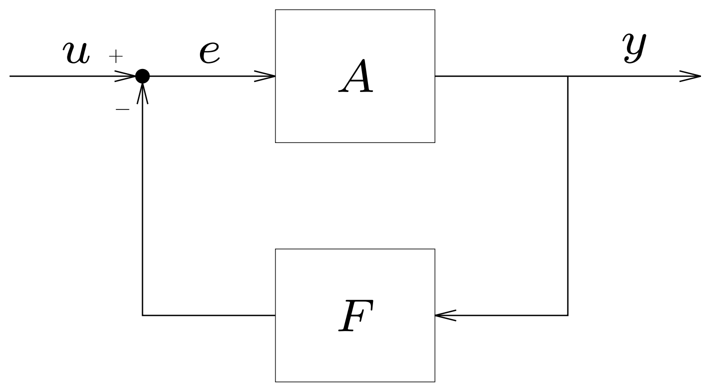
Lecture 11
Feedback: static analysis

- feedback: overview, standard configuration, terms
- static linear case
- sensitivity
- static nonlinear case
- linearizing effect of feedback

Feedback: general

a portion of the output signal is 'fed back' to the input

standard block diagram:



- u is the *input signal*; y is the *output signal*; e is called the *error signal*
- A is called the *forward* or *open-loop* system or *plant*
- F is called the *feedback* system

in equations: $y = Ae$, $e = u - Fy$

- feedback 'loop': e affects y, which affects e . . .
- overall system is called *closed-loop* system
- signals can be analog electrical (voltages, currents), mechanical, digital electrical, . . .
- the $-$ sign is a tradition only

feedback is very widely used

- in amplifiers
- in automatic control (flight control, hard disk & CD player mechanics)
- in communications (oscillators, phase-lock loop)

when properly designed, feedback systems are

- less sensitive to component variation
- less sensitive to some interferences and noises
- more linear
- faster

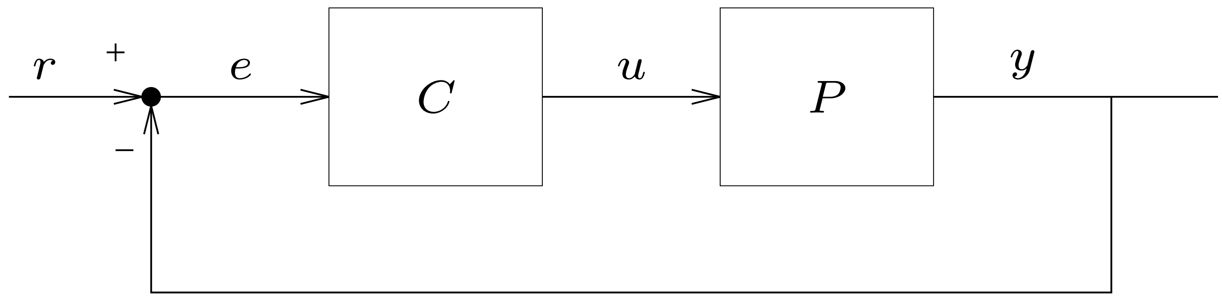
(when compared to similar open-loop systems)

we will also see some disadvantages, $e.g.$

- smaller gain
- possibility of instability

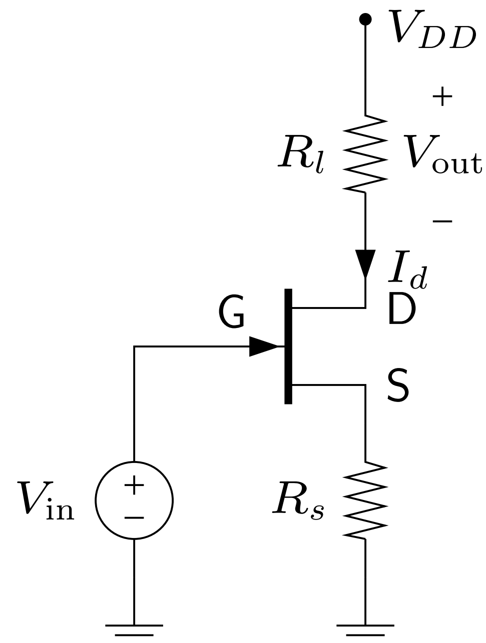
Other feedback configurations

we will also see other feedback configurations, *e.g.*



which is often used in automatic control

for now we stick to the 'standard configuration' (p.11–2) sometimes the 'feedback loop' is not clear ($e.g.$, in amplifier circuits)



