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ON THE PASSIVITY CRITERION FOR LTI $\mathcal{N}$-PORTS

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SUMMARY

Existing proofs of the *passivity criterion* for linear, time-invariant, distributed N-ports are either *incorrect* or too involved, requiring the use of advanced mathematics such as *distribution theory*. This paper presents a simple but completely rigorous proof using only basic real and complex analysis. For the sake of completeness we have included simple proofs of the *Paley–Wiener theorem* and the *Poisson formula for the half plane*. We show that *solvability*, a non-intuitive technical assumption made in rigorous theories of LTI passive networks, is virtually always satisfied. Finally, we give a passivity criterion applicable to N-ports described by general co-ordinates, from which passivity criteria for any specific representation (e.g. impedance, admittance, hybrid, transmission, scattering, etc.) can be trivially derived.

1. INTRODUCTION

In 1954 Raisbeck[1] proposed a general definition of *passivity* which would apply to distributed as well as lumped circuits and gave an informal proof that a linear time-invariant (LTI) N-port is *passive* if and only if its impedance matrix is *positive real.* In 1958 Youla, Castriota, and Carlin published their classic paper on linear passive circuit theory[2] which included the first formal proof of this passivity criterion, but the proof is fairly involved. Wohlers and Beltrami[3,4] and Zemanian[5] gave simpler formal proofs using the theory of distributions.

Despite the 1958 publication of a formal and correct proof[2] attempts persisted for ten years to formalize Raisbeck's original intuitive argument, and in fact several textbooks published *after 1958* use Raisbeck's proof.[6–8] In 1966 Resh[6] pointed out one error in the Raisbeck proof and proposed a correction; two years later Kuo[9] found and corrected an error in Resh's proof. But in fact there were deeper problems with the Raisbeck proof than those addressed by Resh and Kuo. For example Poisson's formula for the half plane is incorrectly used.†

The primary purpose of this paper is to present a formal proof of the passivity criterion which is straightforward, intuitive, and makes the minimum appeal to advanced mathematics; in particular, no distribution theory is used. Much of the advanced mathematics used is condensed into a single theorem which characterizes LTI causal bounded operators in the frequency domain. We have called this theorem the Bochner–Paley–Wiener theorem since it is an easy consequence of their results, and give a self-contained proof in the appendix.

Our second purpose is to discuss some of the intricacies of the problem. We examine the difference between *passive* devices which satisfy $\int_{-\infty}^{T} v^*i \, dt \geq 0$ for all T and devices for which $\int_{-\infty}^{\infty} v^*i \, dt \geq 0$ which we call *weakly passive* (the distinction is due to Wohlers; some authors have used *weak passivity* as their definition of *passivity*). We show that a *solvable* N-port $\mathcal{N}$ is *passive* if and only if it is *weakly passive* and has a *causal scattering operator* and give a weaker criterion for $\mathcal{N}$ to be *weakly passive*. For example, consider a 1-port $\mathcal{N}$ with $v(t) = i(t) + \frac{1}{2}i(t+1)$. This $\mathcal{N}$ has an impedance $Z(j\omega) = 1 + \frac{1}{2}e^{j\omega}$ which has the analytic extension $Z(s) = 1 + \frac{1}{2}e^{s}$ in the whole complex plane. Parseval's relation or direct calculation shows that if i is admissible and $i \in L_2$, then $v \in L_2$ and $\int_{-\infty}^{\infty} vi \, dt \geq 0$, that is, N is *weakly passive*. $Z(s)$, though quite analytic, is far from positive real since $Z(\pi j + 1) = 1 - \frac{1}{2}e < 0$, so this $\mathcal{N}$ is an example showing

† It is stated incorrectly by Guillemen.[17]

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that proofs of the criterion assuming only *weak passivity* are incorrect. We show that a reasonable assumption implies that a device is either *solvable* or exhibits simple nullator behaviour.

Finally, we give a passivity criterion for a device described by general co-ordinates from which specialized criteria in terms of any particular representation (e.g. admittance, hybrid, transmission, etc.) can be trivially derived.

