Convex Optimization in Quantitative Finance

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Overview

convex optimization problems
▶ are a special type of mathematical optimization problem
▶ can be efficiently solved
▶ are easily specified using domain specific languages such as CVXPY
▶ can be used to solve a wide variety of problems arising in finance

these slides give many examples in finance
▶ our examples are simplified, but readily extended
▶ we give code snippets for all of them
▶ full code is available at https://github.com/cvxgrp/cvx-finance-examples
Outline

Convex optimization
Markowitz portfolio construction
Maximum expected utility portfolio construction
Sparse inverse covariance estimation
Worst-case risk analysis
Option pricing
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Optimal execution
Optimal consumption
Alternative investment planning
Blending forecasts
Bond pricing
Model predictive control
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Optimization problem

minimize \( f_0(x) \)
subject to \( f_i(x) \leq 0, \quad i = 1, \ldots, m \)
\( g_i(x) = 0, \quad i = 1, \ldots, p \)

- \( x \in \mathbb{R}^n \) is (vector) variable to be chosen
- \( f_0 \) is the objective function, to be minimized
- \( f_1, \ldots, f_m \) are the inequality constraint functions
- \( g_1, \ldots, g_p \) are the equality constraint functions

- variations: maximize objective, multiple objectives, …
Convex optimization problem

\[
\begin{align*}
\text{minimize} & \quad f_0(x) \\
\text{subject to} & \quad f_i(x) \leq 0, \quad i = 1, \ldots, m \\
& \quad Ax = b
\end{align*}
\]

▷ variable \( x \in \mathbb{R}^n \)

▷ equality constraints are linear

▷ \( f_0, \ldots, f_m \) are convex: for \( \theta \in [0, 1] \),

\[
f_i(\theta x + (1 - \theta)y) \leq \theta f_i(x) + (1 - \theta)f_i(y)
\]

i.e., \( f_i \) have nonnegative (upward) curvature

▷ variations: maximize concave objective, multiple convex objectives, \ldots
for convex optimization problems there are

- effective algorithms
  - get **global solution** (and optimality certificate)
  - theory: polynomial complexity
  - practice: fast and reliable (no need to tune parameters)
  - many open source and commercial implementations

- many applications in machine learning, signal processing, statistics, control, engineering design, and **finance**
Modeling languages

- high level language support for convex optimization
  - describe problem in high level language
  - simple syntax rules to certify problem convexity
  - description automatically transformed to a standard form
  - solved by standard solver, transformed back to original form

- implementations:
  - CVXPY (Python)
  - YALMIP, CVX (Matlab)
  - Convex.jl (Julia)
  - CVXR (R)

- can be coupled with open source or commercial solvers
- work well for problems up to around 100k variables
CVXPY

a modeling language in Python for convex optimization

- developed since 2014
- open source all the way to the solvers
- syntax very similar to NumPy
- used in many research projects, courses, companies
- tens of thousands of users, including many in finance
- over 27,000,000 downloads on PyPI
- many extensions available
Example

regularized least squares problem with bounds:

\[
\begin{align*}
\text{minimize} & \quad \|Ax - b\|^2_2 + \gamma \|x\|_1 \\
\text{subject to} & \quad \|x\|_{\infty} \leq 1
\end{align*}
\]

CVXPY specification:

```python
import cvxpy as cp
x = cp.Variable(n)
cost = cp.sum_squares(A@x-b) + gamma*cp.norm(x,1)
prob = cp.Problem(cp.Minimize(cost),[cp.norm(x,"inf")<=1])
opt_val = prob.solve()
solution = x.value
```

- A, b, gamma are constants, gamma nonnegative
- solve method converts problem to standard form, solves, assigns value attributes
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Mean-variance (Markowitz) optimization

maximize \( \mu^T w \)
subject to \( w^T \Sigma w \leq (\sigma^{\text{tar}})^2, \quad 1^T w = 1 \)

▶ variable \( w \in \mathbb{R}^n \) of portfolio weights
▶ \( \mu \in \mathbb{R}^n \) and \( \Sigma \in \mathbb{S}_+^n \) are asset return mean and covariance
▶ \( \sigma^{\text{tar}} \) is target (per period) volatility
▶ basic form goes back to [Markowitz, 1952]

```python
w = cp.Variable(n)
objective = mu.T @ w
constraints = [cp.quad_form(w, Sigma) <= sigma**2, cp.sum(w) == 1]
prob = cp.Problem(cp.Maximize(objective), constraints)
prob.solve()
```
Adding practical constraints and objective terms

- include cash holdings $c$, previous holdings $w^{\text{pre}}$, trades $z = w - w^{\text{pre}}$
- account for (convex) holding costs $\phi^{\text{hold}}$ and trading costs $\phi^{\text{trade}}$
- limit weights, cash, trades, turnover $T = \|z\|_1$, and leverage $L = \|w\|_1$

$$\begin{align*}
\text{maximize} & \quad \mu^T w - \gamma^{\text{hold}} \phi^{\text{hold}}(w, c) - \gamma^{\text{trade}} \phi^{\text{trade}}(z) \\
\text{subject to} & \quad 1^T w + c = 1, \quad z = w - w^{\text{pre}}, \\
& \quad w^{\min} \leq w \leq w^{\max}, \quad c^{\min} \leq c \leq c^{\max}, \quad L \leq L^{\text{tar}}, \\
& \quad z^{\min} \leq z \leq z^{\max}, \quad T \leq T^{\text{tar}}, \\
& \quad \|\Sigma^{1/2} w\|_2 \leq \sigma^{\text{tar}}
\end{align*}$$

