Linear Matrix Inequalities in System and Control Theory

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Abstract

A wide variety of problems in system and control theory can be formulated (or reformulated) as convex optimization problems involving linear matrix inequalities, that is, constraints requiring an affine combination of symmetric matrices to be positive semidefinite. For a few very special cases, there are "analytical solutions" to these problems, but in general they can be solved numerically very efficiently.

We introduce the reader to Linear Matrix Inequalities or LMIs, provide a brief history of LMIs in system and control theory, and discuss a few problems from systems and control that can be solved via convex optimization over LMIs.

1 Introduction

A number of problems that arise in systems and control such as optimal matrix scaling, digital filter realization, interpolation problems that arise in system identification, and robustness analysis and state-feedback synthesis via Lyapunov functions can be reduced to a handful of standard convex and quasiconvex problems that involve linear matrix inequalities. Extremely efficient interior point algorithms have recently been developed for and tested on these standard problems; further development of algorithms for these standard problems is an area of active research.

Our objective in this paper is to give a brief survey of LMIs in system and control theory: In §2, we present a brief history of LMIs in system and control theory; in §3, we introduce a few "standard" convex and quasiconvex optimization problems involving LMIs; in §4, we present a few problems from system and control theory that may be solved via LMI techniques—scaling a matrix to minimize its condition number, stability analysis of linear differential inclusions and optimization over an affine family of transfer matrices; we make a few brief remarks about solving LMI-based optimization problems in §5, and conclude with §6.

2 A Brief History of LMIs in System and Control Theory

The history of linear matrix inequalities in the analysis of dynamical systems goes back more than 100 years, when Lyapunov published his seminal work introducing what we now call Lyapunov theory. He showed that the differential equation

$$\frac{d}{dt}x(t) = Ax(t) \tag{1}$$

is stable if and only if there exists a positive definite matrix P such that

$$A^T P + PA < 0. \tag{2}$$

The requirement $P > 0$, $A^T P + PA < 0$ is what we now call a Lyapunov inequality on P, which is a special form of a linear matrix inequality. Of course, we can solve this LMI (that is, find a suitable P) by solving a Lyapunov equation.

The next major development was in the 1940's when Lur'e, Postnikov, and others in the Soviet Union applied Lyapunov's methods to some specific practical problems in control engineering, especially, the problem of stability of a control system with a nonlinearity in the actuator [18]. Although they did not explicitly form matrix inequalities, their stability criteria in fact have the form of linear matrix inequalities. These inequalities were reduced to polynomial inequalities which were then checked "by hand" (for, needless to say, small systems).

Then, in the 1960's, Yakubovich, Popov, Kalman, and other researchers succeeded in reducing the solution of the linear matrix inequalities that arose in the problem of Lur'e to simple graphical criteria, using what we now call the Kalman-Yakubovich-Popov (KYP) lemma. This resulted in the celebrated Popov criterion, Circle criterion, Tsypkin criterion, and many variations. These criteria could be applied to higher order systems, but did not gracefully or usefully extend to systems containing more than one nonlinearity. Thus, their contribution may be viewed—in the context of the history of LMIs in control theory—as showing how to solve a certain family of linear matrix inequalities by a graphical method. We should note that the important role of LMIs in control theory was already recognized in the early 1960's, especially by Yakubovich [29].

By 1971, researchers knew several methods for solving special types of LMIs: direct (for very small systems), graphical methods, and by solving Lyapunov or Riccati equations. In Willems' 1971 paper [28] we find the following striking quote:

> The basic importance of the LMI seems to be largely unappreciated. It would be interesting to see whether or not it can be exploited in computational algorithms, for example.

Willems' suggestion that linear matrix inequalities might have some advantages in computational algorithms became closer to being realized with the following observation:

> The LMIs that arise in system and control theory can be formulated as *convex optimization problems*, and hence are amenable to computer solution.

This observation was made explicitly by several researchers: Pyatnitskii and Skorodinskii [24] Horisberger and Belanger [14], to name just a few. This observation, coupled with the development of interior point methods that apply directly to convex problems involving matrix inequalities, by Nesterov and Nemirovsky in 1988, now mean that we can reliably and quickly solve many problems in systems and control for which no "analytical solution" has been found (or is likely to be found), by reducing to them to LMI problems.

