Asymmetry Helps: Eigenvalue and Eigenvector Analyses of Asymmetrically Perturbed Low-Rank Matrices

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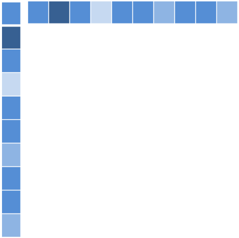
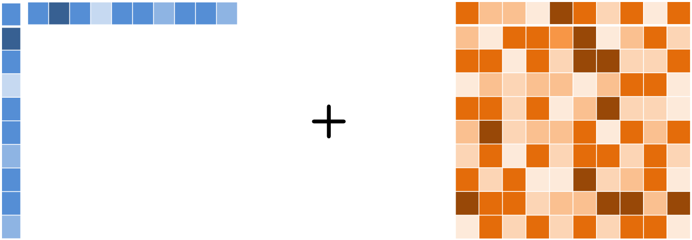
Eigenvalue / eigenvector estimation



$\boldsymbol{M}^\star$: truth

- A rank-1 matrix: $\boldsymbol{M}^\star = \lambda^\star \boldsymbol{u}^\star \boldsymbol{u}^{\star\top} \in \mathbb{R}^{n \times n}$

Eigenvalue / eigenvector estimation



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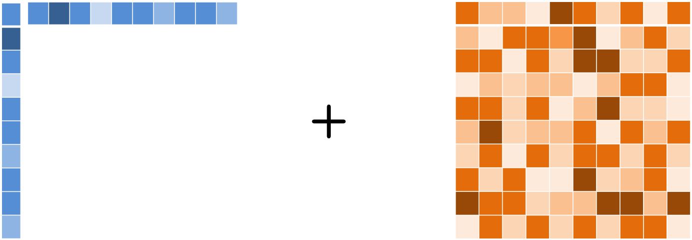
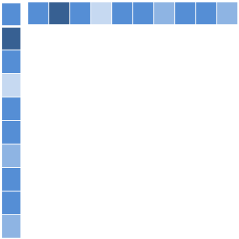


$M^\star$: truth $\qquad$ H: noise

- A rank-1 matrix: $M^\star = \lambda^\star u^\star u^{\star\top} \in \mathbb{R}^{n\times n}$
- Observed noisy data: $M = M^\star + H$
- **Goal:** estimate eigenvalue $\lambda^\star$ and eigenvector $u^\star$

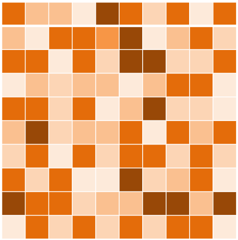
Non-symmetric noise matrix

$\boldsymbol{M} =$



$\boldsymbol{M}^\star = \lambda^\star \boldsymbol{u}^\star \boldsymbol{u}^{\star\top}$

$+$

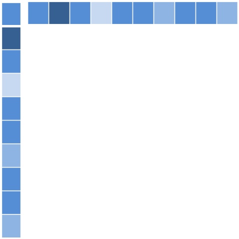


$\boldsymbol{H}$: *asymmetric* matrix

This may arise when, e.g., we have 2 samples for each entry of $\boldsymbol{M}^\star$ and arrange them in an asymmetric manner

A natural estimation strategy: SVD

$$\boldsymbol{M} =$$



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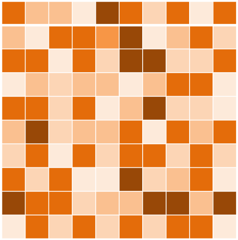
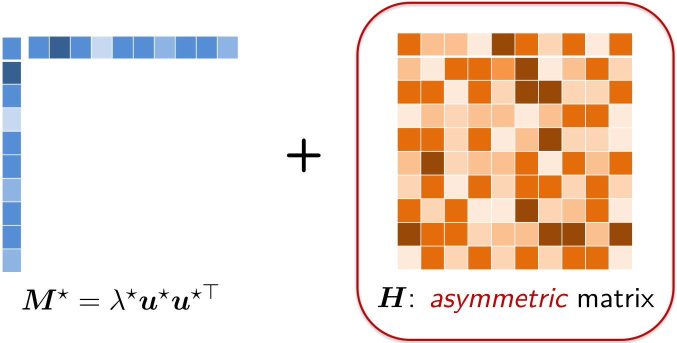


$\boldsymbol{H}$: *asymmetric* matrix

- Use leading singular value λ^{svd} of $\boldsymbol{M}$ to estimate $\lambda^\star$
- Use leading left singular vector of $\boldsymbol{M}$ to estimate $\boldsymbol{u}^\star$

A less popular strategy: eigen-decomposition

$$M = M^\star + H$$



- Use leading ~~singular value λ^{svd}~~ eigenvalue λ^{eigs} of M to estimate $\lambda^\star$
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SVD vs. eigen-decomposition

