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Structures for Nonlinear Systems*

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1. Introduction

A *Volterra Series Operator* is one of the form

$$Nu(t) = h_0 + \sum_{n=1}^{\infty} \int \cdots \int h_n(\tau_1, \ldots, \tau_n) u(t-\tau_1) \ldots u(t-\tau_n) d\tau_1 \ldots d\tau_n$$

and is a generalization of the convolution description of linear time-invariant (LTI) operators to time-invariant (TI) nonlinear operators. The usefulness of Volterra series hinges on their ability to model a very wide class of nonlinear operators. Two general approaches can be taken to establish this.

First, one can demonstrate that many explicitly described systems have input/output (I/O) operators given by Volterra series. Sandberg[1] has established that a wide class of systems have I/O operators which are given by Volterra series, the requirement being, roughly speaking, that the nonlinearities are *analytic*. Thus an op-amp (with transistors modeled by the Ebers-Moll equations, which are analytic) has an I/O operator expressable, at least for small inputs, as a Volterra series.

But many common nonlinear systems are modeled with non-analytic nonlinearities. For example the I/O operator of a control system containing an ideal *saturator*, that is, a memoryless nonlinearity with characteristic

$$\mathrm{SAT}(a) \triangleq \begin{cases} \mathrm{sign}(a) & |a| \geq 1 \\ a & |a| \leq 1 \end{cases}$$

(which of course is not analytic) can easily be shown *not* to have a Volterra series representation valid for any inputs for which the saturator threshold is exceeded. One could reasonably argue that even though the I/O operator of such a control system does not have an *exact* representation as a Volterra series operator, it could be *approximated* by one, for example by replacing the saturator with a polynomial approximation. But exactly what do we mean by *approximate* here? This is one of the questions addressed in this paper.

The second approach to establishing the generality of Volterra series is *axiomatic* in style, and conceptually more satisfying. Here one demonstrates that under only a few physically reasonable assumptions about an operator N (such as causality, time-invariance, and some form of continuity) there is a Volterra series operator $\tilde{N}$ which approximates, in some sense, N. No assumption whatever is made concerning the internal structure or realization of N.

The idea of such an approximation is not new, and in fact is discussed in the original work of Volterra[2], who cites Frechet[3]. Even in this early work one can find the basic idea (clouded by archaic mathematics): there is an analogy between ordinary polynomials and finite Volterra series, and hence some analog of the Weierstrass approximation theorem should apply to approximating general nonlinear operators with finite Volterra series.

Wiener rekindled interest in this problem at MIT in the forties and fifties, [4,5,6] and since then various researchers have considered the problem.[7,8,9,10] A clear discussion of a typical approximation result can be found on pages 34-37 of Rugh's book[11]. Roughly speaking, all of this work has the following problems:

(1) The input space is usually $L^2[0,T]$.
(2) The approximation is always on a *compact* subset of the input space,
(3) The approximation only holds over a finite time interval $[0,T]$.

While demonstrating that Volterra series operators can, at least in a very weak sense, approximate a general causal time-invariant continuous operator, these results are not really satisfying. The choice of L^2 as input space seems more a mathematical convenience than a realistic engineering idealization of actual input signals. (1), (2) and (3) are severe restrictions: we would really like an approximation which allows input signals defined on infinite time intervals and which approximate the general operator over an *infinite time interval*. (1)-(3) preclude, for example, periodic forcing signals. Rugh concludes his discussion with the following comments concerning (2): "...I should point out that the main drawback is in the restrictive input space U. The compactness requirement rules out many of the more natural choices for U."

We will demonstrate that all of these drawbacks can be overcome if the usual continuity assumption on N is strengthened slightly to ensure that N has *fading memory*. The proof is very simple.

2. Fading Memory

The concept of fading memory has a history at least as long as Volterra series themselves. Indeed we find it in Volterra[2, p188]:

> *A first extremely natural postulate is to suppose that the influence of the (input) a long time before the given moment gradually fades out.*

and in Wiener[4, p89]:

> *We are assuming (the output) of the network does not depend on the infinite past. If the response of this apparatus depends on the remote past, then the Brownian motion is not a good approximation because we shall always have to consider the remote past. So we are considering networks in which the output is asymptotically independent of the remote past input...*

and in various other work over the years.[12,5] The fading memory assumption, then, is by no means a new stronger restriction on the operators to be approximated. It is simply an old assumption whose full power has not, until now, been used.

First some notation: $C(R)$ will denote the space of bounded continuous functions on R, with the usual norm $\|u\| \triangleq \sup_{t \in R} |u(t)|$. U_τ will denote the τ-second *delay operator* defined by

$$(U_\tau u)(t) \triangleq u(t-\tau)$$

Definition: N has *Fading Memory* (FM) on a subset K of $C(R)$ if there is a weighting function $w: R_+ \to R_+$, $\lim_{t\to\infty} w(t) = 0$, such that for each $u \in K$ and $\varepsilon > 0$ there is a $\delta > 0$ such that for all $v \in K$

$$\sup_{t \leq 0} |u(t) - v(t)| w(-t) < \delta \;\rightarrow\; |Nu(0) - Nv(0)| < \varepsilon$$

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For example if $w(t)=e^{-\lambda t}$ then we might say N has a λ-exponentially fading memory on K.

3. Approximation Theorem

Approximation Theorem: Let $\varepsilon > 0$ and

$$K \triangleq \left\{ u \;\middle|\; \|u\| \le M_1,\ \|U_\tau u - u\| \le M_2 \tau \right\}$$

Suppose that N is *any* TI operator with fading memory on K. Then there is a finite Volterra series operator $\tilde{N}$ such that for all $u \in K$

$$\|Nu - \tilde{N}u\| < \varepsilon$$

Remark 1: The assumption on N is extremely weak. As mentioned earlier, it does not in any way concern the internal structure or realization of N. For example N could arise from a nonlinear PDE, but even this is not necessary.

Remark 2: K can be described as those signals bounded by M_1 and having Lipschitz constant M_2, that is, *slew-limited* by M_2.†

Remark 3: The signals in K are not "time-limited" (i.e. zero outside of some interval such as $[0,T]$), and the approximation $|Nu(t)-\tilde{N}u(t)| < \varepsilon$ holds *for all* $t \in R$, not just in some interval $[0,T]$.

Remark 4: K is *not* a compact subset of $C(R)$!

Remark 5: In the discrete time case, we can remove the slew limit, that is, K can simply be any ball in l^∞.

4. A Final Comment

The approximating Volterra series $\tilde{N}$ can be realized as a finite-dimensional *linear* dynamical system with a *nonlinear* readout map, that is, by a Wiener structure. This has implications for macro-modeling complicated or large-scale nonlinear systems with such a structure (cf. deFigueiredo and Dwyer[13]). Full details, proofs, and discussion can be found in a forthcoming paper[14].

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