2^{\epsilon_2} \theta_3^{\epsilon_3}\},$$

where $\epsilon_i \in \{1, -1\}$; let $\Theta = \{\theta_1^+ \theta_2^+ \theta_3^+\}$ be the top generator. In Figure 8.4, we have labeled all θ_i^+ by θ^+ and all elements of $\mathbf{a}$ as a and $\mathbf{b}$ as b for simplicity. Since $(\Sigma_0, \alpha_b, \alpha_a)$ represents $\#^3(S^1 \times S^2)$, we have that $\dim \widehat{CF}(\Sigma_0, \alpha_b, \alpha_a) = \dim \widehat{HF}(\#^3(S^1 \times S^2))$, and hence the differential on $\dim \widehat{CF}(\Sigma_0, \alpha_b, \alpha_a)$ must vanish.

Lemma 8.35. *Consider the quadruple diagram $\mathcal{Q}_{a,b} = (\Sigma_0, \alpha_b, \alpha_a, \beta_a, \beta_b, \tau_0)$ shown in Figure 8.4. For $\mathbf{x} \in \mathbb{T}_{\alpha_b} \cap \mathbb{T}_{\alpha_a}$ and $\psi_0 \in \pi_2^R(\mathbf{x}, \mathbf{a}, \tau_0(\mathbf{x}), \mathbf{b})$,*

$$(13) \quad \mu_R(\psi_0) = n_{p_0}(\psi_0) + \mu(\mathbf{x}, \Theta)$$

where $\mu(\mathbf{z}, \Theta)$ is the (ordinary) relative Maslov grading.

Proof. This is the analogue of [JTZ21, Lemma 9.50]. First, assume $\mathbf{x} = \Theta$ and that ψ_0 is equal to the union of the small rectangle and octagon shown in Figure 8.4. The real index of ψ_0 is 0 and $n_{p_0}(\psi_0) = 0$, so the claim holds in this case. The equation respects adding multiples of $[\Sigma_0]$. Every real doubly periodic domain has real index 0 and every real quadruply periodic domain is a sum of these. Hence, the equation is preserved under addition of real quadruply periodic domains.

Finally, if $\mathbf{x}$ is any other point of $\mathbb{T}_{\alpha_b} \cap \mathbb{T}_{\alpha_a}$, then we can choose a (non-invariant) domain $\phi \in \pi_2(\mathbf{x}, \Theta)$. Then $\psi_0 - (\phi - \tau(\phi))$ is an element of $\pi_2^R(\Theta, \mathbf{a}, \tau(\Theta), \mathbf{b})$, and so satisfies (13) by the argument above. However, $\mu(\phi)$ is exactly $\mu(\mathbf{x}, \Theta)$; this does not depend on ϕ since every periodic domain between α_b and α_a has unreal Maslov index 0. $\square$

Proposition 8.36. *Suppose that $\mathcal{Q} = (\Sigma, \alpha', \alpha, \beta, \beta', \tau)$ is an admissible real Heegaard quadruple diagram and let $\mathcal{Q} \# \mathcal{Q}_{a,b}$ be the connected sum where $\mathcal{Q}_{a,b}$ is the quadruple shown in the first frame of Figure 8.4. For a sufficiently stretched real-invariant almost complex structure, the holomorphic rectangle maps satisfy*

$$F_{\mathcal{Q} \# \mathcal{Q}_{a,b}}((\mathbf{z} \times \Theta) \otimes (\mathbf{y} \times \mathbf{a}) \otimes (\tau(\mathbf{z}) \times \tau_0(\Theta))) = F_{\mathcal{Q}}(\mathbf{z} \otimes \mathbf{y} \otimes \tau(\mathbf{z})) \times \mathbf{b}$$

for every $\mathbf{z} \in \mathbb{T}_{\alpha'} \cap \mathbb{T}_{\alpha}$ and $\mathbf{y} \in (\mathbb{T}_{\alpha} \cap \mathbb{T}_{\beta})^R$.

Proof. We follow the proof strategy for handleswaps [JTZ21, Proposition 9.31], pointing out where changes must be made in the real setting. Fix a class $\psi \in \pi_2^R(\mathbf{z}, \mathbf{y}, R(\mathbf{z}), \mathbf{u})$ of rectangles in $\mathcal{Q}$ as well as a class $\psi_0 \in \pi_2^R(\mathbf{x}, \mathbf{a}, \tau_0(\mathbf{x}), \mathbf{b})$ of rectangles in $\mathcal{Q}_{a,b}$ with $n_p(\psi) = n_{p_0}(\psi_0)$ and form $\psi \# \psi_0$. The index formula combined with the previous lemma implies that

$$(14) \quad \mu_R(\psi \# \psi_0) = \mu_R(\psi) + \mu_R(\psi_0) - n_{p_0}(\psi_0) = \mu_R(\psi) + \mu(\mathbf{x}, \Theta).$$

Suppose the class $\psi \# \psi_0$ has holomorphic representatives u_{T_i} for a sequence of neck lengths $T_i \rightarrow \infty$. There is a subsequence converging to a broken rectangle U on $\mathcal{Q}$ representing ψ , a broken rectangle U_0 on $\mathcal{Q}_0$ representing ψ_0 and a collection of curves V in the connected sum regions $S^1 \times \mathbb{R} \times \diamond$ or $S^1 \times \mathbb{R} \times [0, 1] \times \mathbb{R}$. As we are interested in real index 0 rectangles, Equation (14) implies that

$$0 = \mu_R(\psi \# \psi_0) = \mu_R(\psi) + \mu(\Theta, \Theta) = \mu_R(\psi) + \mu(\Theta, \Theta),$$

and therefore that $\mu_R([U]) = 0$.

Following [JTZ21, Proof of Proposition 9.31], we break the remainder of the proof into four claims. The analogous statements are listed in [JTZ21, Equation 9.55].

Claim 1: If U has no constant components, then it consists of a single holomorphic rectangle u of real index 0 satisfying (M1)-(M8).

Proof. The broken curve U has no components mapping into $\Sigma \times [0, 1] \times \mathbb{R}$; by transversality, curves into $\Sigma \times [0, 1] \times \mathbb{R}$ have index at least 1, which would force components mapping into $\Sigma \times \diamond$ to have negative index. Hence, U consists of a single real index 0 rectangle.

We now claim that U satisfies (M1)-(M6). (M1)-(M4) and (M6) are immediate. The fact that property (M5) is preserved follows just as in [JTZ21, Proposition 9.31, Proof of Claim (1)]. Roughly, (M5) holds when the source curve is degenerated along a simple closed curve (although this is impossible by index considerations) and strip breaking is ruled out by the previous paragraph. Hence, it suffices to consider the case that the source curve is pinched along (symmetric pairs) of arcs connecting boundary components of the same type, resulting in boundary degenerations, which increase the real index by at least 2 (compare to [GM, Lemma 4.9]). Hence, by transversality,

boundary degenerations are ruled out. Properties (M7) and (M8) also hold; nodal source curves and boundary degenerations have already been ruled out, and we have assumed U contains no constant curves. $\square$

Claim 2: If U_0 has no constant components, it consists of a single rectangle u_0 of real Maslov index $k := n_{p_0}$ satisfying (M1)-(M8).

Proof. As in [JTZ21, Proposition 9.31, Proof of Claim (2)], we may assume that p is not a branch point of $\pi_\diamond \circ v$ for any index 0 rectangle v in $(\Sigma, \alpha', \alpha, \beta, \beta', \tau)$ and that $\rho^p(v)$ is not an element of $\text{Diag}^k(\diamond)$. This forces curves in V to map to $S^1 \times \mathbb{R} \times \diamond$.

By Equation (14), we have that $\mu_R(\psi_0) = k$. A priori, the curve U_0 might consist of broken rectangles with nodal curves, boundary degenerations, and holomorphic strips. The asymptotics of u and V agree, so there must be a curve u_0 in U_0 which is matched with u , i.e.

$$\rho^p(u) = \rho^{p_0}(u_0).$$

Since $\rho^p(u)$ is in $\text{Sym}^k(\diamond)^R \setminus \text{Diag}^k(\diamond)$, Lemma 8.32 can be applied to guarantee that curve u_0 satisfying the matching condition is somewhere injective on each component, and hence Proposition 8.33 can be applied to show that the moduli space

$$\mathcal{M}_R(\psi_0, \mathbf{d}) = \{u_0 \in \mathcal{M}_R(\psi_0) : \rho^{p_0}(u_0) = \mathbf{d}\}$$

is a smooth manifold of dimension

$$\mu_R(\psi_0) - \text{codim}(\{\mathbf{d}\})$$

for $\mathbf{d} \in \{\rho^{p_0}(u) : u \text{ holomorphic on } \mathcal{Q}, \mu_R(u) = 0\}$. The collection $\{\mathbf{d}\}$ is finite, and therefore, we have that

$$\dim \mathcal{M}_R(\psi_0, \mathbf{d}) = n_{p_0}(\psi_0) - n_p(u) = 0.$$

Repeating the arguments from Claim 1, we see that U_0 must consist of a single rectangle in $\mathcal{M}_R(\psi_0, \mathbf{d})$ mapping into $\Sigma_0 \times \diamond$ satisfying (M1)-(M8). $\square$

Claim 3: V consists entirely of trivial cylinders and there are no constant components of U or U_0 .

Proof. We follow the arguments in [JTZ21, Proposition 9.31, Proof of Claim (3)]. By passing to a subsequence, we can assume each u_{T_i} has the same source curve, $\hat{S}$. This surface can be reconstructed from the curves in U , V , and U_0 and a simple Euler characteristic argument shows that if V did not consist entirely of trivial cylinders, then one would have that

$$\chi(\hat{S}) < \chi(S) + \chi(S_0) - 2k.$$

According to [Lip06, Section 10.2], we have

$$\text{ind}(\psi \# \psi_0, \hat{S}) = \frac{1}{2}(d+2) - \chi(\hat{S}) + 2e(\psi \# \psi_0).$$

This in turn implies

$$\text{ind}(\psi, S) + \text{ind}(\psi_0, S_0) < 2k - 2 + \text{ind}(\psi \# \psi_0, \hat{S}) = 2k - 2.$$

The relation between the real and unreal indices then implies that

$$\text{ind}_R(\psi, S) + \text{ind}_R(\psi_0, S_0) < \frac{1}{2}(2k - 2 + \text{ind}_R(\psi \# \psi_0, \hat{S})) = k - 1.$$

However, since we assumed that $\mathcal{M}_R(\psi)$ and $\mathcal{M}_R(\psi_0, \mathbf{d})$ are smoothly cut out and are of the expected dimension, this is impossible, as one of $\mathcal{M}_R(\psi)$ or $\mathcal{M}_R(\psi_0, \mathbf{d})$ would have

to have negative dimension. The same argument also rules out constant components of U and U_0 . $\square$

From these claims, it follows that the sequence u_{t_i} limits to a pair (u, u_0) with $\mu_R(u) = 0$, $\mu_R(\psi_0) = k$, and $\rho^p(u) = \rho^{p_0}(u_0)$. As Proposition 8.34 describes a neighborhood of (u, u_0) in the compactified moduli space

$$\overline{\bigcup_T \mathcal{M}_{R, J(T)}(\psi \# \psi_0)},$$

it follows that

$$\# \mathcal{M}_{R, J_T}(\psi \# \psi_0) = \#\{(u, u_0) \in \mathcal{M}_R(\psi) \times \mathcal{M}_R(\psi_0) : \rho^p(u) = \rho^{p_0}(u_0)\}.$$

For $\mathbf{x} \in \mathbb{T}_{\alpha_b} \cap \mathbb{T}_{\alpha_a}$ and $\mathbf{d} \in (\text{Sym}^k(\diamond))^R$ a generic divisor, define

$$\mathcal{M}_{R, \mathbf{x}}(\mathbf{d}) = \coprod_{\substack{\psi_0 \in \pi_2^R(\mathbf{x}, \mathbf{a}, \tau_0(\mathbf{x}), \mathbf{b}) \\ n_{p_0}(\psi_0) = k}} \mathcal{M}_R(\psi_0, \mathbf{d}).$$

It therefore suffices to prove that

$$\mathcal{M}_{R, \Theta}(\rho^p(u)) \equiv 1 \pmod{2}.$$

In fact, we have the following.

Claim 4: Let $\mathbf{d} \in \text{Sym}^k(\diamond)^R$ be in the complement of the diagonal. Then, $\mathcal{M}_{R, \Theta}(\mathbf{d})$ is a smoothly cut out zero-dimensional manifold for a generic real almost complex structure. For such a J , we have

$$\mathcal{M}_{R, \Theta}(\mathbf{d}) \equiv 1 \pmod{2}.$$

Proof. For a fixed $\mathbf{d} \in \text{Sym}^k(\diamond)^R$ avoiding the diagonal and a neighborhood of $\partial \diamond \setminus \{v_{\alpha', \alpha}, v_{\alpha, \beta}, v_{\beta, \beta'}\}$, transversality for $\mathcal{M}_{R, \mathbf{x}}(\mathbf{d})$ can be achieved for a generic real almost complex structure satisfying (J1)-(J5), by arguments similar to those above.