We will use the following somewhat standard notation: $\bar{w}$ is the conjugate and w^* the conjugate transpose of $w \in C^N$, $|w| = (w^*w)^{1/2}$; $L_2^N(L_2^N(\mathbb{R}))$ is the set of (measurable) $C^N(R^N)$ valued functions of a real variable $f(t)$ with $\int_{-\infty}^{\infty} f^*(t)f(t)\,dt = \|f\|^2 < \infty$ (Lebesgue integral; functions which differ on a set of measure zero identified); L_2 is L_2^1. For $f \subseteq L_2^N$, $\hat{f}(j\omega)$ is its Fourier transform ($=$l.i.m.$_{A\to\infty} \int_{-A}^{A} f(t)\,e^{-j\omega t}\,dt$); if $f(t)$ is C^N-valued and $T \in \mathbb{R}$, f_T will denote the function which agrees with $f(t)$ for $t \leq T$ and which vanishes for $t \geq T$; L_{2e}^N ('Extended L_2^N') is the set of all f with $f_T \in L_2^N$ for all $T \in \mathbb{R}$; $\mu(E)$ will denote the Lebesgue measure of the (measurable) $E \in \mathbb{R}$, RHP will denote the open right half plane $\{z \in C | \text{Re } z > 0\}$.

We will say that a function $F(j\omega)$ defined only up to sets of measure zero has the *analytic extension* $F(s)$ in the RHP if $F(s)$ is analytic in the RHP and $\lim_{\sigma\to 0+} F(\sigma + j\omega)$ exists and equals $F(j\omega)$ for almost all $\omega \in \mathbb{R}$. *We will routinely drop the qualifier* 'almost' *from* 'almost all', *trusting that the reader familiar with measure theory will be able to supply it where necessary.*

An $\mathcal{N}$-admissible signal or signal pair will mean a *real valued* signal or signal pair in L_{2e}^N which may appear across $\mathcal{N}$.†

2. DEFINITION OF PASSIVITY AND STATEMENT OF THE CRITERION

Following Youla *et al.*[2] we say that $\mathcal{N}$ is *passive* if for all $\mathcal{N}$-admissible port current-voltage pairs (i, v)

$$\text{for all } T \in \mathbb{R}, \qquad \int_{-\infty}^{T} v^*(t)i(t)\,dt \geq 0 \tag{1}$$

This integral exists since $v, i \in L_{2e}^N$ by assumption.

The use of the scattering variables‡ (a, b), where $a(t) = \frac{1}{2}v(t) + \frac{1}{2}i(t)$ and $b(t) = \frac{1}{2}v(t) - \frac{1}{2}i(t)$ is central to our argument, so we reformulate (1) as

$$\text{for all } T \in \mathbb{R}, \qquad \int_{-\infty}^{T} (a^*a - b^*b)\,dt \geq 0 \tag{2}$$

We say that $\mathcal{N}$ is *solvable* if the set of $\mathcal{N}$-admissible a's include all of $L_2^N(R)$ (see Reference 2, assumption p. 4).

Theorem 1 (*Youla et al.*)

A *solvable* N-port $\mathcal{N}$ is LTI and *passive* if and only if

(i) $\mathcal{N}$ has a scattering matrix, i.e. the set of admissible (a, b) with $a \in L_2^N(R)$ is precisely $\{(a, b) | a \in L_2^N(R)$ and $\hat{b}(j\omega) = S(j\omega)\hat{a}(j\omega)\}$.

(ii) $S(j\omega)$ has the analytic extension $S(s)$ in the RHP with $I - S^*(s)S(s)$ positive semidefinite there.

We note that $S(j\omega)$ is defined only up to sets of measure zero, so that statements involving $S(j\omega)$ are to be interpreted as true almost everywhere, whereas statements involving the analytic function $S(s)$ are true everywhere. Our assumption that i, v and therefore a, b are real implies that $S(-j\omega) = \overline{S(j\omega)}$; finally let us note that (ii) implies that $I - S^*(j\omega)S(j\omega)$ is positive semidefinite for almost all $\omega \in \mathbb{R}$ since it is almost everywhere $\lim_{\sigma\to 0+} (I - S^*(\sigma + j\omega)S(\sigma + j\omega))$.