- variation: soften constraints, *i.e.*, penalize violations
- can be implemented in around ten lines in CVXPY
- see [Boyd et al., 2024] for details and reference implementation
Factor covariance model

\[ \Sigma = F \Sigma^f F^T + D \]

- \( F \in \mathbb{R}^{n \times k} \) is matrix of factor loadings
- \( k \) is number of factors, typically with \( k \ll n \)
- \( \Sigma^f \) is \( k \times k \) factor covariance matrix
- \( D \) is diagonal matrix of unexplained (idiosyncratic) variances
- a strong regularizer which can give better return covariance estimates
Exploiting a factor model

- with factor model, cost of portfolio optimization reduced from $O(n^3)$ to $O(nk^2)$ flops [Boyd and Vandenberghe, 2004]
- easily exploited in CVXPY
- timings for Clarabel open source solver:

<table>
<thead>
<tr>
<th>assets $n$</th>
<th>factors $k$</th>
<th>solve time (s)</th>
<th>factor model</th>
<th>full covariance</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>10</td>
<td>0.002</td>
<td>0.002</td>
<td>0.040</td>
</tr>
<tr>
<td>300</td>
<td>20</td>
<td>0.010</td>
<td>0.010</td>
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<td>0.080</td>
<td>25.600</td>
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<tr>
<td>3000</td>
<td>50</td>
<td>0.600</td>
<td>0.600</td>
<td>460.000</td>
</tr>
</tbody>
</table>
Backtesting

- fast solve time enables backtesting of strategy variations
  - what-if analysis
  - sensitivity analysis
  - hyperparameter tuning
- with 1000 assets and 30 factors, we can backtest 3 years of daily trading in a minute
- in one hour, we can carry out 2000 3 year backtests on a 32-core machine
Robustifying Markowitz

- basic mean-variance optimization can be sensitive to estimation errors in $\mu$, $\Sigma$
- replace mean return $\mu^T w$ with worst-case return

$$R^{wc} = \min \{ (\mu + \delta)^T w \mid |\delta| \leq \rho \} = \mu^T w - \rho^T |w|$$

where $\rho \geq 0$ is vector of mean return uncertainties

- replace risk $w^T \Sigma w$ with worst-case risk

$$\left( \sigma^{wc} \right)^2 = \max \left\{ w^T (\Sigma + \Delta) w \mid |\Delta_{ij}| \leq \varrho (\Sigma_{ii} \Sigma_{jj})^{1/2} \right\}$$

$$= \sigma^2 + \varrho \left( \sum_{i=1}^{n} \frac{1}{2} \Sigma_{ii} |w_i| \right)^2$$

where $\varrho \geq 0$ gives covariance uncertainty

- easily handled by CVXPY
Example

- S&P 100, simulated but realistic $\mu$, target annualized risk 10%
- hyper-parameters tuned each year based on previous two years
- out-of-sample portfolio performance for basic Markowitz and robust Markowitz
- Sharpe ratios 0.2 and 4.6 (using the same mean and covariance)
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Expected utility maximization

- asset returns $r \in \mathbb{R}^n$; portfolio weights $w \in \mathbb{R}^n$
- portfolio return $r^T w$; wealth grows by factor $1 + r^T w$
- expected utility is $\mathbb{E} U(1 + r^T w)$, where $U$ is concave increasing utility function
- choose portfolio weights $w \in \mathcal{W}$ (a convex set) to maximize expected utility
- a convex optimization problem [Von Neumann and Morgenstern, 1947]
- reduces to mean-variance in some cases (e.g., exponential utility, Gaussian returns) [Markowitz and Blay, 2014; Luxenberg and Boyd, 2024]
- allows handling of options, nonlinear payoffs, ...
- with $U(x) = \log x$ we get Kelly gambling [Kelly, 1956]; maximizes wealth growth rate
Sample based approximation

- when $E(U(1 + r^T w))$ can’t be expressed analytically, use sample based approximation
- generate $N$ samples $r_1, \ldots, r_N$, with probabilities $\pi_1, \ldots, \pi_N$
- approximate expected utility as $E(U(1 + r^T w)) \approx \sum_{i=1}^{N} \pi_i U(1 + r_i^T w)$
- sample based approximate expected utility maximization:

\[
\begin{align*}
\text{maximize} & \quad \sum_{i=1}^{N} \pi_i U(1 + r_i^T w) \\
\text{subject to} & \quad w \in W
\end{align*}
\]