Further, the count of points in the space $\mathcal{M}_{R, \mathbf{x}}(\mathbf{d})$ is independent of $\mathbf{d}$ for a generic $\mathbf{d}$. This follows as in [JTZ21, Lemma 9.58]. In the standard case, one chooses a path of divisors avoiding the diagonal, and shows that the counts of points in the moduli spaces at the two ends agree. In the unreal case, such paths generically miss the diagonal, which is codimension 2. In the present situation, however, $\text{Diag}^k(\diamond) \cap \text{Sym}^k(\diamond)^R$ is codimension-1: even worse, $\text{Sym}^k(\diamond)^R \setminus \text{Diag}^k(\diamond)$ is disconnected. $\text{Sym}^k(\diamond)^R$ decomposes into connected strata as

$$\text{Sym}^k(\diamond)^R = \bigcup_{\ell=1}^k \text{Sym}^{k-2\ell}(\diamond^R) \times \text{Sym}^\ell(\diamond/R).$$

Each $(\text{Sym}^{k-2\ell}(\diamond^R) \times \text{Sym}^\ell(\diamond/R)) \setminus \text{Diag}^k(\diamond)$ stratum is connected. Indeed,

$$\text{Sym}^q(\diamond^R) \setminus \text{Diag}^q(\diamond^R) = \text{Sym}^q(\{1/2\} \times \mathbb{R}) \setminus \text{Diag}^q(\{1/2\} \times \mathbb{R})$$

is connected as any *unordered* tuple of points can be connected by a path to any other avoiding the diagonal. Similarly for

$$\text{Sym}^q([0, 1/2) \times \mathbb{R}) \setminus \text{Diag}^q([0, 1/2) \times \mathbb{R}).$$

However, the strata $\text{Sym}^{k-2\ell}(\diamond^R) \times \text{Sym}^\ell(\diamond/R)$ are glued along facets contained in $\text{Diag}^k(\diamond)$.

Therefore, we begin by showing that within a given connected component $(\text{Sym}^{k-2\ell}(\diamond^R) \times \text{Sym}^\ell(\diamond/R)) \setminus \text{Diag}^k(\diamond)$, the space $\mathcal{M}_{R,\mathbf{x}}(\mathbf{d})$ is independent of the choice of divisor. To do so, we count the ends of the 1-dimensional moduli space

$$\mathcal{M}_{R,\mathbf{x}}(\mathbf{D}) = \bigcup_{t \in [0,1]} \mathcal{M}_{R,\mathbf{x}}(\mathbf{D}(t)).$$

The space $\mathcal{M}_{R,\mathbf{x}}(\mathbf{D})$ has ends corresponding to $\mathcal{M}_{R,\mathbf{x}}(\mathbf{D}(0))$ and $\mathcal{M}_{R,\mathbf{x}}(\mathbf{D}(1))$ as well as ends corresponding to broken holomorphic rectangles. We claim that ends of the second kind necessarily correspond to index k rectangles breaking into index $k-1$ rectangles matching $\mathbf{D}(t)$ and an index 1 strip with multiplicity zero at p_0 or into an index $k-1$ rectangle and a constant copy of Σ_0 along $\diamond^R$.

Let $u_i : S_0 \rightarrow \Sigma \times \diamond$ be a sequence in $\mathcal{M}_{R,\mathbf{x}}(\mathbf{D})$ representing ψ_0 . Suppose that u_i degenerates along a $\mathbb{Z}/2$ -orbit of disjoint, closed curves in S_0 . Then, the limiting source has at least two interior nodal singularities which map into $\Sigma_0 \times \diamond$. Let S'_0 be the collection of components of the broken curve and let ψ'_0 be its homology class. Since

$$\text{ind}_R(\psi'_0, S'_0) = \mu_R(\psi'_0) - \text{sing}(u),$$

we have that $\text{ind}_R(\psi'_0, S'_0) \leq \mu_R(\psi'_0) - 2 = k - 2$, as S'_0 has at least two nodal singularities. We can compute:

$$\dim \mathcal{M}_R(\psi'_0, S'_0, \mathbf{D}) = \text{ind}_R(\psi'_0, S'_0) - \text{codim}(\mathbf{D}) \leq (k - 2) - (k - 1) = -1.$$

Hence, by transversality, no such degenerations can occur. In the case that the degeneration occurs along an invariant closed curve (appearing in a trivial orbit), the limiting curve contains an interior nodal singularity which is fixed by the involution. In this case, the argument above fails, and we cannot rule out such degenerations. However, just as in [GM, Lemma 4.12], such degenerations necessarily come in pairs. Hence, these ends do not contribute to the total $\mathbb{Z}/2$ -count of points $\partial \mathcal{M}_{R,\mathbf{x}}(\mathbf{D})$. Compare with Proposition 8.26.

Next, we consider curves with boundary nodes. Each such node contributes only $\frac{1}{2}$ to $\text{sing}(u)$, and therefore the index argument above fails. Instead, as in [JTZ21, Lemma 9.58], we rule these out using the matching condition. Were such a curve to form, there would be a limiting curve mapping into $\Sigma_0 \times \{x\}$ with $x \in \partial \diamond$ with nonzero multiplicity on all of Σ_0 . This is impossible. Since the limiting curve has nonzero multiplicity on all of Σ_0 , x must appear as a point of $\rho^{p_0}(u_0)$, but $\mathbf{D}$ misses the boundary of $\diamond$. Hence, $\rho^{p_0}(u_0)$ cannot coincide with any $\mathbf{d}(t)$.

Finally, we consider the case that u_i converges to a broken rectangle and a nontrivial holomorphic strip. The limiting rectangle u represents a class ψ in $\pi_2^R(\mathbf{x}, \mathbf{a}, \tau_0(\mathbf{x}), \mathbf{b})$, for $\mathbf{x} \in \mathbb{T}_{\alpha_b} \cap \mathbb{T}_{\alpha_a}$. If $\mathbf{x} = \Theta$, then by (14), we have that $\mu_R(\psi) = k = \mu_R(\psi_0)$, and the remaining curves must have index zero, miss p_0 , and therefore be constant. By transversality, such u is in the interior of $\mathcal{M}_{R,\mathbf{x}}(\mathbf{D})$, and therefore does not contribute. If $x = \theta_1^+ \theta_2^+ \theta_3^-$, $\theta_1^+ \theta_2^- \theta_3^+$, or $\theta_1^- \theta_2^+ \theta_3^+$ then u has index $k-1$ by (14), so the remaining curves could consist of a pair of holomorphic strips. The count of such ends is trivial, since $\widehat{CF}(\Sigma_0, \alpha_b, \alpha_a, p_0)$ has trivial differential. In the remaining cases, if $\mathbf{x} = \theta_1^+ \theta_2^- \theta_3^-$, $\theta_1^- \theta_2^+ \theta_3^-$, $\theta_1^- \theta_2^- \theta_3^+$, or $\theta_1^- \theta_2^- \theta_3^-$ and $\psi \in \pi_2^R(\mathbf{x}, \mathbf{a}, \tau_0(\mathbf{x}), \mathbf{b})$ with $n_{p_0}(\psi) = k$, then $\mathcal{M}_R(\psi, \mathbf{D})$ is empty. Indeed, applying (14) once more, we have that $\mu_R(\psi) \in \{k-2, k-3\}$. By Proposition 8.33, $\mathcal{M}_R(\psi, S, \mathbf{D})$ is a smooth manifold of dimension

$$\text{ind}_R(\psi, S) - (k-1) \leq \mu_R(\psi) - (k-1) < 0.$$

Hence, for a generic symmetric J , there are no holomorphic curves representing ψ satisfying the matching condition. This completes the proof that $\mathcal{M}_{R,(\Theta,\mathbf{a},\tau_0(\Theta),\mathbf{b})}(\mathbf{d}_0) = \mathcal{M}_{R,(\Theta,\mathbf{a},\tau_0(\Theta),\mathbf{b})}(\mathbf{d}_1) \pmod{2}$.

From here, the proof is the same as in [JTZ21, Lemma 9.58]. A priori, $\mathcal{M}_{R,x}(\mathbf{d})$ may depend on the connected component of $\text{Sym}^k(\diamond) \setminus \text{Diag}^k(\diamond)$, but it is independent of the divisor within the component. Therefore, suppose that $\mathcal{D} : [0, \infty) \rightarrow \text{Sym}^k(\diamond)^R$ is a smooth path in $\text{Sym}^k(\diamond) \setminus \text{Diag}^k(\diamond)$ starting at a divisor $\mathbf{d}$ where points of $\mathbf{D}(T)$ are spaced at least distance T apart and approach $v_{\alpha,\beta}$ as $T \rightarrow \infty$. We consider the space

$$\mathcal{M}_{R,\Theta}(\mathbf{D}) = \bigcup_{T \in [1, \infty)} \mathcal{M}_{R,\Theta}(\mathbf{D}(T)),$$

which has ends corresponding to $\mathcal{M}_{R,\Theta}(\mathbf{d})$, ends corresponding to degenerations of holomorphic curves, and ends corresponding to curves which appear as $T \rightarrow \infty$. The argument above applies here to show that only strips can break off at finite times T , and the mod two count of these contributions is zero.

In the limit, the curve breaks into an index-zero rectangle $\mathbf{r}$ and k curves of real Maslov index 1 satisfying a matching condition. Points of $\mathbf{D}(T)$ separate as they approach $v_{\alpha,\beta}$, so in the limit there must be k such curves on $(\Sigma_0, \alpha_0, \beta_0)$, all of the form $e_a + [\Sigma_0]$, i.e. a constant disk at a and a copy of Σ_0 , each contributing 1 to the index. Hence, the remaining curve $\mathbf{r}$ which maps to $\Sigma_0 \times \diamond$ must be index 0 and for the matching condition to be satisfied, we must have $n_{p_0}(\mathbf{r}) = 0$. As above, one argues the limiting curves satisfy (M1)-(M8).

Hence, by [Lip06], we have that

$$\#\mathcal{M}_{R,\Theta}(\mathbf{d}) \equiv (\#\mathcal{M}_{R,\mathbf{a}}(d))^k \cdot \sum_{\substack{\psi \in \pi_2^R(\Theta,\mathbf{a},\tau_0(\Theta),\mathbf{b}) \\ n_{p_0}(\psi)=0}} \#\mathcal{M}_R(\psi),$$

where $d \in \{1/2\} \times \mathbb{R}$ and $\mathcal{M}_{R,\mathbf{a}}(d)$ is the moduli space of real index 1 curves u on $(\Sigma_0, \alpha_0, \beta_0)$ with $\rho^{p_0} = d$ for a generic symmetric almost complex structure satisfying (J1)-(J4) and (J5'). Furthermore, the count above depended in no way on the component of $\text{Sym}^k(\diamond) \setminus \text{Diag}^k(\diamond)$. $\square$

To count the points of $\mathcal{M}_{R,\mathbf{a}}(d)$, we take a genus 1 diagram for the branched double cover of the two-component unlink, equipped with the covering action. There are two real-invariant bigons which contribute to the differential. By performing three $\{1\}$ -stabilizations inside one of the bigons, we obtain a class which must admit an odd number of holomorphic representatives, by the invariance of $\widehat{HFR}$. See Figure 8.5. By stretching the neck, this count must agree with the count of strips in $\mathcal{M}_R(e_a + [\Sigma_0])$ which match the connected sum point. Hence, $\#\mathcal{M}_{R,\mathbf{a}}(d) \equiv 1 \pmod{2}$. Furthermore, it must be that

$$\sum_{\substack{\psi \in \pi_2^R(\Theta,\mathbf{a},\tau_0(\Theta),\mathbf{b}) \\ n_{p_0}(\psi)=0}} \#\mathcal{M}_R(\psi) \equiv 1 \pmod{2},$$

since this map is a quasi-isomorphism of 1-dimensional chain complexes. This concludes the proof. $\square$

We now turn to the handleslides in Figure A.21; let $\mathcal{Q}_{b,c}$ and $\mathcal{Q}_{c,d}$ be the corresponding real quadruple diagrams shown in Figure 8.6. For both diagrams, $\widehat{CF}(\alpha_c, \alpha_b)$ has a unique generator in the maximal grading, and we will write Θ for both. In $\mathcal{Q}_{b,c}$, we write $\mathbb{T}_{\alpha_b} \cap \mathbb{T}_{\beta_b} = \mathbf{b}$

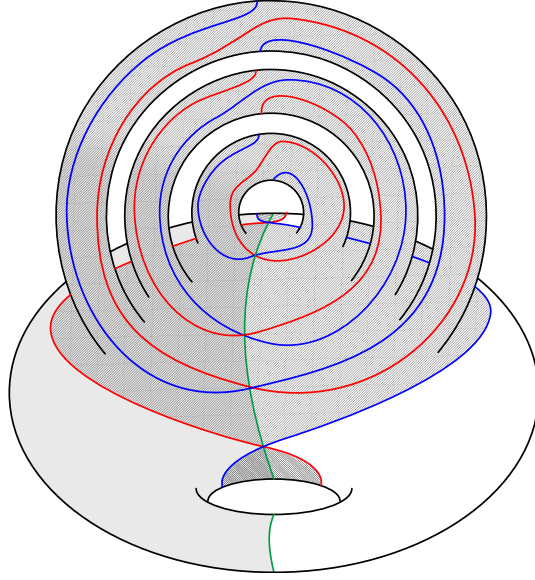


FIGURE 8.5. A triple stabilization of a real Heegaard diagram for $(\Sigma_2(U \cup U), \tau)$. The two domains contributing to the differential are shaded.

and $\mathbb{T}_{\alpha_c} \cap \mathbb{T}_{\beta_c} = \mathbf{c}$. For $\mathcal{Q}_{c,d}$, we write $\mathbb{T}_{\alpha_c} \cap \mathbb{T}_{\beta_c} = \mathbf{c}$ and $\mathbb{T}_{\alpha_d} \cap \mathbb{T}_{\beta_d} = \mathbf{d}$. For these rectangle counting maps, we have the analogue of Proposition 8.36.