† With only minor modification the entire theory may be formulated for complex signals, but we see no advantage.

‡ Here we assume port normalization impedances of 1 Ω; in general

$$a_k = \frac{1}{2R_k}v_k + \frac{R_k}{2}i_k, \qquad b_k = \frac{1}{2R_k}v_k - \frac{R_k}{2}i_k.$$

3. PROOF OF NECESSITY OF (i) AND (ii)

Throughout this section let $\mathcal{N}$ be a *passive solvable* LTI N-port.

Lemma 1

Suppose (a, b) is an admissible signal pair such that $a(t)=0$ for $t<T$. Then $b(t)=0$ for $t<T$.

Proof. Since $\mathcal{N}$ is *passive* $\int_{-\infty}^{T}(a^*(t)a(t)-b^*(t)b(t))\,dt\geq 0$, so under the hypothesis of lemma 1, $-\int_{-\infty}^{T}b^*(t)b(t)\,dt=-\int_{-\infty}^{T}|b(t)|^2\,dt\geq 0$. Thus $b(t)=0$ for $t<T$. □

This simple lemma has extremely profound consequences!

Corollary 1

To each $a\in L_2^N$, there is a *unique* b such that (a, b) is admissible. Furthermore $b\in L_2^N$ and $\|b\|\leq\|a\|$.

Proof. Suppose $a\in L_2^N$. By *solvability*, we know there is at least one b with (a, b) admissible. Suppose that (a, b) and (a, b') are both admissible. Since $\mathcal{N}$ is linear, $(0, b-b')$ is admissible. By lemma 1, $b(t)-b'(t)=0$ for $t<T$ and T arbitrary, so $b=b'$. By *passivity*, we have for all $T\in\mathbb{R}$

$$\|a\|^2\geq\int_{-\infty}^{T}a^*a\,dt\geq\int_{-\infty}^{T}b^*b\,dt$$

which proves $b\in L_2^N$ and $\|b\|\leq\|a\|$. □

Thus we may define a linear operator $\mathcal{S}$ from $L_2^N(R)$ into $L_2^N(R)$ by $\mathcal{S}(a)=b$. The last conclusion of Corollary 1 is that $\mathcal{S}$ is a *bounded* operator.

Corollary 2

$\mathcal{S}$ is a *causal* operator, that is, if $a(t)=a'(t)$ for $t<T$, then $\mathcal{S}a(t)=\mathcal{S}a'(t)$ for $t<T$.

Proof. If $a(t)=a'(t)$ for $t<T$, then $(a-a', \mathcal{S}a-\mathcal{S}a')$ satisfies the hypothesis of lemma 1. Thus $\mathcal{S}a(t)=\mathcal{S}a'(t)$ for $t<T$. □

Thus, $\mathcal{S}\colon L_2^N(R)\to L_2^N(R)$ is a linear time invariant bounded causal operator. It is worth mentioning here that a *causal* operator from L_2^N into L_2^N has a *unique* extension to a *causal* operator from L_{2e}^N to L_{2e}^N.

By the Bochner–Paley–Wiener theorem (see Section 7) $\mathcal{S}$ has a representation as: $\widehat{\mathcal{S}a}(j\omega)=S(j\omega)\hat{a}(j\omega)$ where the $N\times N$ matrix $S(j\omega)$ has the bounded analytic extension $S(s)$ in the RHP. We have shown that $\mathcal{N}$ has a scattering matrix $S(j\omega)$; it remains to show that $I-S^*(s)S(s)$ is positive semidefinite in the RHP.

Lemma 2

For each $c\in C^N$, $c^*(I-S^*(j\omega)S(j\omega))c\geq 0$ for (almost all) $\omega\in\mathbb{R}$.