3 Optimization Problems based on Linear Matrix Inequalities

A linear matrix inequality is a matrix inequality of the form

$$F(x) \triangleq F_0 + \sum_{i=1}^{m} x_i F_i > 0, \tag{3}$$

where $x \in \mathbf{R}^m$ is the variable, and $F_i = F_i^T \in \mathbf{R}^{n\times n}$, $i = 0, \ldots, m$ are given. The set $\{x \mid F(x) > 0\}$ is convex, and need not have smooth boundary. (We've used strict inequality mostly for convenience; inequalities of the form $F(x) \geq 0$ are also readily handled.)

Multiple LMIs $F_1(x) > 0, \ldots, F_n(x) > 0$ can be expressed as the single LMI

$$\mathbf{diag}(F_1(x), \ldots, F_n(x)) > 0.$$

When the matrices F_i are diagonal, the LMI $F(x) > 0$ is just a set of linear inequalities. Nonlinear (convex) inequalities are converted to LMI form using Schur complements. The basic idea is as follows: the LMI

$$\begin{bmatrix} Q(x) & S(x) \\ S(x)^T & R(x) \end{bmatrix} > 0 \tag{4}$$

where $Q(x) = Q(x)^T$, $R(x) = R(x)^T$, and $S(x)$ depend affinely on x, is equivalent to

$$R(x) > 0, \; Q(x) - S(x)R(x)^{-1}S(x)^T > 0. \tag{5}$$

In other words, the set of nonlinear inequalities (5) can be represented as the LMI (4).

The matrix norm constraint $\|Z(x)\| < 1$, where $Z(x) \in \mathbf{R}^{p\times q}$ and depends affinely on x, is represented as the LMI

$$\begin{bmatrix} I & Z(x) \\ Z(x)^T & I \end{bmatrix} > 0$$

(since $\|Z\| < 1$ is equivalent to $I - ZZ^T > 0$). Note that the case $q = 1$ reduces to a general convex quadratic inequality on x.

The constraint

$$\mathbf{Tr}\, S(x)^T P(x)^{-1} S(x) < 1, \;\; P(x) > 0,$$

where $P(x) = P(x)^T \in \mathbf{R}^{n\times n}$ and $S(x) \in \mathbf{R}^{n\times p}$ depend affinely on x, is handled by introducing a new (slack) matrix variable $X = X^T \in \mathbf{R}^{p\times p}$, and the LMI (in x and X):

$$\mathbf{Tr}\, X < 1, \;\; \begin{bmatrix} X & S(x)^T \\ S(x) & P(x) \end{bmatrix} > 0.$$

We often encounter problems in which the variables are matrices, *e.g.*,

$$A^T P + PA < 0 \tag{6}$$

where $A \in \mathbf{R}^{n\times n}$ is given and $P = P^T$ is the variable. In this case we will not write out the LMI explicitly in the form $F(x) > 0$, but instead make clear which matrices are the variables. Leaving LMIs in a condensed form such as (6), in addition to saving notation, leaves open the possibility of more efficient computation.

3.1 LMI feasibility problems

Given an LMI $F(x) > 0$, the corresponding LMI Problem (LMIP) is to find x^{feas} such that $F(x^{\mathrm{feas}}) > 0$ or determine that the LMI is infeasible. (By duality, this means: find a nonzero $G \geq 0$ such that $\mathbf{Tr}\, GF_i = 0$ for $i = 1, \ldots, m$ and $\mathbf{Tr}\, GF_0 \leq 0$.) Of course, this is a convex feasibility problem. We will say "solving the LMI $F(x) > 0$" to mean solving the corresponding LMIP.

3.2 Eigenvalue problems

The eigenvalue problem (EVP) is to minimize the maximum eigenvalue of a matrix, subject to an LMI:

$$\begin{array}{ll} \text{minimize} & \lambda \\ \\ \text{subject to} & \lambda I - A(x) > 0, \quad B(x) > 0. \end{array}$$

Here, A and B are symmetric matrices that depend affinely on the optimization variable x. This is a convex optimization problem.