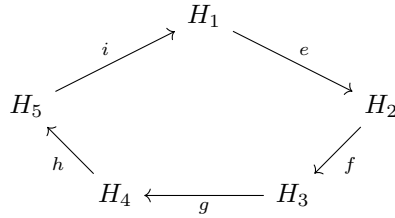
Proposition 8.37. *Suppose that $\mathcal{Q} = (\Sigma, \alpha', \alpha, \beta, \beta', \tau)$ is an admissible real Heegaard quadruple diagram and let $\mathcal{Q} \# \mathcal{Q}_*$ be the connected sum where $\mathcal{Q}_*$ is one of the quadruple diagrams shown in Figure 8.6 for $*$ $\in \{(b, c), (c, d)\}$. For a sufficiently stretched real-invariant almost complex structure, the holomorphic rectangle maps satisfy*

$$\begin{aligned} F_{\mathcal{Q} \# \mathcal{Q}_{b,c}}((\mathbf{z} \times \Theta) \otimes (\mathbf{y} \times \mathbf{b}) \otimes (\tau(\mathbf{z}) \times \tau_1(\Theta))) &\sim F_{\mathcal{Q}}(\mathbf{z} \otimes \mathbf{y} \otimes \tau(\mathbf{z})) \times \mathbf{c} \\ F_{\mathcal{Q} \# \mathcal{Q}_{c,d}}((\mathbf{z} \times \Theta) \otimes (\mathbf{y} \times \mathbf{c}) \otimes (\tau(\mathbf{z}) \times \tau_2(\Theta))) &\sim F_{\mathcal{Q}}(\mathbf{z} \otimes \mathbf{y} \otimes \tau(\mathbf{z})) \times \mathbf{d} \end{aligned}$$

for every $\mathbf{z} \in \mathbb{T}_{\alpha'} \cap \mathbb{T}_{\alpha}$ and $\mathbf{y} \in (\mathbb{T}_{\alpha} \cap \mathbb{T}_{\beta})^R$.

Proof. The proofs are identical to those of Proposition 8.36. □

Let



be a simple trade loop as in Definition 2.25. Recall that edges e and f are $\{1\}$ -stabilizations, g is a real equivalence, h is an equivariant diffeomorphism, and i is a $\mathbb{Z}/2$ -destabilization. The real equivalence is the composition of the three handleslides appearing in Figures 8.4 and 8.6.

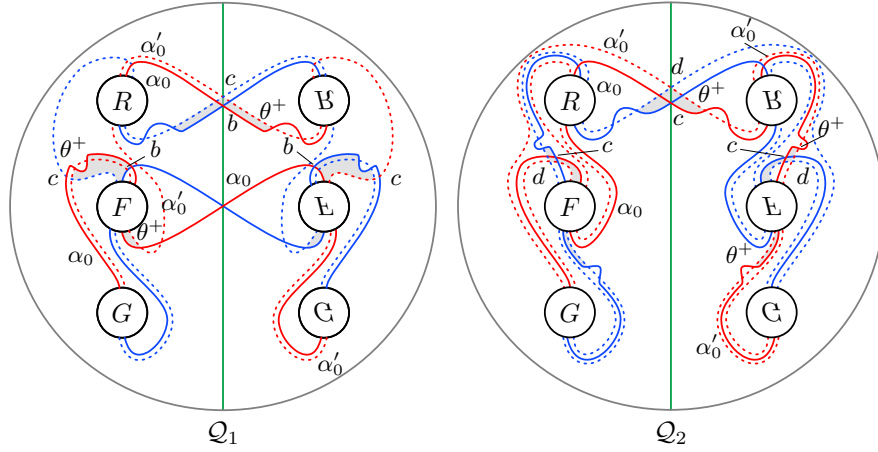


FIGURE 8.6. The quadruple diagrams used for computing the real handleslides in a trade loop.

Therefore, the map Φ_g can be computed as the composition of the three handleslide maps described in ?? 8.36?? 8.37.

To prove invariance under simple trade loops, we must prove the following.

Theorem 17. *The induced maps Φ_* for $*$ $\in \{e, f, g, h, i\}$ satisfy*

$$\Phi_i \circ \Phi_h \circ \Phi_g \circ \Phi_f \circ \Phi_e \sim \text{id}_{CFR^\circ(H_1)}.$$

Proof. Let $\mathcal{H} = (\Sigma, \alpha, \beta, \tau)$ be a real sutured Heegaard diagram. Let $\mathcal{H}_k = \mathcal{H} \#^k E_{\{1\}}$, where $E_{\{1\}}$ is the genus 1 real Heegaard diagram for $\Sigma_2(U)$ such that each $\mathcal{H}_k$ is a representative of $[H_k]$ for $k \in \{1, 2, 3\}$. $\mathcal{H}_4$ is obtained from $\mathcal{H}_3$ by a real equivalence, which is the composition of three real handleslides. Let $\mathcal{H}_3(a) = H_3$, $\mathcal{H}_3(b)$, $\mathcal{H}_3(c)$, and $\mathcal{H}_3(d) = \mathcal{H}_4$ be the intermediate diagrams obtained by the real handleslides shown in Figure A.21. The diagram $\mathcal{H}_5 = \mathcal{H}_1 \# E_{\mathbb{Z}/2}$ is obtained from $\mathcal{H}_4$ by a diffeomorphism, φ , where $E_{\mathbb{Z}/2}$ is the standard genus 2 diagram for $\Sigma_2(U)$. We write e and $\tau_0(e)$ for the two symmetric intersection points of $E_{\mathbb{Z}/2}$. Evidently, $\mathcal{H}_5$ can be $\mathbb{Z}/2$ -destabilized to return to $\mathcal{H}_1$. According to Proposition 8.15, we can fix symmetric almost complex structures on $\mathcal{H}_i$ so that the stabilization maps and holomorphic rectangle maps commute.

In Figure 8.4, there are three intersection points in Σ_0 which (for simplicity) are all labeled a (similarly for b, c, d , and θ^+ in Figure 8.6). Let a_1, a_2 , and a_3 be the intersection points associated to the handles labeled R, F , and G respectively. In this way, if $\mathbf{x} \times a_1$ is a generator for $CFR^\circ(\mathcal{H}_1) \simeq CFR^\circ(\mathcal{H}) \otimes CFR^\circ(E_{\{1\}})$, then $\Phi_f \circ \Phi_e(\mathbf{x} \times a_1) = \mathbf{x} \times \mathbf{a}$, where $\mathbf{a} = \{a_1 a_2 a_3\}$. Let $\mathbf{b}, \mathbf{c}$, and $\mathbf{d}$ be the analogous triples of intersection points in the diagrams $\mathcal{H}_3(b)$, $\mathcal{H}_3(c)$, and $\mathcal{H}_3(d)$ shown in Figure 8.6. The real equivalence Φ_g is the composition of three real handleslides³

$$\Phi_g = \Psi_{\beta \cup \beta_c \rightarrow \beta \cup \beta_d}^{\alpha \cup \alpha_c \rightarrow \alpha \cup \alpha_d} \circ \Psi_{\beta \cup \beta_b \rightarrow \beta \cup \beta_c}^{\alpha \cup \alpha_b \rightarrow \alpha \cup \alpha_c} \circ \Psi_{\beta \cup \beta_a \rightarrow \beta \cup \beta_b}^{\alpha \cup \alpha_a \rightarrow \alpha \cup \alpha_b},$$

where α_i are the alpha curves of $\mathcal{H}_3(i)$ for $i \in \{a, b, c, d\}$. The curves in $\mathcal{H}_3(a)$ and $\mathcal{H}_3(b)$ form a real quadruple diagram $\mathcal{Q} \# \mathcal{Q}_{a,b}$. The map $\Psi_{\beta \cup \beta_a \rightarrow \beta \cup \beta_b}^{\alpha \cup \alpha_a \rightarrow \alpha \cup \alpha_b}$ is defined in terms of holomorphic rectangle counts:

$$\Psi_{\beta \cup \beta_a \rightarrow \beta \cup \beta_b}^{\alpha \cup \alpha_a \rightarrow \alpha \cup \alpha_b}(\mathbf{x} \times \mathbf{a}) = F_{\mathcal{Q} \# \mathcal{Q}_{a,b}}((\Theta_{\alpha', \alpha} \times \Theta) \otimes (\mathbf{x} \times \mathbf{a}) \otimes (\tau(\Theta_{\alpha', \alpha}) \times \tau_0(\Theta))).$$

³Here we abuse notation: we write α for the alpha curves in all four diagrams $\mathcal{H}_3(a)$ through $\mathcal{H}_3(d)$, though implicitly we are applying small Hamiltonian perturbations to pass from one collection to the next.

According to Proposition 8.36, for the appropriate choice of symmetric almost complex structure

$$F_{\mathcal{Q} \# \mathcal{Q}_{a,b}}((\Theta_{\alpha',\alpha} \times \Theta) \otimes (\mathbf{x} \times \mathbf{a}) \otimes (\tau(\Theta_{\alpha',\alpha}) \times \tau_0(\Theta))) = F_{\mathcal{Q}}(\Theta_{\alpha',\alpha} \otimes \mathbf{x} \otimes \tau(\Theta_{\alpha',\alpha})) \times \mathbf{b}.$$

Similarly, by Proposition 8.37,

$$\begin{aligned} \Psi_{\beta \cup \beta_b \rightarrow \beta \cup \beta_c}^{\alpha \cup \alpha_b \rightarrow \alpha \cup \alpha_c}((\Theta_{\alpha',\alpha} \times \Theta) \otimes (\mathbf{x} \times \mathbf{b}) \otimes (\tau(\Theta_{\alpha',\alpha}) \times \tau_1(\Theta))) &\sim F_{\mathcal{Q}}(\Theta_{\alpha',\alpha} \otimes \mathbf{x} \otimes \tau(\Theta_{\alpha',\alpha})) \times \mathbf{c} \\ \Psi_{\beta \cup \beta_c \rightarrow \beta \cup \beta_d}^{\alpha \cup \alpha_c \rightarrow \alpha \cup \alpha_d}((\Theta_{\alpha',\alpha} \times \Theta) \otimes (\mathbf{x} \times \mathbf{c}) \otimes (\tau(\Theta_{\alpha',\alpha}) \times \tau_2(\Theta))) &\sim F_{\mathcal{Q}}(\Theta_{\alpha',\alpha} \otimes \mathbf{x} \otimes \tau(\Theta_{\alpha',\alpha})) \times \mathbf{d}. \end{aligned}$$

Combining these facts, we have that

$$\Phi_g(\mathbf{x} \times \mathbf{a}) \sim \Psi_{\beta \rightarrow \beta'}^{\alpha \rightarrow \alpha'}(\mathbf{x}) \times \mathbf{d},$$

where the curves α' are obtained by a small Hamiltonian perturbation of α ; the curves β and β' are symmetric.