Note that this is weaker than $I-S^*(j\omega)S(j\omega)$ being positive semidefinite for (almost all) $\omega\in\mathbb{R}$.†

Proof. Suppose for some $c\in C^N$ $c^*(I-S^*(j\omega)S(j\omega))c<0$ for ω in some set $\bar{\Delta}$ of positive measure. We may take a subset Δ of $\bar{\Delta}$ with $0<\mu(\Delta)<\infty$ and $\Delta\in[0,\infty)$ or $\Delta\in(-\infty, 0]$ such that $c^*(I-S^*(j\omega)S(j\omega))c<-\varepsilon<0$ for $\omega\in\Delta$. Then for $-\omega\in\Delta$, $\bar{c}^*(I-S^*(j\omega)S(j\omega))\bar{c}=\overline{c^*(I-S^*(-j\omega)S(-j\omega))c}<-\varepsilon$. Define $\hat{a}(j\omega)$ by

$$\hat{a}(j\omega)=\begin{cases}c & \omega\in\Delta\\ \bar{c} & -\omega\in\Delta\\ 0 & \text{elsewhere}\end{cases}$$

Then $\|\hat{a}(j\omega)\|=\sqrt{[2\mu(\Delta)]}|c|<\infty$, so $\hat{a}\in L_2^N$. Consequently $\hat{a}(j\omega)$ is the transform of an $a(t)\in L_2^N$, and since $\hat{a}(-j\omega)=\overline{\hat{a}(j\omega)}$, $a(t)\in L_2^N(R)$ and is thus admissible since all $a(t)\in L_2^N(R)$ are admissible. Intuitively,

† Lemma 2 says $\forall c\in C^N\,\exists N_c\subseteq\mathbb{R}\{\mu(N_c)=0$ and $\omega\notin N_c\Rightarrow c^*(I-S^*(j\omega)S(j\omega))c\geq 0\}$ whereas this statement is $\exists N\subseteq\mathbb{R}\,\forall c\in C^N\{\mu(N)=0$ and $\omega\notin N\Rightarrow c^*(I-S^*(j\omega)S(j\omega))c\geq 0\}$. We shall see later that the stronger statement is true.

$a(t)$ is a signal band limited to the set Δ where $c^*(I-S^*(j\omega)S(j\omega))c<0$. If we apply this signal to $\mathcal{N}$, the Parseval relation yields

$$\int_{-\infty}^{\infty}(a^*a-b^*b)\,\mathrm{d}t=\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{a}(j\omega)^*(I-S^*(j\omega)S(j\omega))\hat{a}(j\omega)\,\mathrm{d}\omega$$

$$=\frac{1}{2\pi}\int_{\Delta}c^*(I-S^*(j\omega)S(j\omega))c\,\mathrm{d}\omega+\frac{1}{2\pi}\int_{-\Delta}\bar{c}^*(I-S^*(j\omega)S(j\omega))\bar{c}\,\mathrm{d}\omega<\frac{-\mu(\Delta)\varepsilon}{\pi}<0$$

Since $\int_{-\infty}^{\infty}(a^*a-b^*b)\,\mathrm{d}t=\lim_{T\to\infty}\int_{-\infty}^{T}(a^*a-b^*b)\,\mathrm{d}t$, there is a T_0 with $\int_{-\infty}^{T_0}(a^*a-b^*b)\,\mathrm{d}t<0$, contradicting $\mathcal{N}$'s passivity. This establishes lemma 2. □

Theorem 2

$I-S^*(s)S(s)$ is positive semidefinite in the RHP.