3.3 Generalized eigenvalue problems

The generalized eigenvalue problem (GEVP) is to minimize the maximum generalized eigenvalue of a pair of matrices that depend affinely on a variable, subject to an LMI constraint. The general form of a GEVP is:

$$\begin{array}{ll} \text{minimize} & \lambda \\ \\ \text{subject to} & \lambda B(x) - A(x) > 0 \\ & B(x) > 0 \\ & C(x) > 0 \end{array}$$

where A, B and C are affine functions of x. This is a quasiconvex problem.

Note that when the matrices are all diagonal, this problem reduces to the general linear fractional programming problem. Many nonlinear quasiconvex functions can be represented in the form of a GEVP with appropriate A, B, and C (see [5]).

4 LMI Problems in Systems and Control

4.1 Minimizing condition number by scaling

The problem of improving the condition number of a matrix via diagonal scaling arises in numerical analysis and control [2, 9, 27]. (The condition number of a matrix M, denoted by $\kappa(M)$, is the ratio of the largest and smallest singular values if M is nonsingular, and ∞ otherwise.)

Given a matrix $M \in \mathbf{R}^{p \times q}$, we consider the problem:

$$\begin{array}{ll} \text{minimize} & \kappa(LMR) \\ \\ \text{subject to} & L \in \mathbf{R}^{p \times p}, \text{ diagonal and nonsingular} \\ & R \in \mathbf{R}^{q \times q}, \text{ diagonal and nonsingular} \end{array} \tag{7}$$

where L and R are the optimization variables. We will show that this problem can be transformed into a GEVP.

Let us assume without loss of generality that $p \geq q$ and M is full-rank.

Let us fix $\gamma > 1$. There exist nonsingular, diagonal $L \in \mathbf{R}^{p\times p}$ and $R \in \mathbf{R}^{q\times q}$ such that $\kappa(LMR) < \gamma$ if and only if

$$\mu I \leq (LMR)^*(LMR) \leq \mu\gamma^2 I$$

for some $\mu > 0$. (M^* stands for the transpose conjugate of M.)

Since we may absorb the factor $1/\sqrt{\mu}$ into L, this is equivalent to the existence of nonsingular, diagonal $L \in \mathbf{R}^{p\times p}$ and $R \in \mathbf{R}^{q\times q}$ such that

$$I \leq (LMR)^*(LMR) \leq \gamma^2 I,$$

which is the same as

$$(RR^*)^{-1} \leq M^*(L^*L)M \leq \gamma^2(RR^*)^{-1}. \tag{8}$$

This is equivalent to the existence of diagonal $P \in \mathbf{R}^{p\times p}$, $Q \in \mathbf{R}^{q\times q}$ with $P > 0$, $Q > 0$ and

$$Q < M^*PM < \gamma^2 Q. \tag{9}$$

To see this, first suppose that $L \in \mathbf{R}^{p\times p}$ and $R \in \mathbf{R}^{q\times q}$ are nonsingular and diagonal, and (8) holds. Then (9) holds with $P = L^*L$ and $Q = (RR^*)^{-1}$. Conversely, suppose that (9) holds for diagonal $P \in \mathbf{R}^{p\times p}$ and $Q \in \mathbf{R}^{q\times q}$ with $P > 0$, $Q > 0$. Then, (8) holds for $L = P^{1/2}$ and $R = Q^{-1/2}$.

Hence we can solve (7) by solving the GEVP:

$$\begin{array}{ll} \text{minimize} & \gamma^2 \\ \text{subject to} & P \in \mathbf{R}^{p\times p} \text{ and diagonal} \\ & Q \in \mathbf{R}^{q\times q} \text{ and diagonal} \\ & 0 < Q < M^*PM < \gamma^2 Q \end{array}$$

We finally note that it is possible to extend the above results to the case of block-diagonal scalings.