- easily handled by CVXPY
Sample based approximation in CVXPY

- returns $r_i$ are columns of $N \times n$ array returns
- probabilities $\pi$ are in array probabilities
- CRRA utility with relative risk aversion $\rho \geq 0$, $U(x) = (x^{1-\rho} - 1)/(1 - \rho)$

```python
def U(x):
    return (x**(1-rho) - 1)/(1-rho)

w = cp.Variable(n)

objective = probabilities @ U(1 + returns @ w)
constraints = [cp.sum(w) == 1]

prob = cp.Problem(cp.Maximize(objective), constraints)
prob.solve()```
Example

- optimize portfolio of one underlying, one call, and one put, both at-the-money
- underlying with \( 1 + r \) log-normal
- CRRA utility with relative risk aversion \( \rho \), \( \mathcal{W} = \{w \mid 1^T w = 1\} \)
- sample approximation with \( N = 10^5 \) samples

![Asset returns](image1)

![Portfolio return distributions](image2)
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Sparse inverse covariance estimation

- model return in period $t$ as $r_t \sim \mathcal{N}(0, \Sigma)$
- log-likelihood
  
  \[
  l_t(\theta) = \frac{1}{2} \left( -n \log(2\pi) + \log \det \theta - r_t^T \theta r_t \right),
  \]

  where $\theta = \Sigma^{-1}$ is the precision matrix
- sparse inverse covariance estimation problem [Friedman et al., 2007]
  
  \[
  \text{maximize} \quad \sum_{t=1}^{T} l_t(\theta) - \lambda \sum_{i<j} |\theta_{ij}|
  
  \text{subject to} \quad \theta \geq 0
  \]

  with variable $\theta$; $\lambda > 0$ is a (sparsity) regularization parameter
- a convex problem; yields matrix with sparse precision matrix $\theta$
- $\theta_{ij} = 0$ means returns $(r_t)_i, (r_t)_j$ are conditionally independent given the others
Sparse inverse covariance estimation in CVXPY

- log_likelihood is sum of log-likelihoods up to positive scaling and additive constant

```python
Theta = cp.Variable((n, n), PSD=True)

log_likelihood = cp.sum(cp.hstack([cp.log_det(Theta) - cp.quad_form(r, Theta) for r in returns]))

mask = np.triu(np.ones((n, n)), k=1).astype(bool)
ojective = log_likelihood - alpha * cp.norm1(Theta[mask])

prob = cp.Problem(cp.Maximize(objective))
prob.solve()
```
Example

- daily returns of US, Europe, Asia, and Africa stock indices from 2009 to 2024
- figure shows yearly sparsity pattern of inverse covariance; white boxes denote zero entries
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Worst-case portfolio risk

- we hold \( n \) assets with weights \( w \in \mathbb{R}^n, 1^T w = 1 \)
- variance of portfolio return is \( w^T \Sigma w \), where \( \Sigma \in \mathbb{R}^{n \times n} \) is the return covariance
- now suppose \( \Sigma \in \mathcal{S} \) but otherwise uncertain
- set of possible covariances \( \mathcal{S} \) is a convex set, e.g.,

\[
\mathcal{S} = \{ \Sigma \geq 0 \mid L_{ij} \leq \Sigma_{ij} \leq U_{ij}, \quad i, j = 1, \ldots, n \}
\]

where \( L \) and \( U \) are lower and upper bounds on entries
- the **worst-case variance** consistent with our belief \( \Sigma \in \mathcal{S} \) is

\[
\sigma_{wc}^2 = \sup \{ w^T \Sigma w \mid \Sigma \geq 0, \ \Sigma \in \mathcal{S} \}
\]
- evaluating \( \sigma_{wc}^2 \) is a convex optimization problem
weights denote portfolio weights
- \( L \) and \( U \) are matrices of lower and upper bounds on covariances

```python
Sigma = cp.Variable((n, n), PSD=True)

objective = cp.Maximize(cp.quad_form(weights, Sigma))
constraints = []
for i in range(n):
    for j in range(i):
        constraints += [L[i, j] <= Sigma[i, j], Sigma[i, j] <= U[i, j]]