here we have

$$V_{\text{out}} = R_l f(V_{GS}), \qquad V_{GS} = V_{\text{in}} - (R_s/R_l)V_{\text{out}},$$

where $I_d = f(V_{GS})$

Static linear case

static case: signals do not vary with time, $i.e.$, signals u, e, y are (constant) real numbers

(dynamic analysis of feedback is *very* important — we'll do it later)

suppose forward and feedback systems are linear, $i.e.$, A and F are numbers ('gains')

eliminate e from $y = Ae$, $e = u - Fy$ to get $y = Gu$ where

$$G = \frac{A}{1 + AF}$$

is called the *closed-loop system gain* (A is called open-loop system gain)

$L = AF$ is called the *loop gain* — it is the gain around the feedback loop, cut at the summing junction observation: if $L = AF$ is large (positive or negative!) then $G \approx 1/F$ and is relatively independent of A

how close is G to $1/F$?

consider *relative error*: $\dfrac{1/F - G}{1/F} = \dfrac{1}{1+AF}$ (after some algebra)

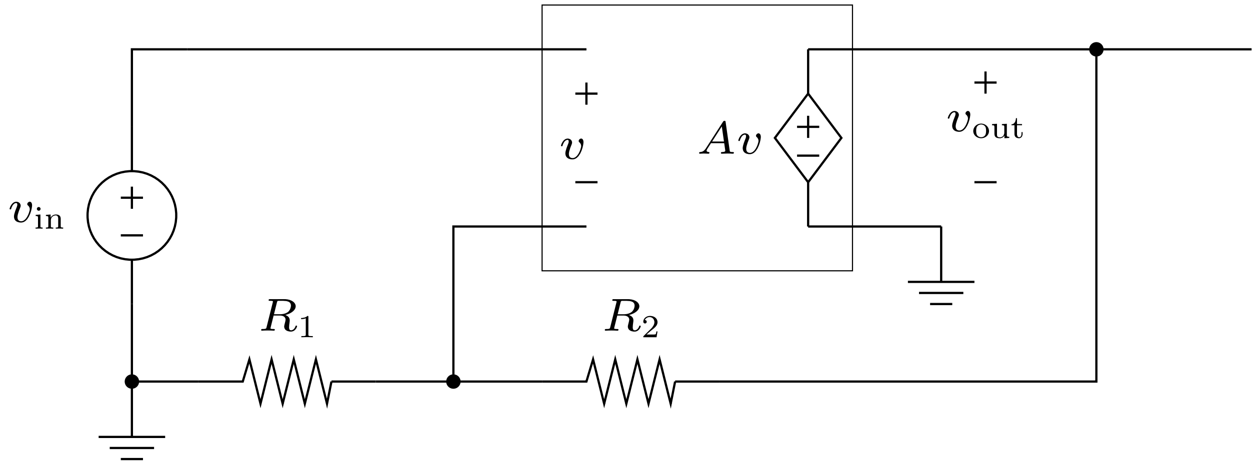
$$S = \frac{1}{1+AF} = \frac{1}{1+L}$$

is called the *sensitivity* (and will come up many times)

for large loop gain, sensitivity $\approx$ 1/loop gain

thus:
for 20dB loop gain, $G \approx 1/F$ within about 10%
for 40dB loop gain, $G \approx 1/F$ within about 1%
etc.