4.2 Lyapunov function search

Consider the differential inclusion (DI)

$$\frac{dx}{dt} = A(t)x(t), \;\; A(t) \in \mathbf{Co}\left\{A_1, \ldots, A_L\right\}, \tag{10}$$

where $\mathbf{Co}$ denotes the convex hull. We ask whether the DI is stable, *i.e.*, whether all trajectories of the system (10) converge to zero as $t \longrightarrow \infty$. A sufficient condition for this is the existence of a quadratic positive function $V(z) = z^T Pz$ such that $dV(x(t))/dt < 0$ for any trajectory of (10). Since

$$\frac{d}{dt}V(x(t)) = x(t)^T \left(A(t)^T P + PA(t)\right) x(t),$$

a sufficient condition for stability is the existence of $P > 0$ such that

$$A(t)^T P + PA(t) < 0, \;\; A(t) \in \mathbf{Co}\left\{A_1, \ldots, A_L\right\}. \tag{11}$$

If there exists such a P, we say the DI (10) is *quadratically stable.*

Condition (11) is equivalent to

$$P > 0, \quad A_i^T P + P A_i < 0, \quad i = 1, \ldots, L,$$

which is a linear matrix inequality in P (see for example [7, 14, 15, 25]). Thus, determining quadratic stability is an LMIP.

V is sometimes called a *simultaneous quadratic Lyapunov function* since it proves stability of each of $A_1, \ldots, A_L$.

4.3 Lyapunov functions and state feedback

Consider the system (10) with state feedback:

$$\frac{dx}{dt} = A(t)x(t) + B(t)u(t), \quad u(t) = Kx(t), \tag{12}$$

where

$$[A(t)\ B(t)] \in \mathbf{Co}\left\{[A_1\ B_1], \ldots, [A_L\ B_L]\right\}.$$

Our objective is to design the matrix K such that such that (12) is quadratically stable. This is the "quadratic stabilizability" problem (see [13], and [23, 12, 10]; related references are [30, 17, 16] and [22]).

System (12) is quadratically stable for some state state-feedback K if there exist $P > 0$ and K such that

$$(A_i + B_i K)^T P + P(A_i + B_i K) < 0, \quad i = 1, \ldots, L.$$

Note that this matrix inequality is *not* convex in P and K. However, with the linear fractional transformation $Y \triangleq P^{-1}$, $W \triangleq KP^{-1}$, we may rewrite it as

$$(A_i + B_i W Y^{-1})^T Y^{-1} + Y^{-1}(A_i + B_i W Y^{-1}) < 0.$$

Multiplying this inequality on the left and right by Y (such a congruence preserves the inequality) we get an LMI in Y and W:

$$Y A_i^T + W^T B_i^T + A_i Y + B_i W < 0, \quad i = 1, \ldots, L.$$

If this LMIP in Y and W has a solution, then the Lyapunov function $V(z) = z^T Y^{-1} z$ proves the quadratic stability of the closed-loop system with state-feedback $u(t) = WY^{-1}x(t)$.

In other words, we can synthesize a linear state feedback for the DI (10) by solving a set of simultaneous Lyapunov inequalities.

Let us also note that by synthesizing a state feedback for the DI (10), we have also synthesized a suitable state feedback for the nonlinear, time-varying, uncertain system

$$\frac{dx}{dt} = f(x, u, t)$$

if, for each x, u and t, there exists $G(x, u, t) \in \mathbf{Co}\left\{[A_1\ B_1], \ldots, [A_L\ B_L]\right\}$ so that

$$f(x, u, t) = G(x, u, t)\begin{bmatrix} x \\ u \end{bmatrix}.$$

4.4 Optimization over an affine family of linear systems

We consider an affine family of stable linear systems of the form

$$\begin{aligned} \frac{d}{dt}x(t) &= Ax(t) + B_w w(t), \\ z(t) &= C_z(\theta)x(t) + D_{zw}(\theta)w(t), \end{aligned} \tag{13}$$

where C_z and D_{zw} depend affinely on a parameter vector θ (the "design" parameter), and we assume that (A, B_w) is controllable. In this section, we show that we may choose θ such that system (13) has certain desirable properties by solving LMIPs which have θ as one of the optimization variables. The results of this section may be applied, for instance, to the problem of Finite Impulse Response filter design.