prob = cp.Problem(objective, constraints)
prob.solve()
```
Example

- portfolio weights and uncertain covariance

\[ w = \begin{bmatrix} 0.5 \\ 0.25 \\ -0.05 \\ 0.3 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} + & + & + & \pm \\ 0.1 & - & - & + \\ + & - & 0.3 & + \\ \pm & - & + & 0.1 \end{bmatrix}, \]

- + means nonnegative, – means nonpositive, and ± means unknown sign
- worst-case risk is 0.18 (volatility 42%)
- risk with diagonal covariance matrix is 0.07 (volatility 26%)
- worst-case covariance is

\[
\begin{bmatrix}
0.20 & 0.14 & -0.24 & 0.14 \\
0.14 & 0.10 & -0.17 & 0.10 \\
-0.24 & -0.17 & 0.30 & -0.17 \\
0.14 & 0.10 & -0.17 & 0.10 \\
\end{bmatrix}
\]
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Investment arbitrage

- invest \( x_j \) in asset \( j \), with prices \( p_1, \ldots, p_n \); initial cost is \( p^T x \)
- at the end of the investment period there are only \( m \) possible outcomes
- \( V_{ij} \) is the payoff of asset \( j \) in outcome \( i \)
- first investment is risk-free (cash): \( p_1 = 1 \) and \( V_{i1} = 1 \) for all \( i \)
- **arbitrage**: there is an \( x \) with \( p^T x < 0 \), \( Vx \geq 0 \)
- \( i.e.\), we receive money up front, and cannot lose

- standard assumption: the prices are such that **there is no arbitrage**
Fundamental theorem of asset pricing

- by Farkas’ lemma, there is no arbitrage $\iff$ there exists $\pi \in \mathbb{R}_+^m$ with $V^T \pi = p$
- first column of $V$ is $1$, so we have $1^T \pi = 1$
- $\pi$ is interpreted as a risk-neutral probability on the outcomes $1, \ldots, m$
- $V^T \pi$ are the expected values of the payoffs under the risk-neutral probability
- $V^T \pi = p$ means asset prices equal their expected payoff under the risk-neutral probability

- fundamental theorem of asset pricing:

  there is no arbitrage $\iff$ there exists a risk-neutral probability distribution under which each asset price is its expected payoff
Check for arbitrage in CVXPY

```python
pi = cp.Variable(m, nonneg=True)
prob = cp.Problem(cp.Minimize(0), [V.T @ pi == p])
prob.solve()

if prob.status == 'optimal':
    print('No arbitrage exists')
elif prob.status == 'infeasible':
    print('Arbitrage exists')
```
Option price bounds

- Suppose $p_1, \ldots, p_{n-1}$ are known, but $p_n$ is unknown.
- Arbitrage-free range for $p_n$ is found by solving

\[
\begin{align*}
\text{minimize/maximize} & \quad p_n \\
\text{subject to} & \quad V^T \pi = p, \quad \pi \geq 0, \quad 1^T \pi = 1
\end{align*}
\]

with variables $p_n \in \mathbb{R}$ and $\pi \in \mathbb{R}^m$.

- Can be solved in CVXPY.
- If the minimum and maximum are equal, the market is complete.
Option price bounds in CVXPY

$p_{\text{known}}$ is vector of known prices, of length $n - 1$

```python
pi = cp.Variable(m, nonneg=True)
p_n = cp.Variable()
p = cp.hstack([p_known, p_n])

prob = cp.Problem(cp.Minimize(p_n), [V.T @ pi == p])
prob.solve()
print(f'Minimum arbitrage-free price: {p_n.value}')

prob = cp.Problem(cp.Maximize(p_n), [V.T @ pi == p])
prob.solve()
print(f'Maximum arbitrage-free price: {p_n.value}')
```
Example

$\n = 7$ assets:
- a risk-free asset with price 1 and payoff 1
- an underlying asset with price 1 and uncertain payoff
- four vanilla options on the underlying with known (market) prices

<table>
<thead>
<tr>
<th>Type</th>
<th>Strike</th>
<th>Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>Call</td>
<td>1.1</td>
<td>0.06</td>
</tr>
<tr>
<td>Call</td>
<td>1.2</td>
<td>0.03</td>
</tr>
<tr>
<td>Put</td>
<td>0.8</td>
<td>0.02</td>
</tr>
<tr>
<td>Put</td>
<td>0.7</td>
<td>0.01</td>
</tr>
</tbody>
</table>

$m = 200$ possible outcomes for the underlying asset, uniformly between 0.5 and 2
- we seek price bounds on a collar option with floor 0.9 and cap 1.15
- solving optimization problem gives the arbitrage-free collar price range $[-0.015, 0.033]$
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Currency exchange problem

- we hold $c^{\text{init}} = (c_1^{\text{init}}, \ldots, c_n^{\text{init}})$ of $n$ currencies, in USD under nominal exchange rates
- want to exchange them to obtain (at least) $c^{\text{req}} = (c_1^{\text{req}}, \ldots, c_n^{\text{req}})$, valued in USD
- $X \in \mathbb{R}^{n \times n}$ is currency exchange matrix; $X_{ij} \geq 0$ the amount of $j$ we exchange for $i$, in USD
- $\Delta_{ij} \geq 0$ is cost of exchanging one USD of currency $j$ for currency $i$, expressed as a fraction
- exchange $X_{ij}$ costs us $X_{ij} \Delta_{ij}$ USD

- optimal currency exchange: find $X$ that minimizes cost

$$\text{minimize} \quad \sum_{i,j=1}^{n} X_{ij} \Delta_{ij}$$
$$\text{subject to} \quad X_{ij} \geq 0, \quad \text{diag}(X) = 0,$$
$$c_i^{\text{init}} + \sum_j X_{ij} - \sum_j X_{ji} \geq c_i^{\text{req}}, \quad i = 1, \ldots, n$$
Currency exchange in CVXPY