Outside of Σ_0 , we assume that φ is isotopic to the identity, taking α' back to the curves α . Hence,

$$\Phi_h(\mathbf{x} \times \mathbf{d}) := \varphi_*(\mathbf{x} \times \mathbf{d}) = (\varphi|_{\Sigma})_*(\mathbf{x}) \times \{a_1 e \tau_0(e)\},$$

where e and $\tau_0(e)$ are the symmetric intersection points in $\mathcal{H}_5 = \mathcal{H}_1 \# E_{\mathbb{Z}/2}$. Finally, we compute the destabilization map:

$$\Phi_i(\mathbf{x} \times \{a_1 e \tau_0(e)\}) = (\sigma_{\mathcal{H}_1 \rightarrow \mathcal{H}_5}^{\mathbb{Z}/2})^{-1}(\mathbf{x} \times \{a_1 e \tau_0(e)\}) = \mathbf{x} \times a_1.$$

Putting everything together, we have

$$\Phi_i \circ \Phi_h \circ \Phi_g \circ \Phi_f \circ \Phi_e(\mathbf{x} \times a_1) \sim ((\varphi|_{\Sigma})_* \circ \Phi_{\beta \rightarrow \beta'}^{\alpha \rightarrow \alpha'})_*(\mathbf{x}) \times a_1.$$

According to Proposition 8.14, we have

$$(\varphi|_{\Sigma})_* \circ \Phi_{\beta \rightarrow \beta'}^{\alpha \rightarrow \alpha'} \sim \text{id}_{CFR^\circ(\mathcal{H})}.$$

Hence,

$$\Phi_i \circ \Phi_h \circ \Phi_g \circ \Phi_f \circ \Phi_e \sim \text{id}_{CFR^\circ(\mathcal{H})} \otimes \text{id}_{CFR^\circ(E_{\{1\}})},$$

completing the proof. $\square$

8.10. Real handleswap loops. Finally, we turn to real handleswaps.

Consider the quadruple diagram $\mathcal{Q}_L = (\Sigma_0, \alpha'_0, \alpha_0, \beta_0, \beta'_0)$ shown in the upper left of Figure 8.7; let $\mathcal{Q}_R$ be the diagram obtained from $\mathcal{Q}_L$ by swapping the roles of alpha and beta. If $\mathcal{Q} = (\Sigma, \alpha', \alpha, \beta, \beta', \tau)$ is any real quadruple diagram, we consider the quadruple

$$\mathcal{Q}_L \# \mathcal{Q} \# \mathcal{Q}_R$$

by taking a connected sum at points $p \neq \tau(p) \in \Sigma$ and points p_L and p_R in Σ_0 . We extend the involution by swapping the two Σ_0 summands. Consider the diagram $\mathcal{Q}_L$, and let $\mathbb{T}_{\alpha_0} \cap \mathbb{T}_{\beta_0} = \mathbf{a}$ and $\mathbb{T}_{\alpha'_0} \cap \mathbb{T}_{\beta'_0} = \mathbf{a}'$ and

$$\mathbb{T}_{\alpha_0} \cap \mathbb{T}_{\alpha'_0} = \{\theta_1^\pm \theta_2^\pm\}.$$

Let $\Theta = \{\theta_1^+ \theta_2^+\}$ be the top generator. Since $\mathcal{Q}_L$ represents $\#^2(S^1 \times S^2)$, we have that $\dim \widehat{CF}(\mathcal{Q}_L) = \dim \widehat{HF}(\#^2(S^1 \times S^2))$, and hence the differential of $\dim \widehat{CF}(\mathcal{Q}_L)$ must vanish.

Proposition 8.38. *Suppose that $\mathcal{Q} = (\Sigma, \alpha', \alpha, \beta, \beta', \tau)$ is an admissible real Heegaard quadruple diagram and let $\mathcal{Q}_L \# \mathcal{Q} \# \mathcal{Q}_R$ be the connected sum where $\mathcal{Q}_L$ and $\mathcal{Q}_R$ are the quadruple diagrams shown in the top of Figure 8.7. For a sufficiently stretched real-invariant almost complex structure, the holomorphic rectangle map $F_{\mathcal{Q}_L \# \mathcal{Q} \# \mathcal{Q}_R}$ is given by*

$$F_{\mathcal{Q}_L \# \mathcal{Q} \# \mathcal{Q}_R}((\Theta \times \mathbf{x} \times \tau(\Theta)) \otimes (\mathbf{a} \times \mathbf{y} \times \tau(\mathbf{a})) \otimes (\tau(\Theta) \times \tau(\mathbf{x}) \times \Theta)) \sim \mathbf{a}' \times F_{\mathcal{Q}}(\mathbf{x} \otimes \mathbf{y} \otimes \tau(\mathbf{x})) \times \tau(\mathbf{a}')$$

for every $\mathbf{x} \in \mathbb{T}_{\alpha'} \cap \mathbb{T}_{\alpha}$ and $\mathbf{y} \in (\mathbb{T}_{\alpha} \cap \mathbb{T}_{\beta})^R$.

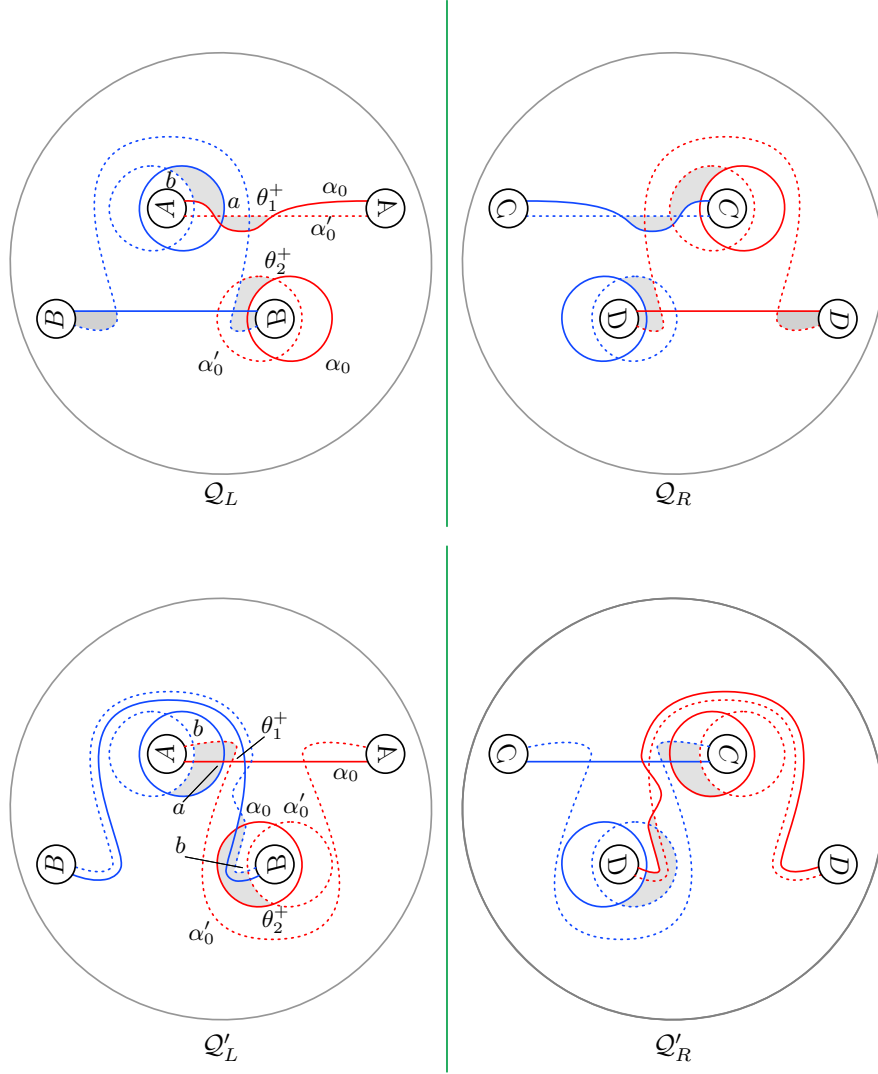


FIGURE 8.7. The quadruple diagrams used for computing the real handleslides in a real handleswap loop.

Proof. The proof follows by an equivariant neck-stretching argument. Since $\mathcal{Q}_L$ and $\mathcal{Q}_R$ are disjoint, the arguments in [JTZ21, Proposition 9.31] apply here. Indeed, pairs of real-invariant rectangles behave exactly like unreal triangles. $\square$

Let

$$\begin{array}{ccc}
 H_1 & \xrightarrow{e} & H_2 \\
 & \nwarrow g \quad \searrow f & \\
 & H_3 &
 \end{array}$$

be a simple real handleswap as in Definition 2.23. Recall that edges e and f are real equivalences and g is an equivariant diffeomorphism. The real equivalences are given by the real handleslides

appearing in Figure 8.7, and therefore the map can be computed as the composition of two real handleslide maps. To prove invariance under a simple real handleswap, we establish the following.

Theorem 18. *The induced maps Φ_* for $* \in \{e, f, g\}$ satisfy*

$$\Phi_g \circ \Phi_f \circ \Phi_e \sim \text{id}_{CFR^\circ(H_1)}.$$

Proof. This follows from Proposition 8.38. The arguments are the same as in [JTZ21, Theorem 9.30] in the unreal setting. $\square$

Proof of Theorem 8. Let CFR° be any of the weak real Heegaard Floer theories from Theorems 5, 6, or 7. Axioms (1)-(3) were verified in Section 8.5 and Axioms (4) and (5) were verified in ?? 17?? 18. $\square$

9. INVOLUTIVE REAL HEEGAARD FLOER HOMOLOGY

Heegaard Floer homology has an intrinsic symmetry, given by Spin^c -conjugation. In [HM17], Hendricks and the second author used this symmetry to define *involutive Heegaard Floer homology*. We review their construction.

For simplicity, fix $(Y, \mathfrak{s})$, where $\mathfrak{s}$ is a self-conjugate Spin^c -structure, i.e., $\bar{\mathfrak{s}} = \mathfrak{s}$. Choose a pointed Heegaard diagram $\mathcal{H} = (\Sigma, \alpha, \beta, z)$ for Y . There is a canonical isomorphism

$$\eta : CF^\circ(\mathcal{H}, \mathfrak{s}) \rightarrow CF^\circ(\bar{\mathcal{H}}, \mathfrak{s}),$$

where $\bar{\mathcal{H}}$ is the diagram $(-\Sigma, \beta, \alpha, z)$. Since $\mathcal{H}$ and $\bar{\mathcal{H}}$ both represent Y , a choice of Heegaard moves connecting the two diagrams gives a homotopy equivalence

$$\Phi_{\bar{\mathcal{H}} \rightarrow \mathcal{H}} : CF^\circ(\bar{\mathcal{H}}, \mathfrak{s}) \rightarrow CF^\circ(\mathcal{H}, \mathfrak{s}),$$

which is well-defined by naturality. Let

$$\iota := \Phi_{\bar{\mathcal{H}} \rightarrow \mathcal{H}} \circ \eta : CF^\circ(\mathcal{H}, \mathfrak{s}) \rightarrow CF^\circ(\mathcal{H}, \mathfrak{s}).$$

The involutive Heegaard Floer homology $HFI(Y, \mathfrak{s})$ is defined to be

$$H_*(\text{Cone}(CF^\circ(\mathcal{H}, \mathfrak{s}) \xrightarrow{Q(1+\iota)} CF^\circ(\mathcal{H}, \mathfrak{s}))[-1]),$$

where Q is a formal variable and $[-1]$ denotes a degree shift.

The same construction can be carried out in the real context. Given a real Heegaard diagram $(\mathcal{H}, \tau) = (\Sigma, \alpha, \beta, z, \tau)$ for (Y, τ) and a self-conjugate real Spin^c -structure $\mathfrak{s}^R$, there is, again, a canonical map

$$\eta : CFR^\circ(\mathcal{H}, \tau, \mathfrak{s}^R) \rightarrow CFR^\circ(\bar{\mathcal{H}}, \tau, \mathfrak{s}^R).$$

The diagrams $(\mathcal{H}, \tau)$ and $(\bar{\mathcal{H}}, \tau)$ both represent (Y, τ) , so a sequence of real Heegaard moves between them induces a homotopy equivalence

$$\Phi_{(\bar{\mathcal{H}}, \tau) \rightarrow (\mathcal{H}, \tau)} : CFR^\circ(\bar{\mathcal{H}}, \tau, \mathfrak{s}^R) \rightarrow CFR^\circ(\mathcal{H}, \tau, \mathfrak{s}^R).$$

This gives rise to a homotopy involution

$$\iota := \Phi_{(\bar{\mathcal{H}}, \tau) \rightarrow (\mathcal{H}, \tau)} \circ \eta : CFR^\circ(\mathcal{H}, \tau, \mathfrak{s}^R) \rightarrow CFR^\circ(\mathcal{H}, \tau, \mathfrak{s}^R).$$

We define $HFRI^\circ(Y, \tau, \mathfrak{s}^R)$, the *involutive real Heegaard Floer homology of $(Y, \tau, \mathfrak{s}^R)$* , to be the homology of the mapping cone

$$\text{Cone}(CFR^\circ(\mathcal{H}, \tau, \mathfrak{s}^R) \xrightarrow{Q(1+\iota)} CFR^\circ(\mathcal{H}, \tau, \mathfrak{s}^R))[-1],$$

over the ring $\mathbb{F}[Q, v]/(Q^2)$ with v and Q both in degree -1 .