For *asymmetric* matrices:

- Numerical stability

$$\text{SVD} \quad > \quad \text{eigen-decomposition}$$

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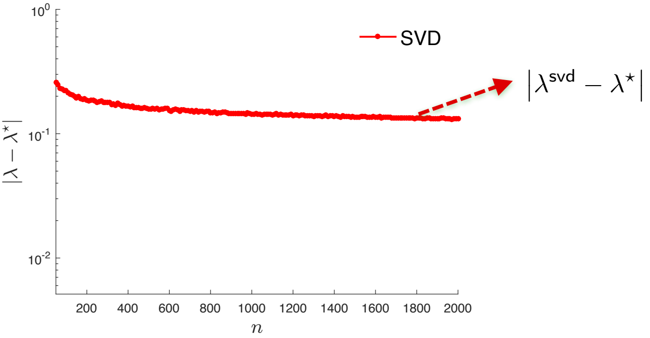
Shall we always prefer SVD over eigen-decomposition?

A curious numerical experiment: Gaussian noise

$$M = \underbrace{u^\star u^{\star\top}}_{M^\star} + H; \qquad \{H_{i,j}\}: \text{ i.i.d. } \mathcal{N}(0,\sigma^2), \sigma = \frac{1}{\sqrt{n \log n}}$$



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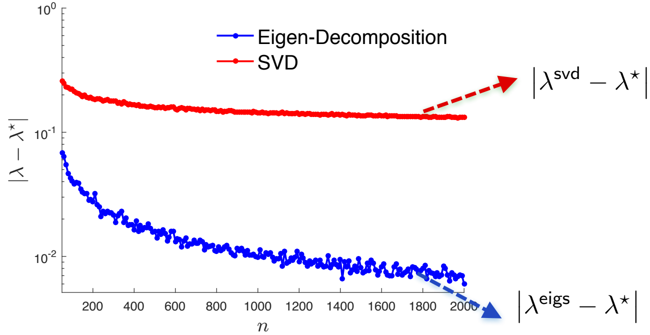

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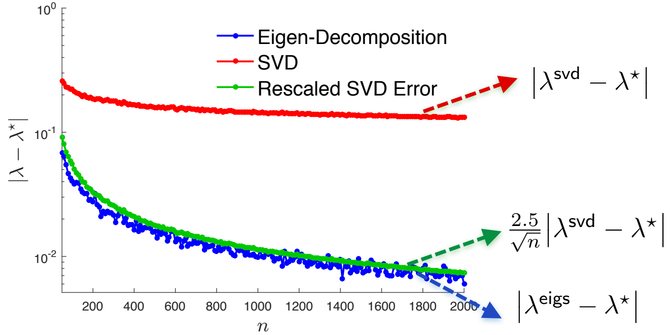

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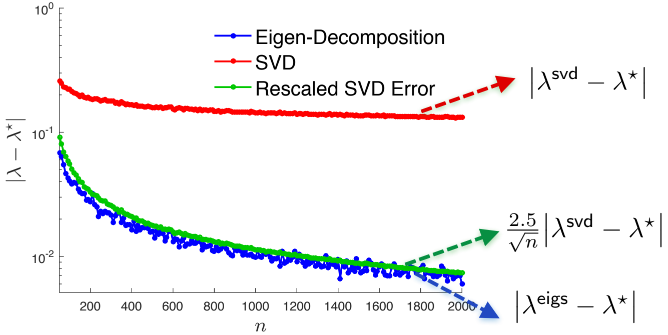

empirically, $|\lambda^{\mathsf{eigs}} - \lambda^\star| \approx \frac{2.5}{\sqrt{n}}|\lambda^{\mathsf{svd}} - \lambda^\star|$

Another numerical experiment: matrix completion

$$\boldsymbol{M}^\star = \boldsymbol{u}^\star \boldsymbol{u}^{\star\top}; \qquad M_{i,j} = \begin{cases} \frac{1}{p} M^\star_{i,j} & \text{with prob. } p, \\ 0, & \text{else,} \end{cases} \qquad p = \frac{3\log n}{n}$$

$$\begin{bmatrix} \checkmark & ? & ? & ? & \checkmark & ? \\ ? & ? & \checkmark & \checkmark & ? & ? \\ \checkmark & ? & ? & \checkmark & ? & ? \\ ? & ? & \checkmark & ? & ? & \checkmark \\ \checkmark & ? & ? & ? & ? & ? \\ ? & \checkmark & ? & ? & \checkmark & ? \end{bmatrix}$$

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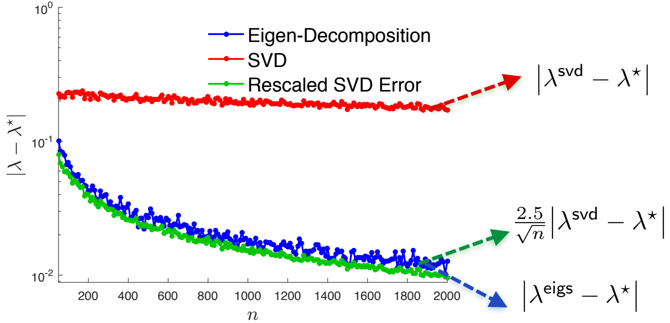

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Why does eigen-decomposition work so much better than SVD?