```python
X = cp.Variable((n, n), nonneg=True)

objective = cp.sum(cp.multiply(X, Delta))
constraints = [
    cp.diag(X) == 0,
    c_init + cp.sum(X, axis=1) - cp.sum(X, axis=0) >= c_req
]

prob = cp.Problem(cp.Minimize(objective), constraints)
prob.solve()
```
Example

- USD, EUR, CAD, SEK, with initial and required holdings (in $10^6$)
  \[c^{\text{init}} = (1, 1, 1, 1), \quad c^{\text{req}} = (2.1, 1.5, 0.3, 0.1)\]

- exchange rates in basis points (bps) \((10^{-4})\)

\[
\begin{array}{cccc}
\text{USD} & \text{EUR} & \text{CAD} & \text{SEK} \\
\text{USD} & 0.0 & 0.1 & 4.4 & 4.8 \\
\text{EUR} & 0.1 & 0.0 & 5.0 & 5.7 \\
\text{CAD} & 2.8 & 6.9 & 0.0 & 8.5 \\
\text{SEK} & 1.1 & 7.9 & 7.6 & 0.0 \\
\end{array}
\]

- cheap to trade USD and EUR, expensive to trade CAD and SEK
- optimal exchanges (in $10^6$)
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Purchase execution and risk

- want to purchase $Q$ shares over $T$ periods $t = 1, \ldots, T$
- $q \in \mathbb{R}^T$ is purchase schedule; $q \geq 0$, $1^T q = Q$
- (bid-ask midpoint) price dynamics:

$$p_t = p_{t-1} + \xi_t, \quad t = 2, \ldots, T,$$

with $p_1$ known, $\xi_t$ IID $\mathcal{N}(0, \sigma^2)$
- nominal cost is random variable $p^T q$, with

$$\mathbb{E}(p^T q) = p_1 Q, \quad \text{var}(p^T q) = q^T \Sigma q$$

where $\Sigma_{kl} = \sigma^2 \min(k - 1, l - 1)$
- $q^T \Sigma q$ is the risk
Market impact

Transaction (market impact) cost, in USD (‘squareroot model’):

\[
\sum_{t=1}^{T} \sigma \pi_t^{1/2} q_t = \sigma \sum_{t=1}^{T} q_t^{3/2} / v_t^{1/2}
\]

- \( v_t \) is market volume, \( \pi_t = q_t / v_t \) is participation rate in period \( t \)
- Actual cost of execution is nominal mean cost \( p_1 Q \) plus transaction cost
Optimal execution

- trade off risk and transaction cost, with participation rate limit

\[
\begin{align*}
& \text{minimize} & & \sigma \sum_{t=1}^{T} \left( q_t^{3/2} / v_t^{1/2} \right) + \gamma q^T \Sigma q \\
& \text{subject to} & & q \geq 0, \quad 1^T q = Q, \quad q_t / v_t \leq \pi_{\text{max}}, \quad t = 1, \ldots, T,
\end{align*}
\]

- \( \gamma > 0 \) is a risk aversion parameter
- \( \pi_{\text{max}} \) participation rate limit
- a convex problem [Almgren and Chriss, 2001]
- an alternate formulation reduces computational complexity from \( O(T^3) \) to \( O(T) \)
- without risk term and participation contraint, constant participation is optimal
Optimal execution in CVXPY

```python
q = cp.Variable(T, nonneg=True)
pi = q / v

risk = cp.quad_form(q, Sigma)
transaction_cost = sigma * cp.power(q, 3 / 2) @ cp.power(v, -1 / 2)

objective = cp.Minimize(transaction_cost + gamma * risk)
constraints = [cp.sum(q) == Q, pi <= pi_max]

prob = cp.Problem(objective, constraints)
prob.solve()
```
Example

- purchase 10 million Apple shares over 10 trading days (Feb 8–22, 2024)
- participation rate limit $\pi_{\text{max}} = 5\%$

![Graphs showing volume, optimal purchase schedule, and participation rate over different periods.](image-url)
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Merton consumption-investment dynamics

- plan consumption and investment at times \( t = 0, \ldots, T \)
- in period \( t \), wealth is \( k_t \), consumption is \( c_t \), labor income is \( y_t \) (all in inflation adjusted USD)
- remaining wealth invested in \( n \) assets with return mean \( \mu \) and covariance \( \Sigma \)
- \( r_t \in \mathbb{R}^n \) is asset return in period \( t \)
- \( h_t \in \mathbb{R}^n \) denotes amounts invested in period \( t \) in USD
- \( r_t^T h_t \) is the portfolio return in USD
- wealth dynamics given by
  \[
  k_{t+1} = k_t - c_t + y_t + r_t^T h_t
  \]
Merton consumption-investment problem

- maximize expected utility of consumption and bequest

\[ E \left( \frac{\beta}{\rho} k_T^\rho + \frac{1}{\rho} \sum_{t=0}^{T-t} c_t^\rho \right) \]

- \( \beta > 0 \) sets relative importance of bequest; \( \rho < 1 \) sets risk aversion
- a stochastic control problem, solved in [Merton, 1975]
Deterministic wealth dynamics

- replace stochastic wealth dynamics \( k_{t+1} = k_t - c_t + y_t + r^T_t h_t \) with deterministic dynamics

\[
k_{t+1} = k_t - c_t + y_t + \mu^T h_t - \frac{(1 - \rho) h^T_t \Sigma h_t}{2k_t + v_t}
\]

- \( v_t \) is the present value of future labor income discounted at the risk-free rate \( \mu^{rf} \)

\[
v_t = \sum_{\tau=t}^{T-1} y_{\tau} \exp(-\mu^{rf} (\tau - t))
\]

- we require \( k_t + v_t > 0 \), i.e., wealth plus future labor income is positive
- last term is a pessimistic adjustment for risk derived in [Moehle and Boyd, 2021]
Certainty equivalent convex optimization formulation

- yields deterministic convex optimization problem

\[
\begin{align*}
\text{maximize} & \quad \frac{\beta}{\rho} k_T + \frac{1}{\rho} \sum_{t=0}^{T-1} c_t^\rho \\
\text{subject to} & \quad k_{t+1} \leq k_t - c_t + y_t + \mu^T h_t - \frac{(1-\rho)}{2} \frac{h_t^T \Sigma h_t}{k_{t+1}} \\
& \quad k_t = 1^T h_t, \quad c_t \geq 0
\end{align*}
\]

(dynamic equality is replaced by inequality constraint, which is tight at solution)

- this certainty equivalent problem also solves stochastic problem

- can be extended to include mortality, liabilities, taxes, portfolio constraints, …
Certainty equivalent Merton problem in CVXPY