Proof of Theorem 4. This follows just as in the proof of [HM17, Theorem 1.1], making use of the fact that $\Phi_{(\overline{\mathcal{H}}, \tau) \rightarrow (\mathcal{H}, \tau)}$ is well defined up to chain homotopy. This last fact is naturality at the chain level, which was established in the proof of Theorem 1. $\square$

Proposition 9.1. *For $\circ \in \{-, \infty, +, \wedge\}$, there is an exact triangle of $\mathbb{F}[v]$ -linear maps:*

$$(15) \quad \begin{array}{ccc} HFR^\circ(Y, \tau, \mathfrak{s}^R) & \xrightarrow{Q(1 + \iota_*)} & Q \cdot HFR^\circ(Y, \tau, \mathfrak{s}^R)[-1] \\ & \nwarrow \quad \nearrow & \\ & HFRI^\circ(Y, \tau, \mathfrak{s}^R) & \end{array}$$

Proof. This is clear from the definition. Compare [HM17, Proposition 4.6]. $\square$

Example 9.2. Suppose (Y, τ) is the double branched cover over a knot $K \subset S^3$ and that Y is an L -space (with $\mathbb{F}$ -coefficients). The latter condition means that $\widehat{HF}(Y, \mathfrak{s}; \mathbb{F}) \cong \mathbb{F}$ for every Spin^c structure $\mathfrak{s}$. Then, by the work of Hendricks [Hen, Theorem 1.2], we also have $\widehat{HFR}(Y, \tau, \mathfrak{s}^R) \cong \mathbb{F}$, for every real Spin^c structure $\mathfrak{s}^R$. If $\mathfrak{s}^R$ is self-conjugate, we obtain

$$\begin{aligned} HFRI^-(Y, \tau, \mathfrak{s}^R) &\cong \mathbb{F}[Q, v]/(Q^2), \\ \widehat{HFRI}(Y, \tau, \mathfrak{s}^R) &\cong \mathbb{F}[Q]/(Q^2). \end{aligned}$$

The proof follows that of [HM17, Corollary 4.8], making use of the exact triangle (15). To show that the extension of HFR° by HFR° splits, we observe that these are free modules over the respective base rings.

Example 9.3. (This was suggested to us by Kristen Hendricks.) Let Y be the Brieskorn sphere $\Sigma(2, 3, 7)$, viewed as the double cover of S^3 branched along the torus knot $T(3, 7)$, and let τ be the deck transformation. The real manifold (Y, τ) admits a unique real Spin^c structure, which we drop from the notation. By [Hen, Theorem 1.2], there is a spectral sequence whose E_1 page is $\widehat{HF}(Y) \otimes \mathbb{F}[\theta, \theta^{-1}]$, which converges to $\widehat{HFR}(Y, \tau) \otimes \mathbb{F}[\theta, \theta^{-1}]$, and whose d_1 differential is $(1 + \iota_* \tau_*)\theta$. By the calculations in [OS03, Equation (27)], [HLS20, Proposition 6.26], and [HM17, Section (6.8)], we have that $\dim \widehat{HF}(Y; \mathbb{F}) = 3$, with τ_* being the identity and ι_* exchanging two generators, so that $1 + \iota_* \tau_*$ has rank 1. From here it follows that

$$\dim \widehat{HFR}(Y, \tau) = 1.$$

The involution ι_* must act as the identity on $\widehat{HFR}(Y, \tau)$. Using the exact triangle (15), we deduce that $\widehat{HFRI}(Y, \tau)$ has dimension 2. On the other hand, from [HM17, Section (6.8)] we see that $\dim \widehat{HFI}(Y) = 4$. Thus, unlike in Example 9.2, in this case we have

$$\dim \widehat{HFRI}(Y, \tau) < \dim \widehat{HFI}(Y).$$

APPENDIX A. RESOLUTIONS OF CODIMENSION-2 BIFURCATIONS

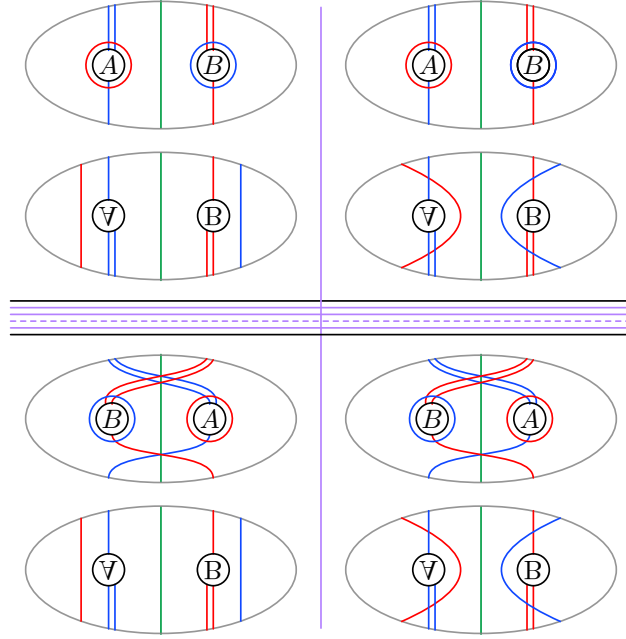


FIGURE A.1. Resolving an (A4b) bifurcation.

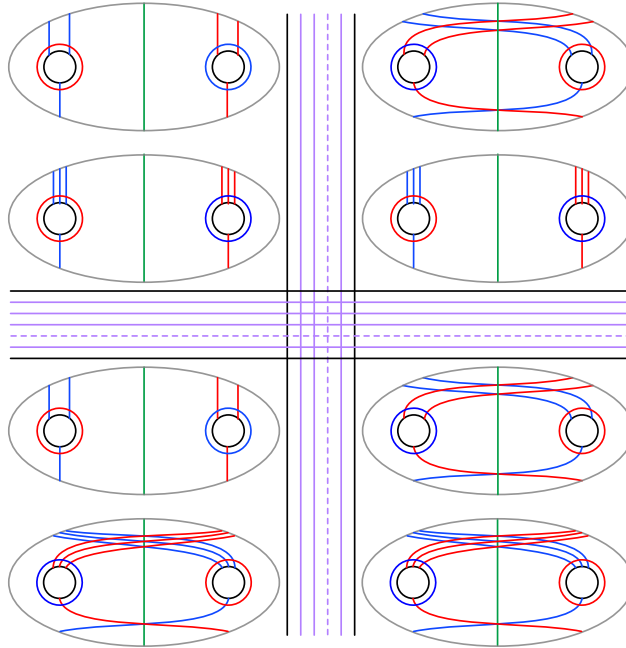


FIGURE A.2. Resolving an (A3) bifurcation.

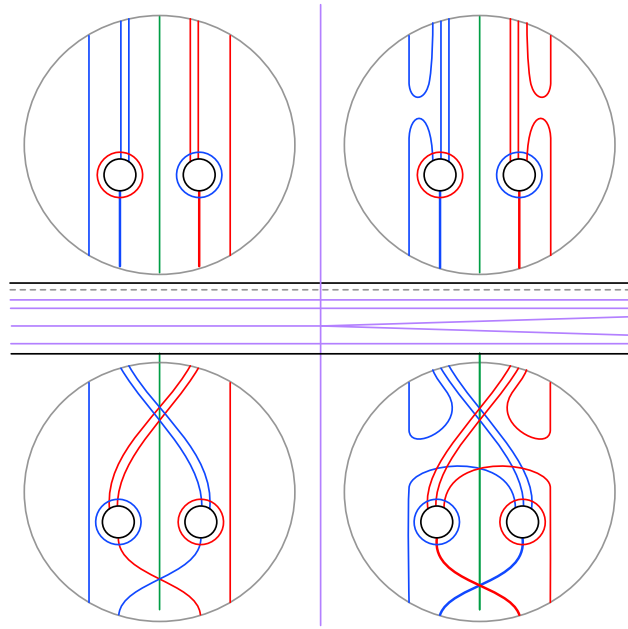


FIGURE A.3. Resolving an (A4a) bifurcation.

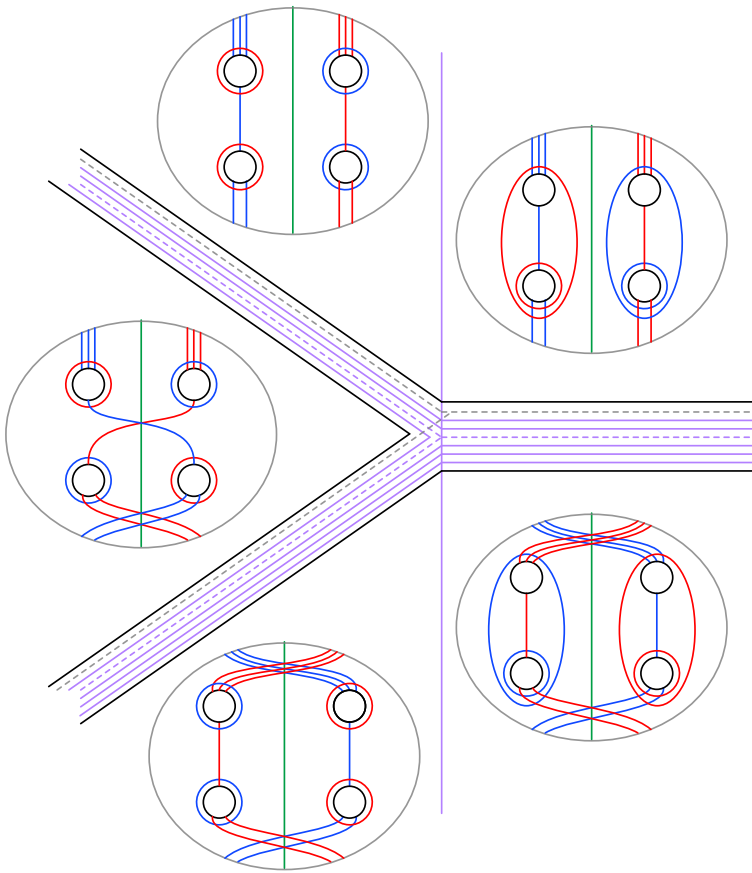


FIGURE A.4. Resolving a simple (A4c) bifurcation.

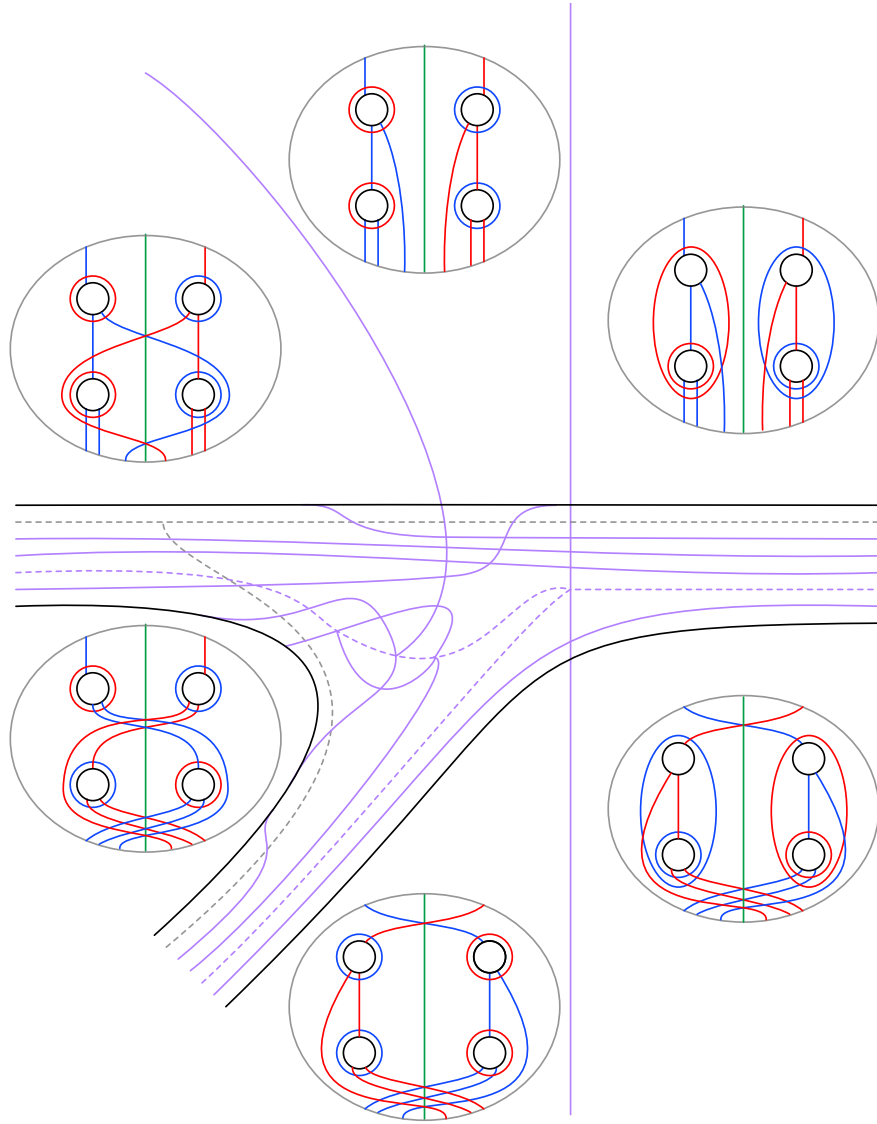


FIGURE A.5. Resolving a more complicated (A4c) bifurcation.