Example: feedback amplifier



described by: $v_{\text{out}} = Av$, $v = v_{\text{in}} - (R_1/(R_1 + R_2))v_{\text{out}}$

- v_{in} is the input u; v_{out} is the output y
- v is the 'error signal' e
- open-loop gain is A
- feedback gain is $F = R_1/(R_1 + R_2)$

$v_{\text{out}} = Gv_{\text{in}}$, where closed-loop gain is $G = \dfrac{A}{1 + AF}$

example: for $F = 0.1$ and $A \geq 100$, $G \approx 10$ within 10%

as A varies from, say, 100 to 1000 (20dB variation),
G varies about 10% (around 1dB variation)

in this example, large variations in open-loop gain lead to much smaller variations in closed-loop gain

Sensitivity to small changes in A

how do small changes in the open-loop gain A affect closed-loop gain G?

$$\frac{\partial G}{\partial A} = \frac{\partial}{\partial A} \frac{A}{1+AF} = \frac{1}{(1+AF)^2}$$

so for small change δA, we have

$$\delta G \approx \frac{1}{(1+AF)^2} \delta A$$

express in terms of *relative* or *fractional* gain changes:

$$(\delta G/G) \approx \frac{1}{1+AF}(\delta A/A) = S(\delta A/A)$$

hence the name 'sensitivity' for S

for small fractional changes in open-loop gain,

$$S \approx \frac{\text{fractional change in closed-loop gain}}{\text{fractional change in open-loop gain}}$$

(so 'sensitivity *ratio*' is perhaps a better term for S)

for large loop gain (positive or negative), $|S| \ll 1$, so small fractional changes in A yield *much smaller* fractional changes in G:

feedback has *reduced* the sensitivity of the gain G w.r.t. changes in the gain A

we can relate (small) relative changes to changes in dB:

$$\delta(20\log_{10}X) = \frac{20}{\log 10}\delta\log X \approx \frac{20}{\log 10}(\delta X/X)$$

($20/\log 10 \approx 9$, *i.e.*, 10% relative change ≈ 0.9dB)

hence we have (for small changes in A),

$$\delta(20\log_{10}G) \approx S\ \delta(20\log_{10}A)$$

thus (for small changes in open-loop gain),

$$S \approx \frac{\text{dB change in closed-loop gain}}{\text{dB change in open-loop gain}}$$

Example: ± 2dB variation in A, with $L \approx 10$, yields approximately ± 0.2dB variation in G

Summary:

for loop gain $|L| \gg 1$,

- gain is reduced by about $|L|$
- sensitivity of gain w.r.t. A is reduced by about $|L|$

thus, feedback allows us to trade gain for reduced sensitivity

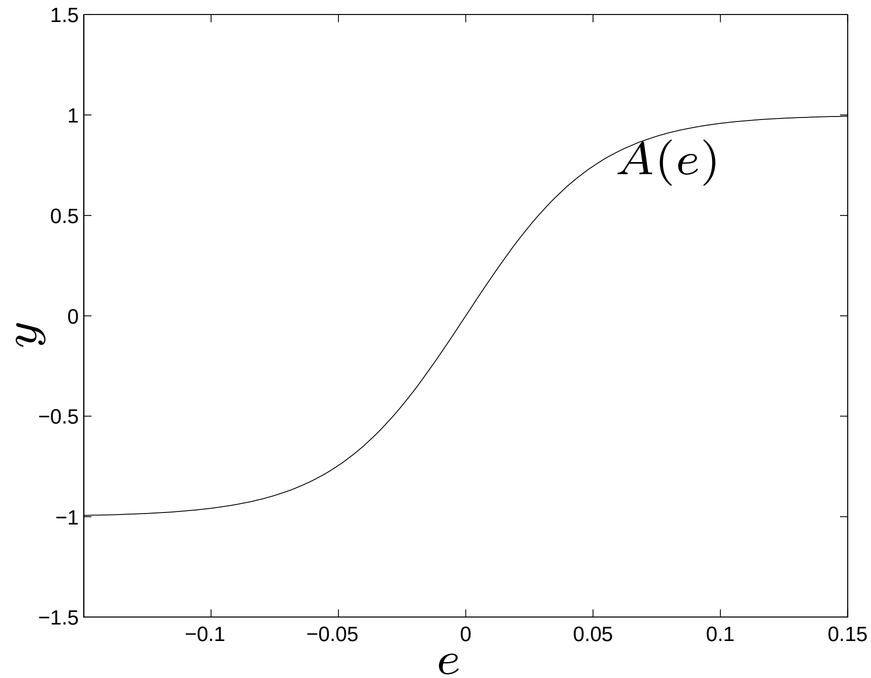
$e.g.$, convert amplifier with gain 30 ± 2dB to one with gain 20 ± 0.7dB or 10 ± 0.2dB