Problem setup

$$\boldsymbol{M} = \underbrace{\boldsymbol{u}^\star \boldsymbol{u}^{\star\top}}_{\boldsymbol{M}^\star} + \boldsymbol{H} \in \mathbb{R}^{n\times n}$$

- $\boldsymbol{H}$: noise matrix
    - **independent entries:** $\{H_{i,j}\}$ are independent
    - **zero mean:** $\mathbb{E}[H_{i,j}] = 0$
    - **variance:** $\mathsf{Var}(H_{i,j}) \leq \sigma^2$
    - **magnitudes:** $\mathbb{P}\{|H_{i,j}| \geq B\} \lesssim n^{-12}$

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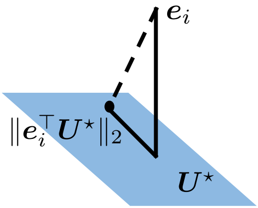
- $\boldsymbol{M}^\star$ obeys incoherence condition

$$\max_{1\leq i\leq n} \left|\boldsymbol{e}_i^\top \boldsymbol{u}^\star\right| \leq \sqrt{\frac{\mu}{n}}$$



Classical linear algebra results

$$\begin{aligned} \left|\lambda^{\mathsf{svd}} - \lambda^\star\right| &\le \|\boldsymbol{H}\| && \text{(Weyl)} \\ \left|\lambda^{\mathsf{eigs}} - \lambda^\star\right| &\le \|\boldsymbol{H}\| && \text{(Bauer-Fike)} \end{aligned}$$

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$$\Downarrow \quad \text{matrix Bernstein inequality}$$

$$|\lambda^{\mathsf{svd}} - \lambda^\star| \lesssim \sigma\sqrt{n\log n} + B\log n$$

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Main results: eigenvalue perturbation

Theorem 1 (Chen, Cheng, Fan '18)

With high prob., leading eigenvalue λ^{eigs} *of* $\boldsymbol{M}$ *obeys*

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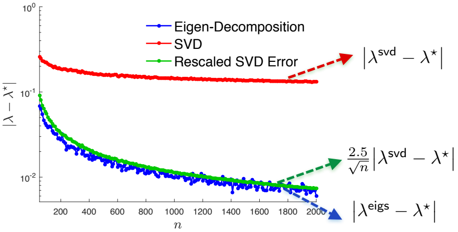

- Eigen-decomposition is $\sqrt{\frac{n}{\mu}}$ *times better* than SVD!

  — recall $\left|\lambda^{\mathsf{svd}} - \lambda^\star\right| \lesssim \sigma\sqrt{n\log n} + B\log n$

Main results: entrywise eigenvector perturbation

Theorem 2 (Chen, Cheng, Fan '18)

With high prob., leading eigenvector $\boldsymbol{u}$ of $\boldsymbol{M}$ obeys

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- entrywise eigenvector perturbation is well-controlled

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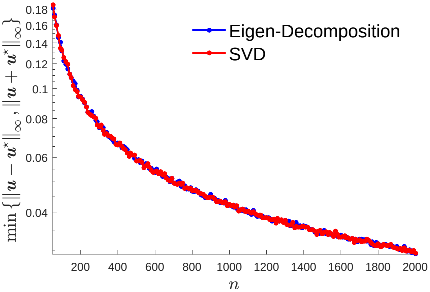

$\{H_{i,j}\}$: i.i.d. $\mathcal{N}(0, \sigma^2)$; $\sigma^2 = \frac{1}{n \log n}$

Main results: perturbation of linear forms of eigenvectors

Theorem 3 (Chen, Cheng, Fan '18)

Fix any unit vector $\boldsymbol{a}$*. With high prob., leading eigenvector* $\boldsymbol{u}$ *of* $\boldsymbol{M}$ *obeys*

$$\min\left\{\left|\boldsymbol{a}^\top(\boldsymbol{u} \pm \boldsymbol{u}^\star)\right|\right\} \lesssim \max\left\{\left|\boldsymbol{a}^\top\boldsymbol{u}^\star\right|, \sqrt{\frac{\mu}{n}}\right\}\left(\sigma\sqrt{n\log n} + B\log n\right)$$

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- perturbation of an *arbitrary* linear form of leading eigenvector is well-controlled