Proof. From our remarks after Corollary 2 we know that $S(s)$ is bounded in the RHP. Hence Poisson's representation is valid (Reference 10; see Section 7 for proof): For $s_0=\sigma_0+j\omega_0$, $\sigma_0>0$

$$S(s_0)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sigma_0}{(\omega-\omega_0)^2+\sigma_0^2}S(j\omega)\,\mathrm{d}\omega$$

Let $c\in C^N$. Then

$$|S(s_0)c|=\left|\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sigma_0}{(\omega-\omega_0)^2+\sigma_0^2}S(j\omega)\,\mathrm{d}\omega\right|\leqslant\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sigma_0}{(\omega-\omega_0)^2+\sigma_0^2}|S(j\omega)c|\,\mathrm{d}\omega$$

By lemma 2, $c^*(I-S^*(j\omega)S(j\omega))c=|c|^2-|S(j\omega)c|^2\geqslant 0$ for (almost all) $\omega\in\mathbb{R}$, so $|c|\leqslant|S(j\omega)c|$ for (almost all) $\omega\in\mathbb{R}$ and

$$|S(s_0)c|\leqslant|c|\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sigma_0}{(\omega-\omega_0)^2+\sigma_0^2}\,\mathrm{d}\omega=|c|$$

Thus $|c|^2-|S(s_0)c|^2=c^*(I-S^*(s_0)S(s_0))c\geqslant 0$ establishing Theorem 2 and the necessity of (i) and (ii) in Theorem 1. Incidentally, this last argument is equivalent to the following: $\|S(s)\|$ is a *bounded subharmonic* function in RHP which is bounded by 1 on the $j\omega$-axis. Consequently it is bounded by 1 in the whole RHP, and this means $I-S^*(s)S(s)$ is positive semidefinite in the whole RHP.

4. SUFFICIENCY OF (i) AND (ii)

We assume now (i) and (ii), that is $\mathcal{N}$ has a scattering matrix $S(j\omega)$ which has the analytic extension $S(s)$ in the RHP, and that $I-S^*(s)S(s)$ is positive semidefinite in the RHP. (i) includes the assumption that $\mathcal{N}$ is *solvable*. (ii) implies that $S(s)$ is bounded in the RHP, for if e_k is the kth standard basis vector $(0,\ldots,1,\ldots 0)^*$, $e_k^*(I-S^*(s)S(s))e_k=1-\sum_{j=1}^{N}|S_{jk}(s)|^2\geqslant 0$, so that $|S_{ij}(s)|\leqslant 1$ for $s\in$ RHP. By the Bochner–Paley–Wiener theorem, S is the frequency domain representation of an LTI bounded causal operator $\mathcal{S}$: $L_2^N(R)\to L_2^N(R)$ (see Section 7). It remains only to establish (2). If $a\in L_{2e}^N(R)$ then

$$\int_{-\infty}^{T}(a^*a-(\mathcal{S}a)^*(\mathcal{S}a))\,\mathrm{d}t=\int_{-\infty}^{\infty}(a_T^*a_T-(\mathcal{S}a)_T^*(\mathcal{S}a)_T)\,\mathrm{d}t$$

$$=\int_{-\infty}^{\infty}(a_T^*a_T-(\mathcal{S}a_T)_T^*(\mathcal{S}a_T)_T)\,\mathrm{d}t$$

$$=\int_{-\infty}^{\infty}(a_T^*a_T-(\mathcal{S}a_T)^*(\mathcal{S}a_T))\,\mathrm{d}t+\int_{T}^{\infty}(\mathcal{S}a_T)^*(\mathcal{S}a_T)\,\mathrm{d}t$$

since $\mathcal{S}$ is causal. Note that the second integral exists since $a_T \in L_2^N(R)$ and $\mathcal{S}: L_2^N(R) \to L_2^N(R)$

$$\geqslant \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{a_T^*}(j\omega)(I - S^*(j\omega)S(j\omega))\widehat{a_T}(j\omega)\,\mathrm{d}\omega \geqslant 0$$

since $I - S^*(j\omega)S(j\omega)$ is positive semidefinite for (almost all) $\omega \in \mathbb{R}$. This proves $\mathcal{N}$ is *passive* and completes the proof of Theorem 1. □