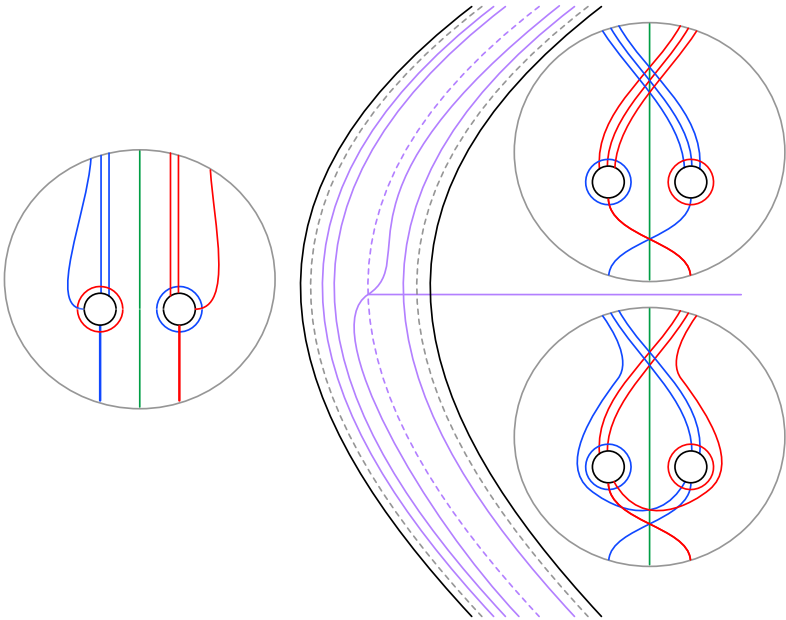


FIGURE A.6. Resolving an (E2) bifurcation.

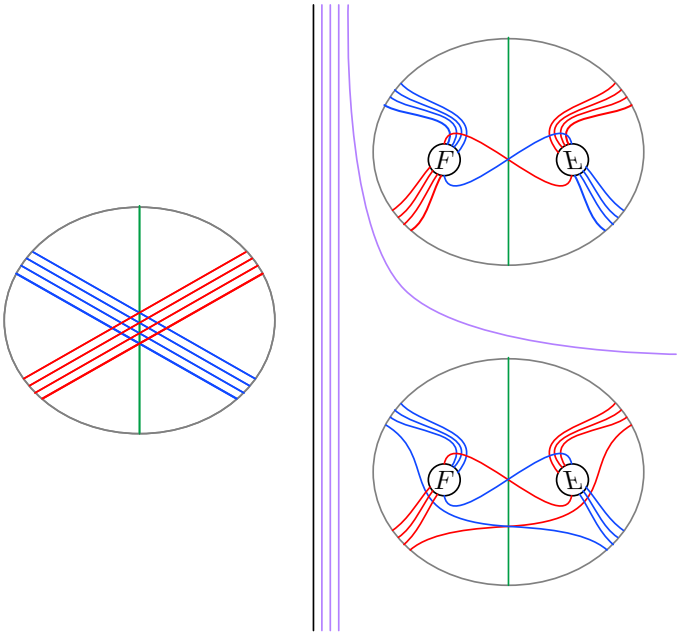


FIGURE A.7. Resolving a $\{1\}$ -orbit (B3) bifurcation.

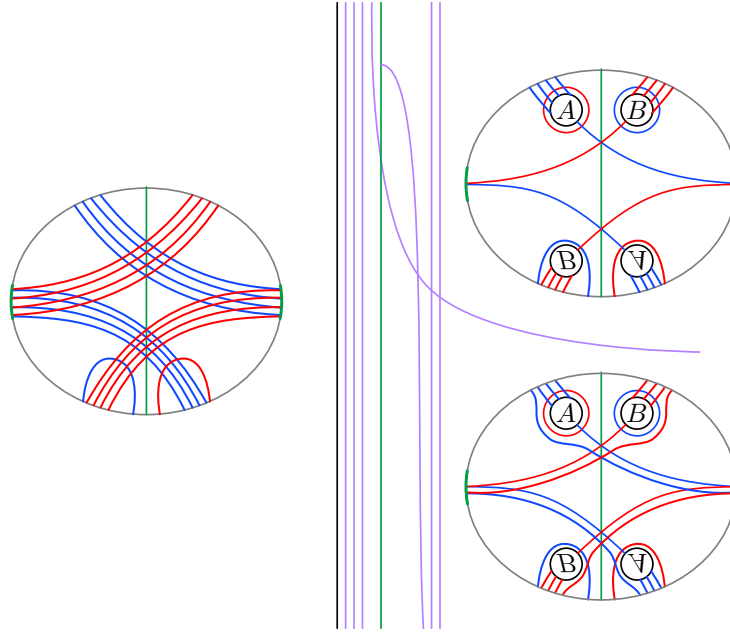
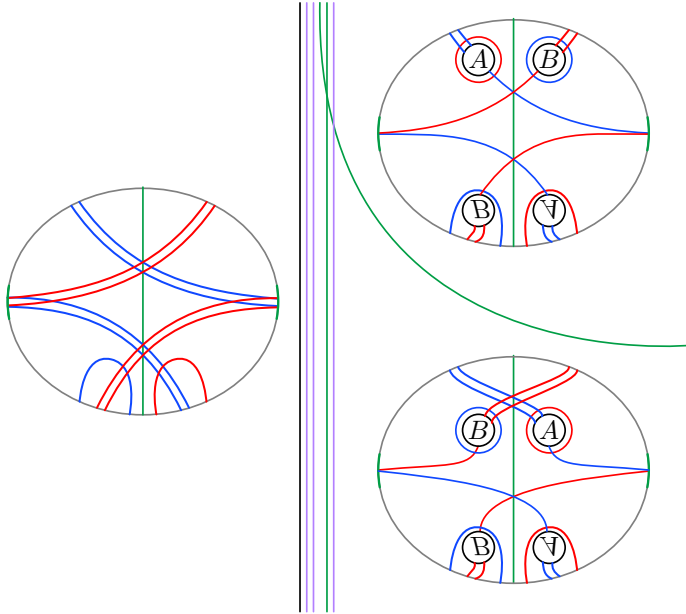

 FIGURE A.8. Resolving a $\mathbb{Z}/2$ -orbit (B3) bifurcation.


FIGURE A.9. Resolving a (B5) bifurcation.

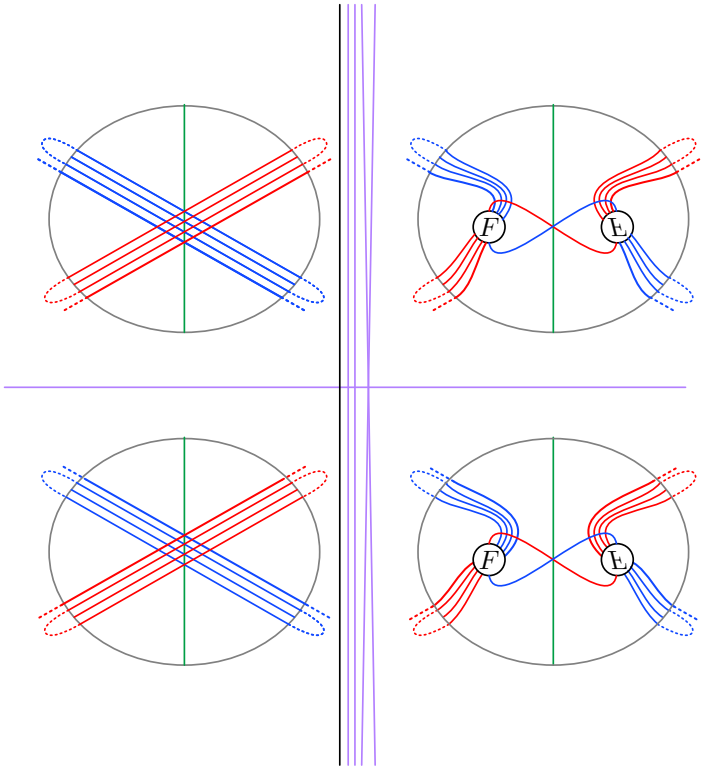
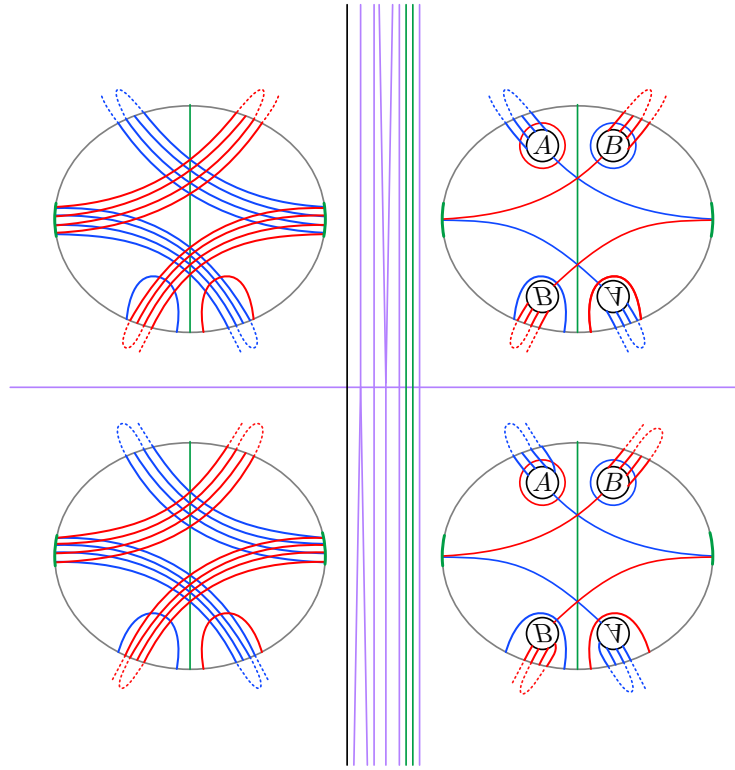
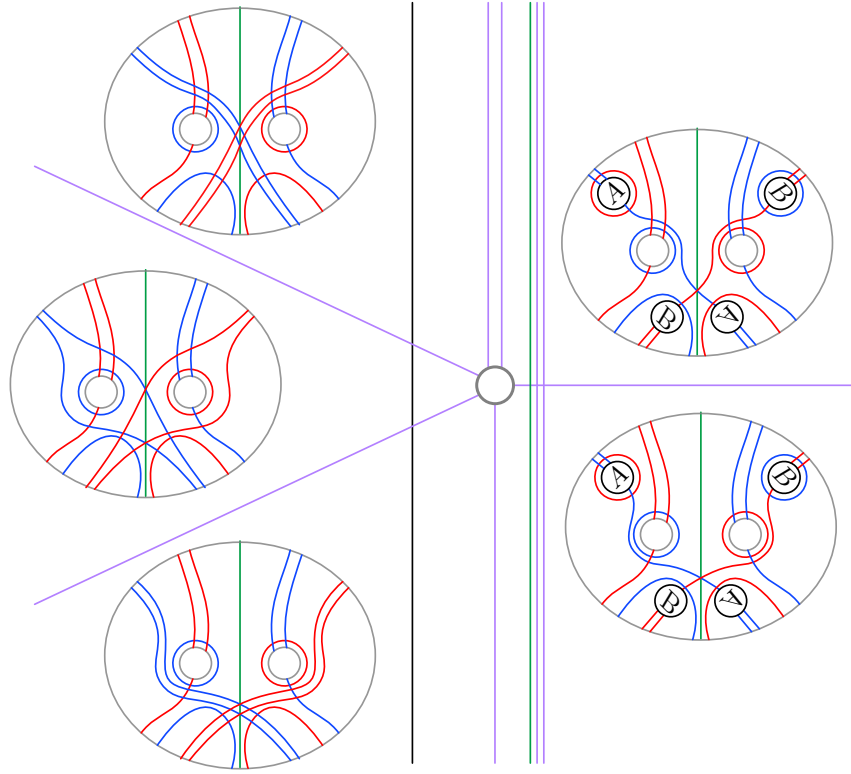
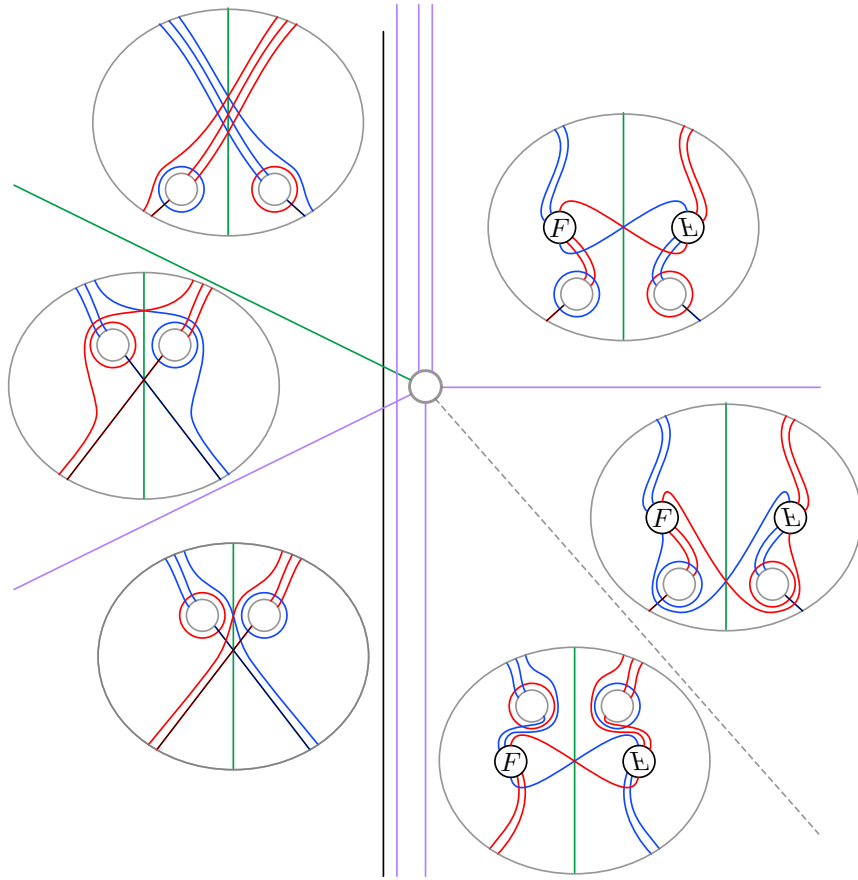


