Mathematical Control Theory: Deterministic Finite-Dimensional Systems,

by Eduardo D. Sontag

reviewed by Stephen P. Boyd for IEEE Transactions Automatic Control

The title of this book gives a very good description of its contents and style, although I might have added "Introduction to" at the beginning. The style is mathematical: precise, clear statements (*i.e.*, theorems) are asserted, then carefully proved. The book covers many of the key topics in control theory, except — as the subtitle has warned us — those involving stochastic processes or infinite-dimensional systems. The level is appropriate for a senior undergraduate majoring in Mathematics (I'll say more about this below) or a graduate student in Electrical Engineering who has been exposed to the style of Mathematics (in, say, a first course on analysis or algebra). This book fills an important niche, and I think can play a key role in increasing the awareness and appreciation of control theory among mathematicians.

I will start with the traditional list of topics. The first chapter is a nicely written extended discussion about what control theory is. Sontag introduces some of the key concepts (*e.g.*, system, feedback, state-space, and nonlinearity) via simple examples, using an informal style. For an engineering graduate student it will be a review of some undergraduate courses; for the mathematics student it provides some cultural background for the rest of the book. This chapter is not really in the same style as the remainder of the book, so I might have numbered it Chapter 0. There are no theorems or proofs, and Sontag keeps the discussion at the level of informal ideas, for example (p5) "Designs based on linearizations work locally for the original system". Pity the mathematics student who then searches backwards for the precise definition of the terms "design" and "work"!

The book proper starts at Chapter 2, which covers the abstract idea of a system. The style jumps from the informality of Chapter 1 to a fairly heavy barage of long mathematical definitions. Even Sontag admits the chapter contains some "abstract nonsense" (p.ix), *e.g.*, very general and abstract definitions of a dynamical system. On this topic he points out a very good analogy from the foundations of mathematics. A student of mathematics first thinks of a function in an informal way, as as a "mapping" or "subroutine". The student then encounters the formal definition of a function from A to B as a subset of $A \times B$ that satisfies certain properties. Not long thereafter the student reverts to the more informal model of a function as a "mapping" or "subroutine". The difference is that now, the student really knows what a function is! In a similar way, chapter 2 introduces the student to the formal definition of a dynamical system. After this introduction, the student can revert to a less formal idea of a dynamical system (*e.g.*, "the state summarizes the effects of the past inputs on the future outputs"). But now, these ideas rest on a firm footing. Sontag punctuates the heavy definitions with nice examples, which makes the medicine of a formal foundation easier to take.

Chapter 3 covers various topics involving reachability and controllability. Of course Sontag examines these topics in some depth for the important special case of linear, time-invariant (LTI) systems. My feeling is that the treatment might be a bit brief for the student who has had no previous exposure to LTI systems. This is no problem for the engineering graduate student, who will surely have had one or possibly two courses that use these ideas.

Chapter 4 covers (state) feedback, with a bit on Lyapunov stability. In this chapter Sontag gives a precise statement of the linearization principle described informally in chapter 1 (and quoted above). Chapter 5 covers the dual notion of observability, as well as an introduction to realization theory and minimality. Although §5.8, Abstract Realization Theory*, is marked as skippable, it shows the advantage of the abstract approach: generality. As an example Sontag constructs a minimal realization of a parity check system.

In chapter 6 Sontag considers observers and dynamic feedback. He cannot give a detailed discussion of the design of observers (*e.g.*, the tradeoff between convergence rate and noise sensitivity) since he assumes no knowledge of stochastic processes, but he does give a nice deterministic version of the Kalman filter in the next (and final) chapter, which covers optimal control. Two sections in chapter 6 are, in my opinion, too brief for students who have never seen the topics before: Frequency-domain Considerations (§6.4) and Parametrization of Stabilizers (§6.5). I would have marked these sections as optional, or even decided to not cover them (as Sontag did with stochastic processes).

Chapter 7 covers optimal control, taking Bellman's dynamic programming approach. One of the advantages of this approach is that it applies very gracefully to more abstract systems, *e.g.*, finite state systems, so I would like to have seen an example of this type before launching into the traditional case of a linear system with quadratic cost. This chapter ends with a nice deterministic development of the Kalman filter. Here the goal is to find the initial state that minimizes an integral quadratic measure of the difference between the output we have observed, and the output we would predict, given the initial state.

The appendices cover very useful material, *e.g.*, singular value decomposition. It is a pity that some of these topics are not covered in the traditional mathematics curriculum.

The book contains about 400 references. A mere list of references does not have great added value in these days of computerized literature searches. Sontag, however, ends each chapter with a very useful Notes and Comments section with annotated pointers to the references. I noticed that the references seem to be concentrated in Western journals (*e.g.*, IEEE Trans. Autom. Control, Systems and Control Letters); only a handful are from the (former) Soviet Union, which has a very strong tradition in mathematical control theory tracing back to Pontriagin and indeed, Lyapunov. I must admit that this comment applies equally well to the books I have written.

I envision several uses for this book. It would work very well as the text for an undergraduate senior-level mathematics course. This course could replace, or at least complement, the tired traditional course on 19th century calculus of variations that we often find as a senior-level mathematics elective. The course could also serve graduate students in engineering.

The second use I see for the book is as a reference or supplementary text for engineering graduate students, particularly PhD students. These students have already been exposed to many of the ideas in Sontag's book in engineering-oriented courses. Often in their first year or two of PhD study they shore up their mathematics background by taking mathematics courses such as analysis, functional analysis, abstract algebra, and differential geometry. Sontag's book serves as an excellent bridge between the two disciplines: it covers topics traditionally treated in engineering courses, but in a mathematical style.

One similar book is Vidyasagar[Vid92], which is oriented more towards the advanced engineering student, and I think more appropriate for a engineering course on advanced control theory. The book by Delchamps[Del88] could serve as a secondary text with Sontag's book, since it covers more linear systems theory, also in a precise mathematical style. The book by Wonham[Won85] covers linear system and control theory in the most elegant mathematical style. It could serve as the text for a sequel to the introductory mathematical control theory course based on Sontag's book. The recent text by J. Zabczyk [Zab92], *Mathematical Control Theory: An Introduction,* published by Birkhauser in 1992, is similar to Sontag's book is style, level, and coverage. Zabczyk's choice of more advanced topics differs from Sontag's: Zabczyk includes more on optimal control as well as a fairly complete introduction to infinite-dimensional systems.

Sontag clearly put much thought and effort into this book, and it shows. The book succeeds in conveying the important basic ideas of mathematical control theory, with appropriate level and style, to seniors in mathematics.

References

[Del88] D. F. Delchamps. *State-Space and Input-Output Linear Systems.* Springer-Verlag, New York, 1988.

[Son90] E. D. Sontag. *Mathematical Control Theory*, volume 6 of *Texts in Applied Mathematics*. Springer-Verlag, 1990.

[Vid92] M. Vidyasagar. *Nonlinear Systems Analysis*. Prentice-Hall, Englewood Cliffs, second edition, 1992.

[Won85] W. M. Wonham. *Linear Multivariable Control.* Springer-Verlag, New York, 1985.

[Zab92] J. Zabczyk. *Mathematical Control Theory: An Introduction*. Systems & Control: Foundations & Applications. Birkhauser, 1992.