5. DISCUSSION

In this section we examine the definition of passivity we have used, the restriction imposed by *solvability*, and our proof of the passivity criterion. Let us first consider the energy integral (2). Several authors use the alternative integral

$$\int_{-\infty}^{\infty} (a^*a - b^*b)\,\mathrm{d}t \geqslant 0 \tag{3}$$

where $a, b \in L_2^N(R)$ instead of the extended spaces $L_{2e}^N(R)$. Let us call an N-port $\mathcal{N}$ *weakly passive* if it satisfies (3) and is *solvable*. Wohlers[3,4] points out that *weak passivity* has the advantage of being independent of causality. We can prove a theorem analogous to Theorem 1 for *weakly passive* N-ports:

Theorem 3

A *solvable* N-port $\mathcal{N}$ is LTI and *weakly passive* if and only if
(i) $\mathcal{N}$ has a scattering matrix $S(j\omega)$.
(ii) $I - S^*(j\omega)S(j\omega)$ is positive semidefinite for (almost all) $\omega \in \mathbb{R}$.

The difference between this and Theorem 1 is that $S(j\omega)$ need not have an analytic extension into the RHP, and when it does $I - S^*(s)S(s)$ need not be positive semidefinite there (cf. example in Section 1).

Proof. Corollary 1 is easily checked for a LTI *weakly passive* $\mathcal{N}$. Bochner's theorem applies directly and we conclude that $\mathcal{N}$ has a scattering matrix $S(j\omega)$. If $I - S^*(j\omega)S(j\omega)$ were negative definite in some set $\bar{\Delta}$ of positive measure, we could construct a (measurable) $\hat{a}(j\omega)$ supported on $\Delta \cup -\Delta$ with $\hat{a}(j\omega)^*\hat{a}(j\omega) = 1$ and $\hat{a}(j\omega)^*(I - S^*(j\omega)S(j\omega))a(j\omega) < -\varepsilon < 0$ for $\omega \in \Delta \cup -\Delta$ where $\mu(\Delta) < \infty$ and $\Delta \subseteq [0, \infty)$ or $(-\infty, 0]$, and $\hat{a}(-j\omega) = \overline{\hat{a}(j\omega)}$ as in Lemma 2. Then $\hat{a}(j\omega) \in L_2^N$ and corresponds to $a(t) \in L_2^N(R)$ for which

$$\int_{-\infty}^{\infty} (a^*a - b^*b)\,\mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{a}^*(j\omega)(I - S^*(j\omega)S(j\omega))\hat{a}(j\omega)\,\mathrm{d}\omega$$

$$\leqslant \frac{-\varepsilon\mu(\Delta)}{\pi} < 0$$

which contradicts *weak passivity* (3).

The converse is easily proved, for suppose $\mathcal{N}$ has a scattering matrix $S(j\omega)$ with $I - S^*(j\omega)S(j\omega)$ positive semidefinite for $\omega \in \mathbb{R}$. Then $\mathcal{N}$ is clearly LTI and $S(j\omega)$ is bounded so if $a \in L_2^N(R)$, $\hat{b}(j\omega) = S(j\omega)\hat{a}(j\omega) \in L_2^N$, so $b \in L_2^N$. Furthermore

$$\int_{-\infty}^{\infty} (a^*a - b^*b)\,\mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{a}^*(j\omega)(I - S^*(j\omega)S(j\omega))\hat{a}(j\omega)\,\mathrm{d}\omega \geqslant 0$$

So that $\mathcal{N}$ is *weakly passive*. □

An example of a *weakly passive* but not *passive* (solvable) 1-port is $\mathcal{N}$ given by

$$b(t) = \int_{-\infty}^{\infty} \frac{\sin^2 \pi\tau}{\pi^2\tau^2} a(t - \tau)\,\mathrm{d}\tau$$

for which

$$S(j\omega)=\begin{cases}1-|\omega|, & |\omega|\leq 1\\ 0, & |\omega|>1\end{cases}$$

Note that $S(j\omega)$ has no analytic extension into the RHP and that $\mathcal{S}$ is not a causal operator. Another example is a -1H inductor, which has $S(j\omega)=(1+j\omega)/(1-j\omega)$. It is *weakly passive* but not *passive*; however it is not *solvable*.