FIGURE A.10. Resolving a $\{1\}$ -orbit (B1) bifurcation.


 FIGURE A.11. Resolving a $\mathbb{Z}/2$ -orbit (B1) bifurcation.

FIGURE A.12. Resolving a $\mathbb{Z}/2$ -orbit (B2) bifurcation.


 FIGURE A.13. Resolving a $\{1\}$ -orbit (B2) bifurcation.

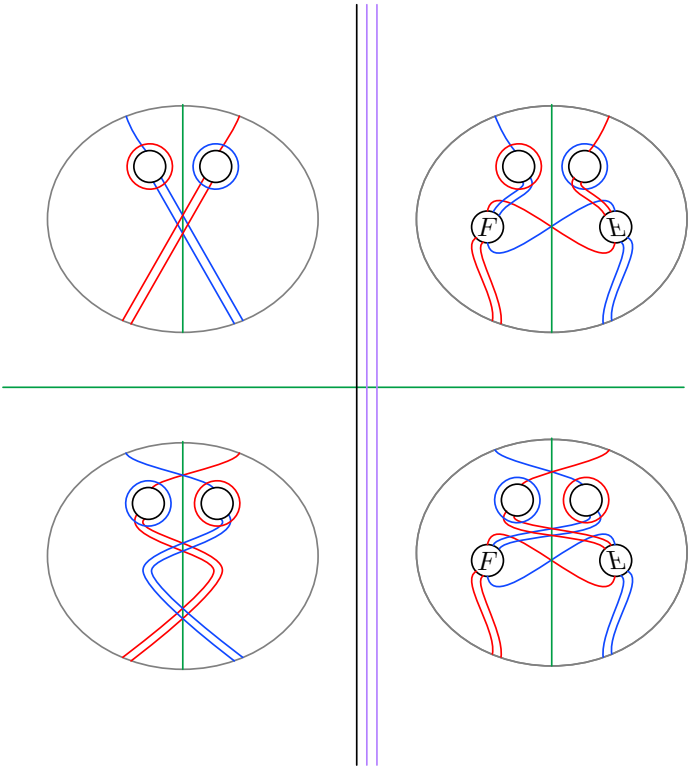
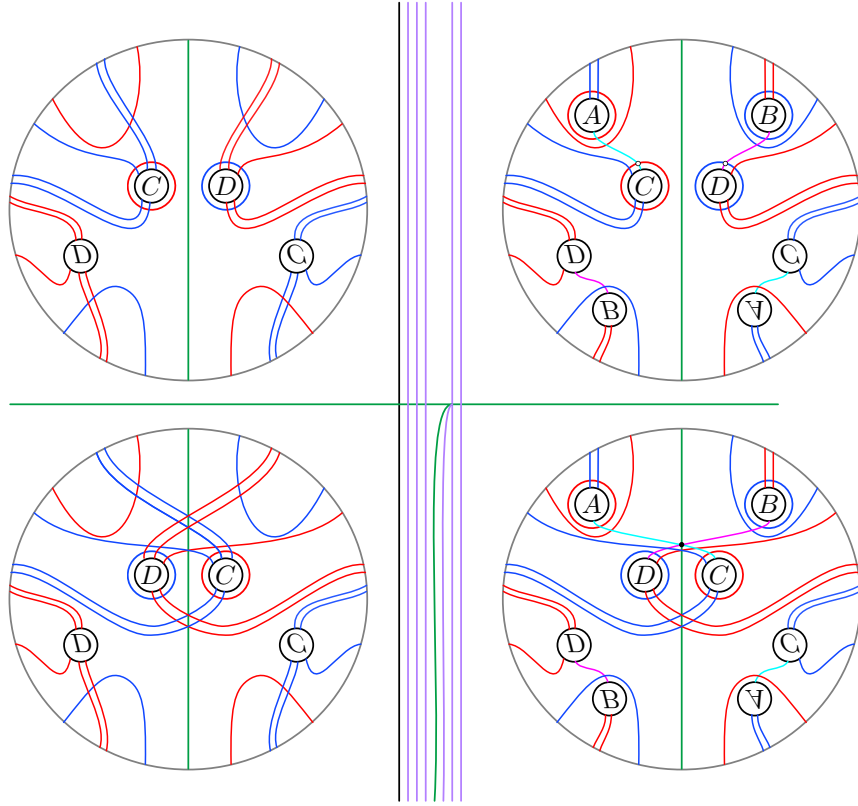
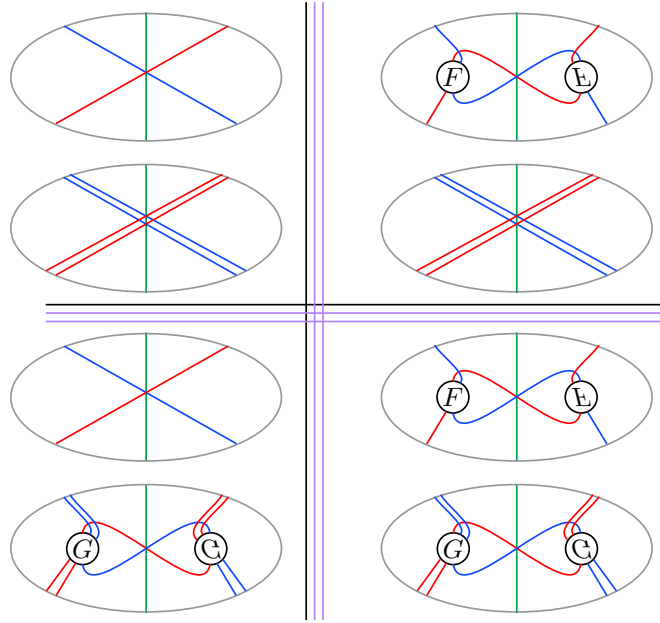
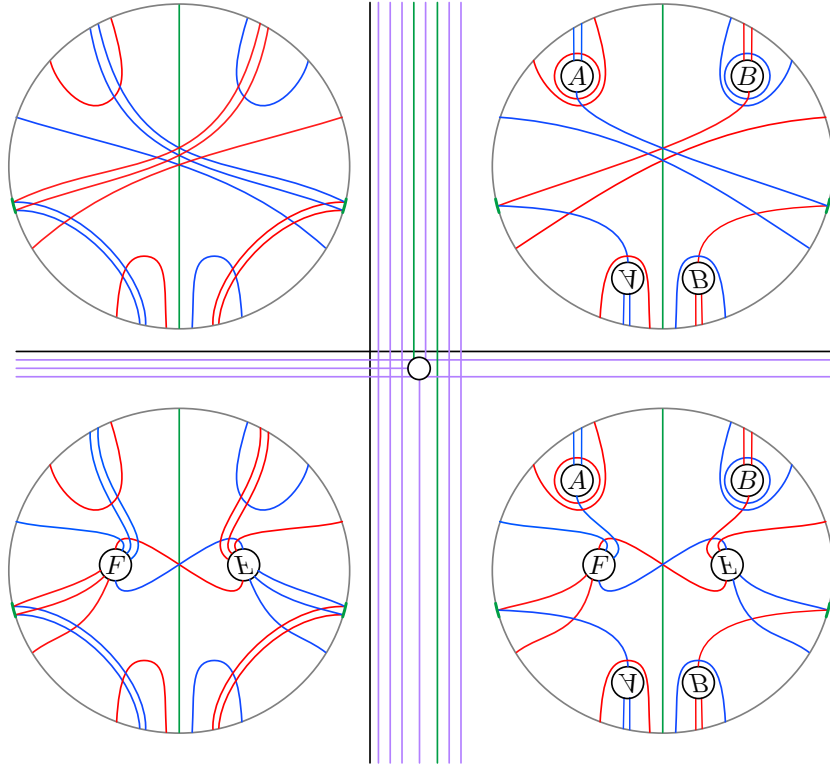
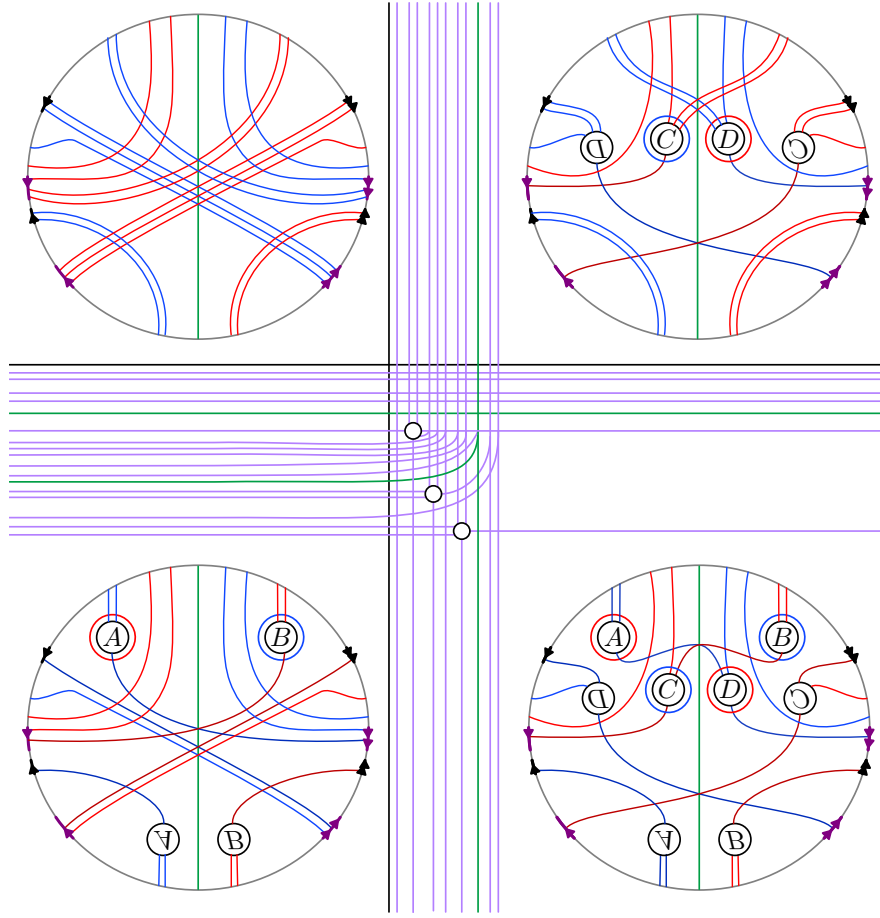


FIGURE A.14. Resolving a $\{1\}$ -orbit (B4) bifurcation.