```python
k = cp.Variable(T + 1)
h = cp.Variable((n, T))
c = cp.Variable(T, nonneg=True)

Sigma_half = np.linalg.cholesky(Sigma)

objective = beta / rho * k[T] ** rho + 1 / rho * cp.sum(c**rho)
constraints = [k[0] == k0, k[:-1] == cp.sum(h, axis=0)]
constraints += [
    k[t + 1] <= k[t] - c[t] + y[t] + mu.T @ h[:, t]
    - (1 - rho) / 2 * cp.quad_over_lin(Sigma_half.T @ h[:, t], k[t] + v[t])
    for t in range(T)
]

prob = cp.Problem(cp.Maximize(objective), constraints)
prob.solve()
```
Example

- plan over 80 years (age 25–105), with initial wealth $k_0 = 10,000$ USD
- $n = 5$ assets, utility parameter $\rho = -4$, bequest parameter $\beta = 10$
- five asset classes, with long only portfolio constraint $h_t \geq 0$
- salary grows until age 50, then is constant, then drops to 50% at age 65

optimal consumption

optimal investments
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Alternative investments

- investor makes **commitments** to an alternative investment in each quarter \( t = 1, \ldots, T \)
- over time she puts committed money into the investment in response to **capital calls**
- she receives money through **distributions**
- examples: private equity, venture capital, infrastructure projects, …
Alternative investment dynamics

- $c_t, p_t, d_t \geq 0$ are commitments, capital calls, and distributions
- $n_t \geq 0$ is net asset value (NAV), $r_t$ is investment return
- $u_t \geq 0$ is total uncalled previous commitments
- dynamics:
  \[ n_{t+1} = n_t (1 + r_t) + p_t - d_t, \quad u_{t+1} = u_t - p_t + c_t \]
  with $n_0 = 0, u_0 = 0$
- simple model of calls and distributions:
  \[ p_t = \gamma_{\text{call}} u_t, \quad d_t = \gamma_{\text{dist}} n_t \]
- $\gamma_{\text{call}}, \gamma_{\text{dist}} \in (0, 1)$ are call and distribution intensities or rates
Alternative investment planning

- choose commitments $c_1, \ldots, c_T$ to minimize

$$
\frac{1}{T+1} \sum_{t=1}^{T+1} (n_t - n_{\text{des}})^2 + \lambda \frac{1}{T-1} \sum_{t=1}^{T-1} (c_{t+1} - c_t)^2,
$$

where $n_{\text{des}}$ is the desired NAV and $\lambda > 0$ is a smoothing parameter

- penalizes deviation from desired NAV, and encourages smooth commitment schedule

- can add constraints such as

$$
c_t \leq c_{\text{max}}, \quad u_t \leq u_{\text{max}}
$$

- yields convex problem

- can be extended to uncertain parameters, multiple illiquid investments, and mixed with liquid investments [Luxenberg et al., 2022]
Alternative investment planning in CVXPY

\begin{verbatim}
import cvxpy as cp

T = 10
n = cp.Variable(T+1, nonneg=True); u = cp.Variable(T+1, nonneg=True)
p = cp.Variable(T, nonneg=True); d = cp.Variable(T, nonneg=True)
c = cp.Variable(T, nonneg=True)

tracking = cp.mean((n-n_des)**2)
smoothing = lmbda * cp.mean(cp.diff(c)**2)

constraints = [c <= c_max, u <= u_max, n[0] == 0, u[0] == 0]
for t in range(T):
    constraints += [n[t+1] == (1+r)*n[t]+p[t]-d[t]]
    constraints += [u[t+1] == u[t]-p[t]+c[t]]
    constraints += [p[t] == gamma_call*u[t], d[t] == gamma_dist*n[t]]
prob = cp.Problem(cp.Minimize(tracking+smoothing), constraints)
prob.solve()
\end{verbatim}
Example

- $T = 32$ (eight years), $r_t = 0.04$ (4% quarterly return), $\gamma^{\text{call}} = .23$, $\gamma^{\text{dist}} = .15$
- planning parameters: $c^{\text{max}} = 4$, $u^{\text{max}} = 10$, $n^{\text{des}} = 15$, and $\lambda = 5$
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Blending forecasts

- we observe $x_1, \ldots, x_t \in \mathbb{R}^d$ and seek a forecast $\hat{x}_{t+1}$ of $x_{t+1}$
- we have $K$ forecasts $\hat{x}^1_{t+1}, \ldots, \hat{x}^K_{t+1}$
- called $K$ experts [Hastie et al., 2009]
- we blend them using weights $\pi^1_t, \ldots, \pi^K_t$ with $\pi_t \geq 0, 1^T \pi_t = 1$
  \[ \hat{x}_{t+1} = \sum_{k=1}^{K} \pi^k_t \hat{x}^k_{t+1} \]

- the weights can vary over time
- we may want them smoothly varying, i.e., $\pi_{t+1} \approx \pi_t$
- we may want them close to some prior or baseline weights $\pi^{\text{pri}}$
- examples: return mean, return covariance, …
Blending forecasts using convex optimization

- find weights $\pi_t$ as solution of convex optimization problem

$$\begin{align*}
\text{minimize} & \quad \frac{1}{M} \sum_{\tau=t-M+1}^{t} \ell(\hat{x}_\tau, x_\tau) + r^{sm}(\pi, \pi_{t-1}) + r^{pri}(\pi, \pi^{pri}) \\
\text{subject to} & \quad \hat{x}_\tau = \sum_{k=1}^{K} \pi_k \hat{x}^k_\tau, \quad \pi \geq 0, \quad 1^T \pi = 1
\end{align*}$$

with variable $\pi \in \mathbb{R}^K$

- $\ell$ is prediction loss, $r^{sm}$ penalizes weight change, $r^{pri}$ penalizes deviation from prior
- we assume $\ell, r^{sm}, r^{pri}$ are convex in $\pi$
- idea: use blending weights that would have worked well over the last $M$ periods
Blending forecasts in CVXPY

- $X\_\text{hat}$ is an $M \times K$ matrix of expert forecasts over the last $M$ periods $\tau = t - M + 1, \ldots, t$
- $x$ is an $M$-vector of observed quantities over the same period