4.4.1 $\mathbf{H}_2$ norm

The $\mathbf{H}_2$ norm of system (13), if $D_{zw}(\theta) = 0$, is defined to be $\mathbf{Tr}\, C_z(\theta)W_c C_z(\theta)^T$, where $W_c \geq 0$ is the controllability Gramian which satisfies

$$AW_c + W_c A^T + B_w B_w^T = 0. \tag{14}$$

For nonzero $D_{zw}(\theta)$, the $\mathbf{H}_2$ norm is defined to be ∞. Therefore, the $\mathbf{H}_2$ norm of system (13) is less than or equal to γ if and only if the following LMI in θ and γ^2 is satisfied:

$$D_{zw}(\theta) = 0, \quad \mathbf{Tr}\, C_z(\theta)W_c C_z(\theta)^T \leq \gamma^2.$$

4.4.2 $\mathbf{H}_\infty$ norm

The $\mathbf{H}_\infty$ norm of system (13) is defined to be

$$\max_{\|w\|_2 \leq 1} \|z\|_2,$$

with $x(0) = 0$. From an extension of the Kalman-Yakubovich-Popov lemma, due to Willems [28], the constraint that the $\mathbf{H}_\infty$ norm of system (13) does not exceed γ is equivalent to an LMI in θ, P and γ^2:

$$P \geq 0, \begin{bmatrix} A^T P + PA & PB_w \\ B_w^T P & -\gamma^2 I \end{bmatrix} + \begin{bmatrix} C_z(\theta)^T \\ D_{zw}(\theta)^T \end{bmatrix} \begin{bmatrix} C_z(\theta) & D_{zw}(\theta) \end{bmatrix} \leq 0.$$

4.4.3 Minimum dissipation

Suppose that w and z are vectors with the same number of components. We define the *minimum dissipation* η of as system (13) as the largest η such that

$$\int_0^T w(t)^T z(t) - \eta w(t)^T w(t)\, dt \geq 0$$

holds for all $T \geq 0$, with $x(0) = 0$. Thus passivity corresponds to nonnegative minimum dissipation.

Once again, from Willems [28], the constraint that the minimum dissipation of system (13) be at least η is equivalent to

$$P \geq 0, \quad \begin{bmatrix} A^T P + PA & PB_w - C_z(\theta)^T \\ B_w^T P - C_z(\theta) & 2\eta I - D_{zw}(\theta) - D_{zw}(\theta)^T \end{bmatrix} \leq 0,$$

an LMI in P, θ and η.

5 Solving LMI-based Problems

The most important point is:

LMIPs, EVPs and GEVPs are tractable

in a sense that can be made precise from a number of theoretical and practical viewpoints. (This is to be contrasted with much less tractable problems, *e.g.*, the general problem of robustness analysis for a system with real parameter perturbations.)

From a theoretical standpoint:

- We can immediately write down necessary and sufficient optimality conditions.
- There is a well-developed duality theory (for GEVPs, in a limited sense).
- These problems can be solved in polynomial time (indeed with a variety of interpretations of the term "polynomial-time").

The most important practical implication is that there are effective and powerful algorithms for the solution of these problems, that is, algorithms that rapidly compute the global optimum, with non-heuristic stopping criteria. Thus, on exit, the algorithms can prove that the global optimum has been obtained to within some prespecified accuracy.

There are a number of general algorithms for the solution of these problems, for example, the ellipsoid algorithm (see *e.g.*, [4, 3]). The ellipsoid method has polynomial-time complexity, and works in practice for smaller problems, but can be slow for larger problems. Other algorithms specifically for LMI-based problems are discussed in, *e.g.*, [8, 21].

Recently, various researchers [19, 5, 26, 1] have developed interior point methods for solving LMI-based problems, based on the work of Nesterov and Nemirovsky [20]. Numerical experience shows that these algorithms solve LMI problems with great efficiency. In some specific cases (see for example, [26]), these methods can solve LMI-based problems with computational effort that is comparable to that required to "evaluate" the "analytic" solutions of similar problems.

6 Conclusion

We have shown that many problems in systems and control can be cast as convex optimization problems involving LMIs. These problems do not have "analytic solutions" but can be solved extremely efficiently. The list of problems we have presented is by no means exhaustive. Other problems include:

- synthesis of gain-scheduled state-feedback
- linear controller design via Q-parametrization
- multi-criterion LQG
- interpolation problems involving scaling
- synthesis of Lyapunov functions with Popov terms for nonlinearities
- synthesis of multipliers for analysis of systems with unknown constant parameters
- analysis and design for randomly varying systems

- synthesis of quadratic Lyapunov functionals for delay systems
- problems in robust identification

We refer the reader to the forthcoming monograph [6] for details.

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