 FIGURE A.15. Resolving a $\mathbb{Z}/2$ -orbit (B4) bifurcation.

 FIGURE A.16. Resolving a $\{1\}$ - $\{1\}$ -orbit bifurcation of type (C).

FIGURE A.17. Resolving a $\mathbb{Z}/2\text{-}\{1\}$ -orbit bifurcation of type (C).


 FIGURE A.18. Resolving a $\mathbb{Z}/2$ - $\mathbb{Z}/2$ -orbit bifurcation of type (C).

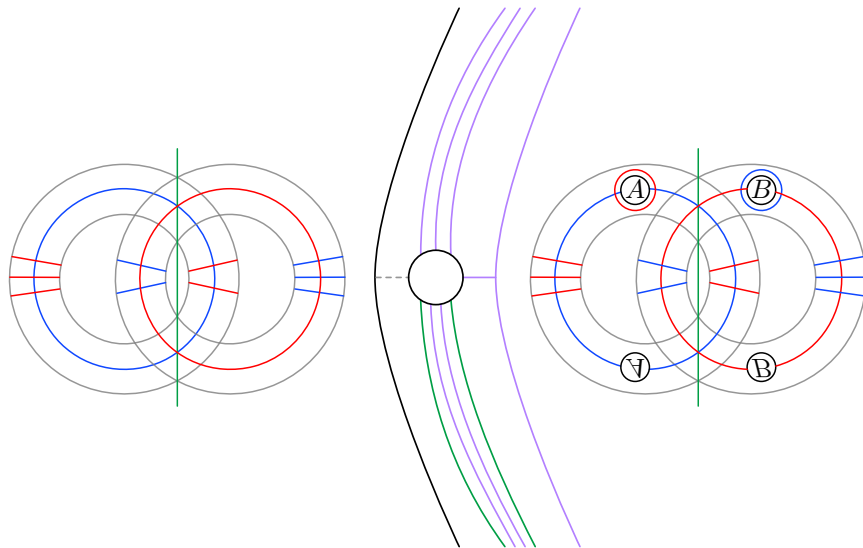


FIGURE A.19. Resolving a bifurcation of type (D1).

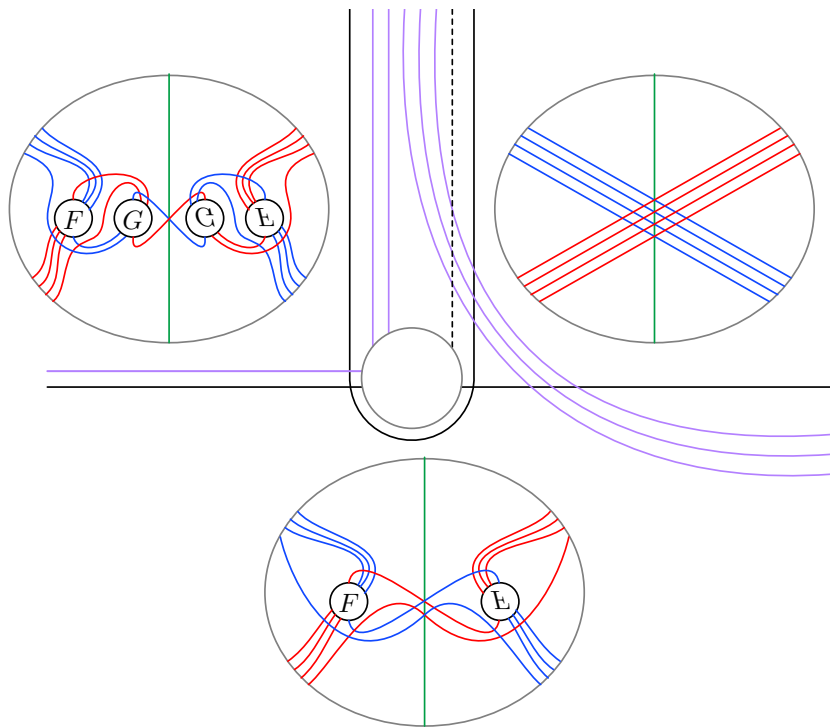


FIGURE A.20. Resolving a bifurcation of type (D2).

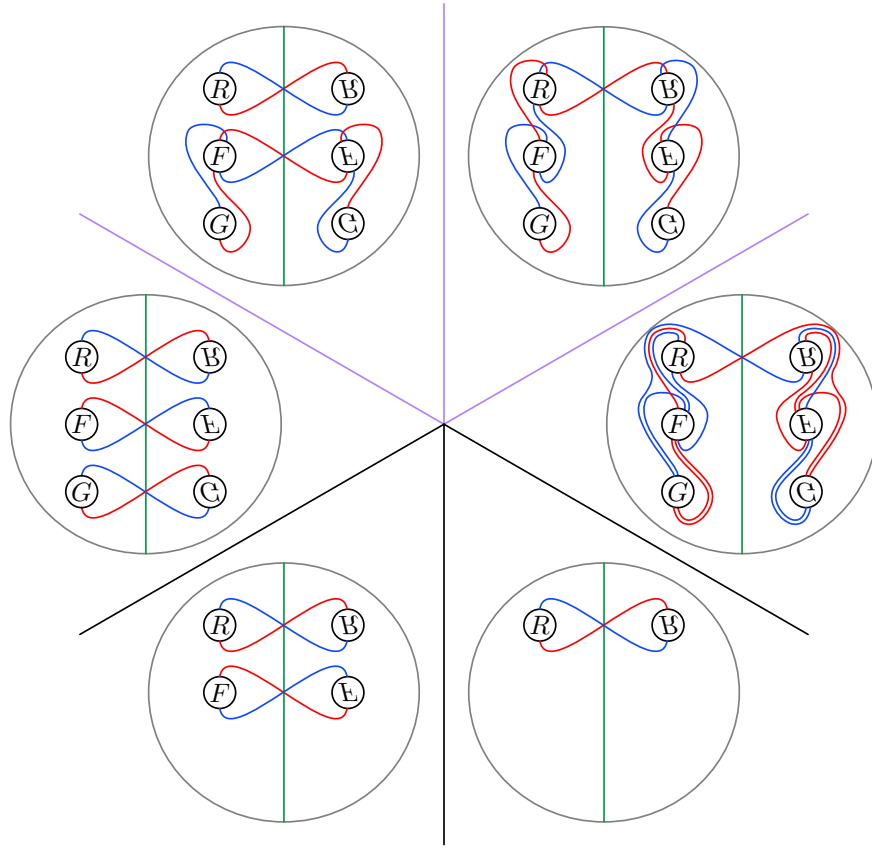


FIGURE A.21. Resolving a bifurcation of type (D2).

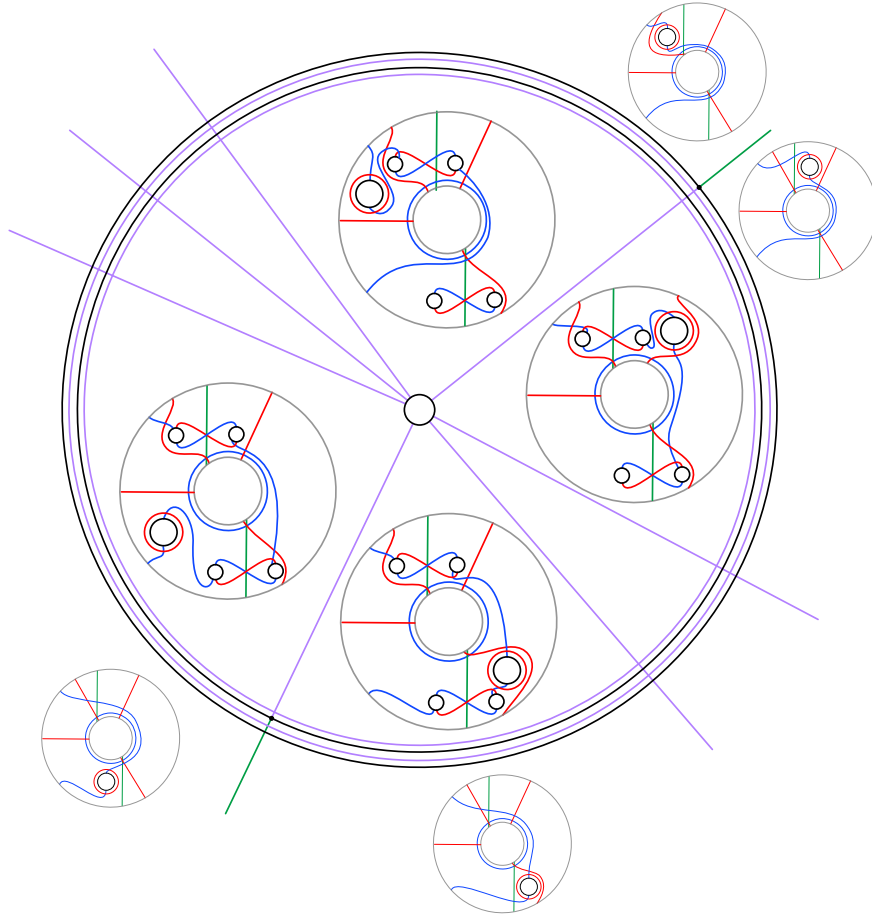


FIGURE A.22. Simplifying a real handleswap of type (k, ℓ, r) to one of type $(k, \ell, 0)$.

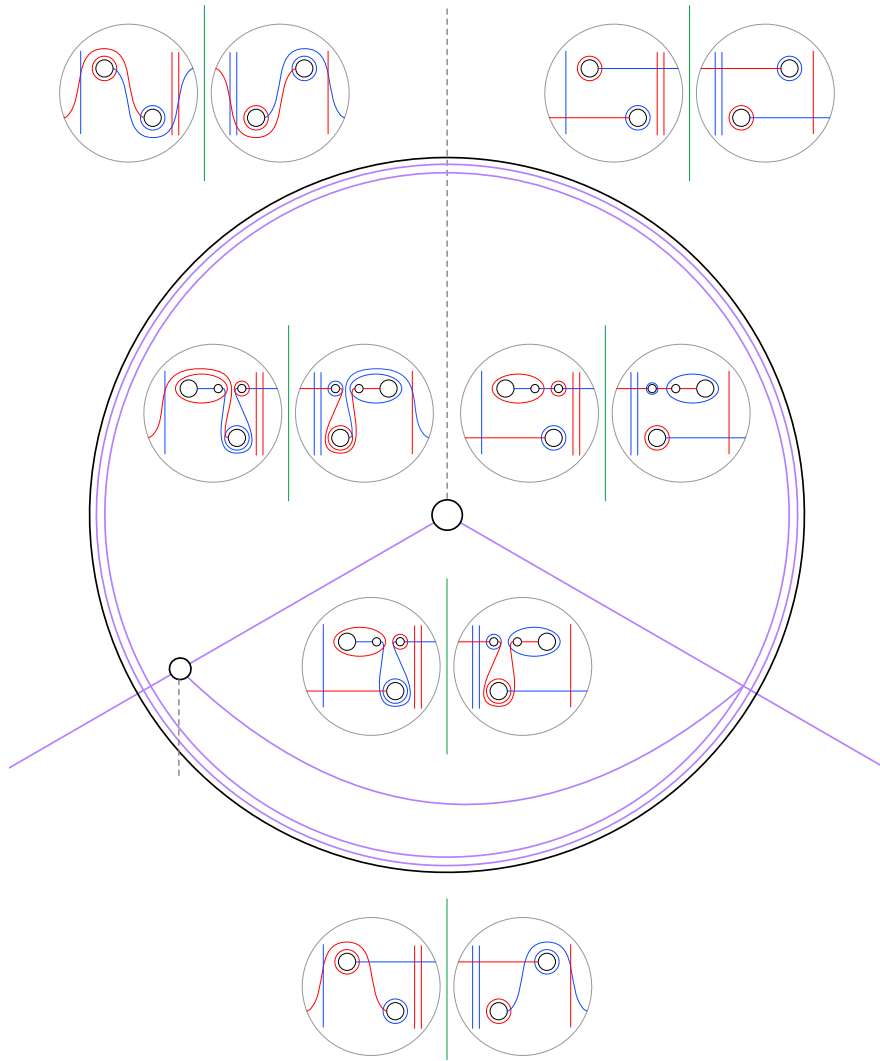


FIGURE A.23. Reducing to a pair of unreal handleswaps. Compare with [JTZ21, Figure 54].

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