```python
pi = cp.Variable(K, nonneg=True)
x\_hat = X\_hat @ pi

objective = cp.Minimize(cp.mean(loss(x\_hat, x)))
constraints = [cp.sum(pi) == 1]

prob = cp.Problem(objective, constraints)
prob.solve()
```
Example

- predict log of daily trading volume of Apple, 1982–2024
- \( K = 3 \) predictors: 5-day (fast), 21-day (medium), and 63-day (slow) moving medians
- absolute loss \( \ell(\hat{x}_\tau, x_\tau) = |\hat{x}_\tau - x_\tau|; \ M = 250 \) trading days

<table>
<thead>
<tr>
<th>error</th>
<th>fast</th>
<th>median</th>
<th>slow</th>
<th>blend</th>
</tr>
</thead>
<tbody>
<tr>
<td>median</td>
<td>0.25</td>
<td>0.27</td>
<td>0.29</td>
<td>0.24</td>
</tr>
<tr>
<td>90th percentile</td>
<td>0.72</td>
<td>0.74</td>
<td>0.82</td>
<td>0.68</td>
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<tr>
<td>10th percentile</td>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
<td>0.04</td>
</tr>
</tbody>
</table>

250-day rolling median absolute error weights (\( \pi \))
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Bond price model

- Bond pays holder $c_t$ in periods $t = 1, \ldots, T$ (coupons and principal)
- Price of bond is discounted present value of payments

$$p = \sum_{t=1}^{T} c_t \exp(-t(y_t + s)),$$

where $y = (y_1, \ldots, y_T) \in \mathbb{R}^T$ is the yield curve, and $s \geq 0$ the spread

- We assume $y = Ya$ where $Y$ are basis functions and $a$ are coefficients
- Can use principal component analysis to fit $Y$ to historical data
- Spread depends on bond rating (readily extended to depend on other attributes)
Matrix pricing problem

- we are given market prices \( p_i \) and ratings \( r_i \in \{1, \ldots, K\} \) of \( n \) bonds, \( i = 1, \ldots, n \)
- we want to fit a yield curve \( y \in \mathbb{R}^T \) and spreads \( s \in \mathbb{R}^K \) to this data
- we add constraint \( 0 \leq s_1 \leq \cdots \leq s_K \) (higher ratings have lower spreads)
- using square error we fit \( y \) and \( s \) by solving problem

\[
\begin{align*}
\text{minimize} & \quad \sum_{i=1}^{n} \left( p_i - \sum_{t=1}^{T} c_{i,t} \exp(-t(y_t + s_{r_i})) \right)^2 \\
\text{subject to} & \quad 0 \leq s_1 \leq \cdots \leq s_K
\end{align*}
\]

with variables \( y \) and \( s \)
- not convex, but can be solved (approximately) as a sequence of convex problems
- linearize exponential term and iteratively fit yields and spreads
Matrix pricing in CVXPY

- use CVXPY to automatically linearize the exponential term as $p_{\text{current}} + \Delta_{\text{hat}}$
- would add trust penalty to the iterates in practice
- code below computes first iteration