The relation between *weak passivity* and *passivity* is simple:

Theorem 4

A solvable N-port $\mathcal{N}$ is *passive* if and only if it is *weakly passive* and its scattering operator $\mathcal{S}$ is *causal.*

Proof. If $\mathcal{N}$ is *passive*, then it is clearly *weakly passive* and we have seen in corollary 2 that its $\mathcal{S}$ is causal. Conversely, if $\mathcal{N}$ is *weakly passive* then its $\mathcal{S}$ is a bounded operator and if it is causal then $(\mathcal{S}(a_T))_T=(\mathcal{S}a)_T$.

Following the argument in Section 4, if $a\in L_{2e}^N$ then $a_T\in L_2^N$ and

$$\int_{-\infty}^{T}(a^*a-b^*b)\,\mathrm{d}t=\int_{-\infty}^{\infty}(a_T^*a_T-(\mathcal{S}a_T)^*(\mathcal{S}a_T))\,\mathrm{d}t+\int_{T}^{\infty}(\mathcal{S}a_T)^*(\mathcal{S}a_T)\,\mathrm{d}t\geq 0$$

Thus $\mathcal{N}$ is *passive.*

One final remark concerning *weak passivity* is in order. Any proof of the passivity criterion, Theorem 1, which has as hypothesis only *weak passivity* without the auxiliary assumption that $\mathcal{S}$ is causal is incorrect. Mere analyticity of Z or S is not enough, though boundedness of S is (see Section 7, Bochner–Paley–Wiener theorem; cf. References 7–9). Nor is the assumption that $\mathcal{Z}$ is causal adequate as the -1H inductor shows.†

We now turn to the technical assumption of *solvability* which, informally, says that there are 'enough', admissible a's. The obvious example of a *passive* but not *solvable* device is the 1-port nullator characterized by $v\equiv i\equiv 0$. We will show now that the nullator is the *only* reasonable *passive* 1-port which is not *solvable.*

Suppose the 1-port $\mathcal{N}$ is *passive* but not *solvable.* We assume that the set M of admissible a's $\in L_2(R)$ is closed.‡ M is a closed, translation invariant subspace of $L_2(R)$ which by a theorem of Bochner and Wiener[11] may be described by

$$M=\{a\in L_2(R)|\hat{a}(j\omega)=0\text{ for (almost all) }|\omega|\in E\}$$

for some $E\subseteq[0,\infty)$. Thus the admissible a's in L_2 are simply those whose spectrum vanishes on a certain set E of frequencies; it can be shown that the corresponding b's, also have spectra vanishing on E. It follows that $\hat{v}=\hat{i}=0$ on E, that is, $\mathcal{N}$ acts as a *frequency selective nullator.*

We now make the observation that if a signal $a(t)$ which is not identically zero satisfies $a(t)=0$ for $t<0$ (let us call such a signal *positively supported*) then $\hat{a}(j\omega)$ vanishes for ω in a set of measure zero. This is easily seen from the fact that $\hat{a}(j\omega)$ has an analytic extension in the RHP which would vanish identically if $\hat{a}(j\omega)$ vanished on a set of positive measure, or from the well known version of the Paley–Wiener theorem which asserts

$$\int_{-\infty}^{\infty}\frac{|\ln|\hat{a}(j\omega)||}{1+\omega^2}\,\mathrm{d}\omega<\infty$$

We conclude that the only *positively supported* $a\in M$ is 0. If there is no other $a\in M$, then $\mathcal{N}$ is simply a nullator. All other $a\in M$ have the curious property of having started in the infinite past (and in fact they

† It is interesting to note that Raisbeck's original definition of passivity is what we call *weak passivity* together with the additional assumption that Z is causal, so that his criterion is not quite right.

‡ This is not a deep assumption since lemma 1 shows that $\mathcal{S}$ exists and is bounded, hence extends to the closure of its domain uniquely and boundedly. An $\mathcal{N}$ with a non-closed domain is similar to the anlaytic function $s-1/s-1$ defined on $C-\{1\}$.

must continue into the infinite future). This precludes any testing of the device in the laboratory (a pathology shared by many non-causal devices). It is very natural, if not philosophically necessary, to assume this cannot happen. Specifically, if we make the assumption:

$$\text{There is a positively supported } a \in M,\ a \neq 0. \tag{4}$$

then we may conclude that $\mathcal{N}$ is *solvable*.