Intuition: asymmetry reduces bias

From $\underbrace{\text{Neumann series}}_{\text{some sort of Taylor expansion}}$ one can verify

$$|\lambda - \lambda^\star| \asymp \left| \frac{\boldsymbol{u}^{\star\top} \boldsymbol{H} \boldsymbol{u}^\star}{\lambda} + \frac{\boldsymbol{u}^{\star\top} \boldsymbol{H}^2 \boldsymbol{u}^\star}{\lambda^2} + \frac{\boldsymbol{u}^{\star\top} \boldsymbol{H}^3 \boldsymbol{u}^\star}{\lambda^3} + \cdots \right|$$

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To develop some intuition, let's look at 2nd order term

- if $\boldsymbol{H}$ is symmetric,

$$\mathbb{E}[\boldsymbol{u}^{\star\top} \boldsymbol{H}^2 \boldsymbol{u}^\star] = \mathbb{E}[\|\boldsymbol{H}\boldsymbol{u}^\star\|_2^2] = n\sigma^2$$

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- if $\boldsymbol{H}$ is asymmetric,

$$\underbrace{\mathbb{E}[\boldsymbol{u}^{\star\top}\boldsymbol{H}^2\boldsymbol{u}^\star] = \mathbb{E}[\langle \boldsymbol{H}^\top\boldsymbol{u}^\star, \boldsymbol{H}\boldsymbol{u}^\star\rangle] = \sigma^2}_{\text{much smaller than symmetric case}}$$

What happens if $M^\star$ is also not symmetric?

- A rank-1 matrix: $\boldsymbol{M}^\star = \lambda^\star \boldsymbol{u}^\star \boldsymbol{v}^{\star\top} \in \mathbb{R}^{n_1 \times n_2}$
- Suppose we observe 2 independent noisy copies

$$\boldsymbol{M}_1 = \boldsymbol{M}^\star + \boldsymbol{H}_1, \qquad \boldsymbol{M}_2 = \boldsymbol{M}^\star + \boldsymbol{H}_2$$

- **Goal:** estimate $\lambda^\star$, $\boldsymbol{u}^\star$ and $\boldsymbol{v}^\star$

Asymmetrization + dilation

Compute leading eigenvalue / eigenvector of

$$\begin{bmatrix} \mathbf{0} & \boldsymbol{M}_1 \\ \boldsymbol{M}_2^\top & \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \boldsymbol{M}^\star + \boldsymbol{H}_1 \\ \boldsymbol{M}^{\star\top} + \boldsymbol{H}_2^\top & \mathbf{0} \end{bmatrix}$$

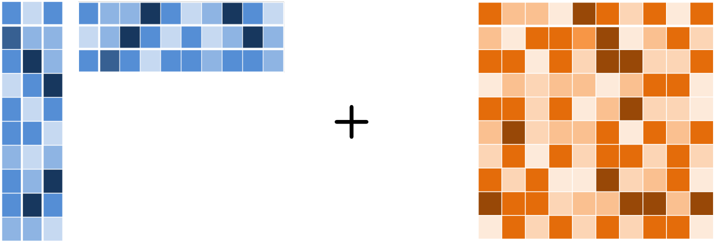
- Our findings (eigenvalue / eigenvector perturbation) continue to hold for this case!

Rank-r case

$\boldsymbol{M}^{\star}$: truth

- A rank-r and well-conditioned matrix: $\boldsymbol{M}^{\star} = \sum_{i=1}^{r} \lambda_i^{\star} \boldsymbol{u}_i^{\star} \boldsymbol{u}_i^{\star\top}$
- Observed noisy data: $\boldsymbol{M} = \boldsymbol{M}^{\star} + \boldsymbol{H}$, where $\{H_{i,j}\}$ are independent
- **Goal:** estimate $\lambda^{\star}$

Rank-r case



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Eigenvalue perturbation: rank-r case

Theorem 4 (Chen, Cheng, Fan '18)

With high prob., ith largest eigenvalue λ_i $(1 \leq i \leq r)$ of $\boldsymbol{M}$ obeys

$$|\lambda_i - \lambda_j^\star| \lesssim \sqrt{\frac{\mu r^2}{n}}\left(\sigma\sqrt{n \log n} + B \log n\right)$$

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- Eigen-decomposition is $\sqrt{\frac{n}{\mu r^2}}$ times better than SVD!
- Might be improvable to $\sqrt{\frac{\mu r}{n}}\left(\sigma\sqrt{n \log n} + B \log n\right)$?

Concluding remarks

Eigen-decomposition could be much more powerful than SVD when dealing with non-symmetric data matrices

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Future directions:

- Eigenvector perturbation for rank-r case
- Beyond i.i.d. noise

Y. Chen, C. Cheng, J. Fan, "Asymmetry helps: Eigenvalue and eigenvector analyses of asymmetrically perturbed low-rank matrices", arXiv:1811.12804, 2018