```python
y = cp.Variable(T, value=y_init)
s = cp.Variable(K, value=S_init, nonneg=True)
a = cp.Variable(a.size, values=a_init)

do\text{iscount} = \text{cp.exp}(\text{cp.multiply}(-t, y.\text{reshape}((1, -1)) + s[\text{ratings}].\text{reshape}((-1, 1))))
p_{\text{current}} = \text{cp.sum}(\text{cp.multiply}(C, \text{discount}), \text{axis}=1)

\Delta_{\text{hat}} = p_{\text{current}}.\text{grad}[y].T \odot (y-y.\text{value}) + p_{\text{current}}.\text{grad}[s].T \odot (s - s.\text{value})
objective = \text{cp.norm2}(p - (p_{\text{current}}.\text{value} + \Delta_{\text{hat}}))
constraints = [cp.diff(s) >= 0, y == Y @ a]

problem = cp.Problem(cp.Minimize(objective), constraints)
problem.solve()
```
Example

- consider $n = 1000$ bonds, with a maturity of up to 30 years
- bonds are rated AAA, AA, A, BBB, BB
- used data from 1990 to 2024 to fit basis functions, latest yields, and spread to price bonds
- fit yields and spreads to bond prices gives $0.03$ RMSE (2.9 bps)

iteratively fitted yields for rating AAA

rating spreads (vs. AAA)
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Stochastic control

- dynamics \( x_{t+1} = f(x_t, u_t, w_t), \ t = 0, 1, \ldots, T - 1 \)
- \( x_t \in X \) is the state, \( u_t \in U \) is the input or action, \( w_t \in W \) is the disturbance
- \( x_0, w_0, \ldots, w_{T-1} \) are independent random variables
- state feedback policy \( u_t = \pi_t(x_t), \ t = 0, 1, \ldots, T - 1 \)
- stochastic control problem: choose policy to minimize

\[
J = \mathbb{E} \left( \sum_{t=0}^{T-1} g_t(x_t, u_t) + g_T(x_T) \right)
\]

- stage cost \( g_t(x_t, u_t) \); terminal cost \( g_T(x_T) \)
- examples: investing, execution, consumption, \ldots
Solution via dynamic programming

- exact solution from Bellman [1954]
- only practical in special cases
  - $X, U$ finite
  - linear dynamics and quadratic cost
  - $x_t \in \mathbb{R}^n$ with $n$ very small, like 2 or 3
  - a few other special cases (e.g., Merton problem)

- but several heuristics and approximations work very well
Model predictive control

to evaluate $\pi_t^{\text{mpc}}(x_t)$:

▶ **forecast**: predict stochastic future values $w_t, \ldots, w_{T-1}$ as $\hat{w}_t|t, \ldots, \hat{w}_{T-1}|t$

▶ **plan**: solve certainty equivalent problem assuming forecasts are correct

$$\begin{align*}
\text{minimize} & \quad \sum_{\tau=t}^{T-1} g_t(\hat{x}_\tau|t, u_\tau) + g_T(\hat{x}_T|t) \\
\text{subject to} & \quad \hat{x}_{\tau+1}|t = f(x_\tau|t, u_\tau|t, \hat{w}_\tau|t), \quad \tau = t, \ldots, T-1, \quad \hat{x}_t|t = x_t
\end{align*}$$

with variables $u_t|t, \ldots, u_{T-1}|t, \hat{x}_t|t, \ldots, \hat{x}_T|t$

▶ **execute**: $\pi_t^{\text{mpc}}(x_t) = u_t|t$ (i.e., take first action in plan)
Model predictive control

- when $f$ is linear in $x, u$ and $g$, are convex, planning problem is convex, hence tractable
- MPC is optimal in a few special cases, but often performs extremely well
- used in many industries, e.g., guiding SpaceX's Falcon first stages to their landings [Blackmore, 2016]

- **receding horizon MPC**
  - a variation for when there is no terminal time $T$
  - solve planning problem over $H$-period horizon $\tau = t$ to $\tau = t + H$
  - can include terminal cost or constraint
Example: Order execution via MPC

- purchase 10 million Apple shares over 10 trading days (Feb 8–22, 2024)
- participation rate limit $\pi^{\text{max}} = 5\%$
- forecasts:
  - $\hat{v}_{\tau|t}$ is 5-day trailing median of volumes, $\tau = t, \ldots, T$
  - $\sigma_{\tau|t}$ is 21-day trailing standard deviation, $\tau = t, \ldots, T$
- transaction cost is 1.881B USD for fixed schedule and 1.877B USD for MPC
- MPC saves us 4M USD, about 20 bps
Example: Order execution via MPC