Computation of the Worst-Case Covariance for Linear Systems with Uncertain Parameters

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Abstract

For a class of linear systems with unknown parameters that lie in intervals, we present a branch and bound algorithm for computing the worst-case covariance of the state.

1 Introduction

We consider the family of linear time-invariant systems described by

$$\begin{array}{rcl} \dot{x} & = & Ax + B_u u + B_w w, \quad x(0) = 0, \\ y & = & C_y x, \\ z & = & C_z x, \\ u & = & \Delta\, y, \end{array} \tag{1}$$

where $x(t) \in \mathbf{R}^n$, $w(t) \in \mathbf{R}^{n_i}$, $z(t) \in \mathbf{R}^{n_o}$, $u(t), y(t) \in \mathbf{R}^p$, and A, B_u, B_w, C_y and C_z are real matrices of appropriate sizes. Δ is a diagonal matrix, parametrized by a vector of parameters $q = [q_1, q_2, \ldots, q_m]$, and is given by

$$\Delta = \mathrm{diag}(q_1 I_1, q_2 I_2, \ldots, q_m I_m), \tag{2}$$

where I_i is an identity matrix of size p_i. Of course, $\sum_i^m p_i = p$. The rectangle in which q lies is given by $\mathcal{Q}_{\rm init} = [l_1, u_1] \times [l_2, u_2] \times \cdots \times [l_m, u_m]$.

Eliminating u and y from equations (1) yields the closed-loop system equations:

$$\begin{array}{rcl} \dot{x} & = & \mathcal{A}(q)x + B_w w, \\ z & = & C_z x, \end{array} \tag{3}$$

where $\mathcal{A}(q) = A + B_u \Delta C_y$. We note that the entries of $\mathcal{A}(q)$ are *affine functions* of the parameter vector q.

Loosely speaking, the above framework describes a class of linear systems with fixed, unknown gains that lie in intervals. Many important questions arise for such systems: robust stability, stability margin, minimum stability degree *etc.* (see [1] for a brief discussion of such questions). In this paper, we will describe the computation of the of largest possible trace of the state covariance, when w is unit-intensity white noise, *i.e.*,

$$\mathcal{C}(\mathcal{Q}_{\rm init}) = \max_{q \in \mathcal{Q}_{\rm init}} \lim_{t \to \infty} \mathrm{Tr}\ \mathbf{E}\ \ x_q(t) x_q(t)^T, \tag{4}$$

where x_q is the solution to the state equations corresponding to the parameter vector q, $\mathbf{E}$ stands for the expected value and $\mathrm{Tr}\, M$ is the trace (sum of diagonal entries) of the square matrix M. We assume that the system (1) is robustly stable, that is $\mathcal{A}(q)$ has eigenvalues with negative real part for all $q \in \mathcal{Q}_{\rm init}$. For convenience, we let $X(q) = \lim_{t\to\infty} \mathbf{E}\ \ x_q(t)^T x_q(t)$.

For a fixed value of q, $X(q)$ can be computed as the unique solution to the Lyapunov equation

$$\mathcal{A}(q)X(q) + X(q)\mathcal{A}(q)^T + B_w B_w^T = 0. \tag{5}$$

We may therefore rewrite equation (4) as

$$\mathcal{C}(\mathcal{Q}_{\rm init}) = \max_{q \in \mathcal{Q}_{\rm init}} \left\{ \mathrm{Tr}\ (X(q)) \;\middle|\; \begin{array}{r} \mathcal{A}(q)X(q) + X(q)\mathcal{A}(q)^T \\ + B_w B_w^T = 0 \end{array} \right\}.$$

$\mathcal{C}(\mathcal{Q}_{\rm init})$ is the maximum possible sum of the covariance of the state components when the system is driven by unit-intensity white noise, and serves as a measure of the robustness of the system.

There are no known analytic methods that compute $\mathcal{C}(\mathcal{Q}_{\rm init})$ exactly. However, for any rectangle $\mathcal{Q}$, it is possible to compute upper and lower bounds for $\mathcal{C}(\mathcal{Q})$. These bounds may be used with a branch and bound technique to compute $\mathcal{C}(\mathcal{Q}_{\rm init})$ to within any given accuracy $\epsilon > 0$. We first describe a branch and bound algorithm, and then describe the computation of simple upper and lower bounds for $\mathcal{C}(\mathcal{Q})$.

2 The Branch and Bound Algorithm

The branch and bound algorithm we present here is a minor variation on the one presented in [2]. It finds the maximum of a function $f : \mathbf{R}^m \to \mathbf{R}$ over an m-dimensional rectangle $\mathcal{Q}_{\rm init}$ (the subscript "init" stands for *initial* rectangle).

For a rectangle $\mathcal{Q} \subseteq \mathcal{Q}_{\rm init}$ we define

$$\Phi_{\max}(\mathcal{Q}) = \max_{q \in \mathcal{Q}} f(q).$$

Then, the algorithm computes $\Phi_{\max}(\mathcal{Q}_{\rm init})$ to within an absolute accuracy of $\epsilon > 0$, using two functions $\Phi_{\rm lb}(\mathcal{Q})$ and $\Phi_{\rm ub}(\mathcal{Q})$ defined over $\{\mathcal{Q} : \mathcal{Q} \subseteq \mathcal{Q}_{\rm init}\}$ (which, presumably, are easier to compute than $\Phi_{\max}(\mathcal{Q})$). These two functions must satisfy the following conditions:

(R1) $\Phi_{\rm lb}(\mathcal{Q}) \le \Phi_{\max}(\mathcal{Q}) \le \Phi_{\rm ub}(\mathcal{Q})$.

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