A generalization to N-ports may be made even though the closed translation invariant subspaces of L_2^N are quite complicated.[14]

$$\text{There is a } \textit{positively supported } a \neq 0 \text{ such that}$$
$$a(t)e_k \in M, \qquad k = 1, 2, \ldots, N \tag{5}$$

(5) implies that $\mathcal{N}$ is solvable, for our argument above indicates that each of the N sets $\{0\} \times \ldots \times L_2(R) \times \ldots \{0\} \subseteq M$, so by linearity $L_2^N(R) = M$. We should mention here that when (5) is not satisfied, $\mathcal{N}$ can be more complicated than a nullator—we might have for example $M = \{a \in L_2^2(R) | \hat{a}_1(j\omega) + \hat{a}_2(j\omega) = 0$ for $|\omega| \in [0, 1]\}$. Our point is that the reasonable assumption (5) implies *solvability*.

The reader may have wondered why we have used the scattering representation as opposed to the more common impedance representation, used for example in Raisbeck's original informal argument. There are two reasons: certain *passive* devices such as open circuits do not have an impedance representation, and more important, for a *passive* device the scattering operator $\mathcal{S}$ is *bounded* whereas the impedance operator $\mathcal{Z}$ need not be. The recognition of the importance of the scattering representation for *passive* networks is of course due to Youla *et al.*

The boundedness of $\mathcal{S}$ is crucial to our proof. First it allows us to use the Bochner–Paley–Wiener theorem to show that a *passive* $\mathcal{N}$ has a scattering matrix $S(s)$. Distribution theory must be used to prove that $\mathcal{N}$ has an impedance matrix $Z(s)$ (assuming it *has* an impedance representation). Even assuming the existence of $Z(s)$, as Raisbeck and Kuo do, it may be unbounded.

We can have $\mathcal{Z}i \notin L_2^N$, even if $i \in L_2^N$, so that Parseval's relation must be used with care. Furthermore Poisson's representation is not valid. For example, if $\mathcal{N}$ is a (quite *passive*) 1 H inductor, $\sigma_0 j\omega/((\omega - \omega_0)^2 + \sigma_0^2)$ is not even integrable (e.g. reference 1, line 16, reference 9, line 5). In the sufficiency proof we considered a_T, admissible since all of $L_2^N(R)$ was known to be admissible; this too was a consequence of the boundedness of $\mathcal{S}$. The same argument fails for $\mathcal{Z}$, since its domain may be a proper subset of $L_2^N(R)$. With the inductor above, i_T need not be admissible since i_T is generally not differentiable. This is only a partial list, but we can say that arguments using $\mathcal{Z}$ instead of $\mathcal{S}$ cannot be made formal without considerable trouble.

6. PASSIVITY CRITERION WITH GENERAL CO-ORDINATES

In this section we consider the use of variables other than the scattering variables. Specifically, we consider the variables ξ and η related to v and i by

$$\begin{bmatrix} v \\ i \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix}, \qquad \begin{bmatrix} \xi \\ \eta \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} v \\ i \end{bmatrix} \tag{6}$$

where

$$\Omega = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is a real invertible $2N \times 2N$ matrix. We shall say a LTI N-port $\mathcal{N}$ has an Ω-representation if for each $\mathcal{N}$-admissible $\xi(t)$ there is a unique $\mathcal{N}$-admissible $\eta(t)$, in other words there is an (LTI) operator Λ with $\xi = \Lambda\eta$. We assume neither that the domain of Λ includes $L_2^N(R)$ nor that Λ is bounded. For example an inductor has an $\Omega = I_2$ representation with $\Lambda\eta = L\dot{\eta}$; we call this the *impedance representation* and Λ the *impedance operator*. By suitable choice of Ω, this general framework includes the scattering, impedance,