Remarks:

- feedback critical with vacuum tube amplifiers
(gains varied substantially with age . . .)
- get benefits for 'negative' ($AF > 0$) or 'positive' ($AF < 0$) feedback — makes little difference in static case
- sensitivity w.r.t. F is *not* small — need accurate, reliable feedback components
- can also trade sensitivity for more gain, by setting $AF \approx -1$

Nonlinear static feedback

We suppose now that the forward system is nonlinear static, $i.e.$, A is a function from **R** into **R**, $e.g.$,

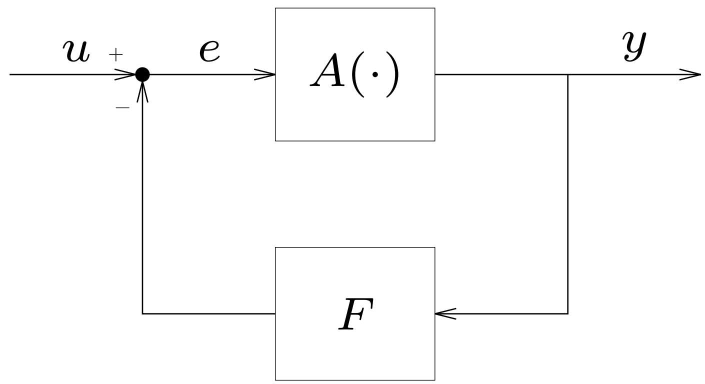


very common for amplifiers, transducers, etc. to be at least a bit nonlinear

A is called the *nonlinear transfer characteristic* of the forward system

(*never* to be confused with transfer function!)

we'll keep the feedback system F linear for now



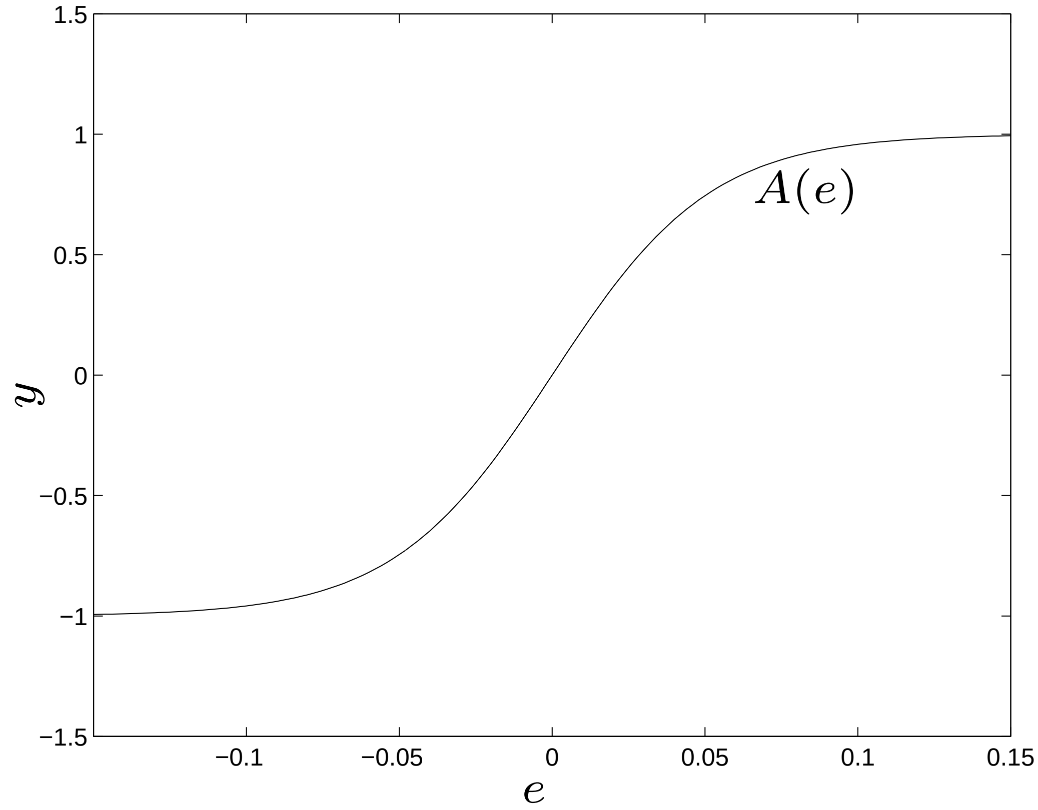
feedback system is described by $y = A(e)$, $e = u - Fy$

these are coupled *nonlinear* equations:

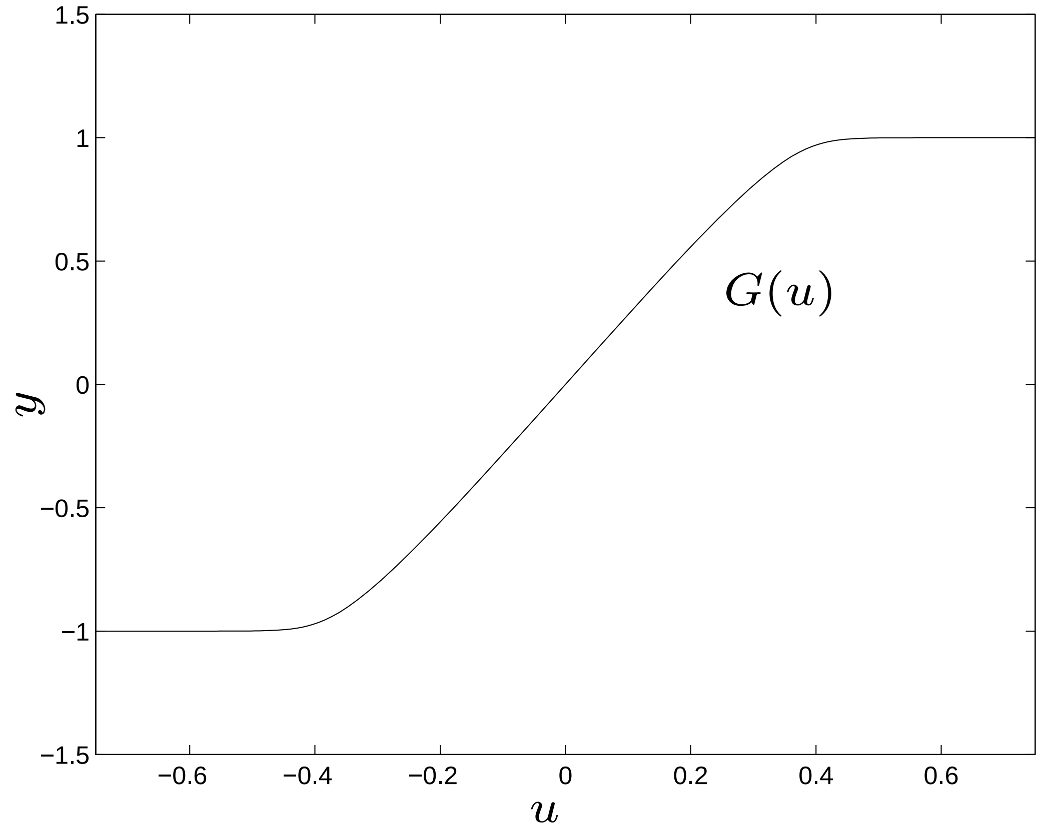
- maybe *multiple* solutions; maybe *no* solutions
- usually impossible to solve analytically
- can be solved graphically, or by computer

usually for each $u \in \mathbf{R}$ there is one solution y, so we can express the *closed-loop transfer characteristic* as a function: $y = G(u)$

Example: open-loop characteristic A:



with feedback gain $F = 0.2$, yields closed-loop characteristic



(you should check a few points!)

Observations: with feedback

- 'gain' is lower (note different horizontal scales)
- characteristic is more linear (for $|y| < 1$)

these phenomena are general . . .

closed-loop transfer characteristic function G satisfies

$$G(u) = y = A(e), \qquad e = u - FG(u)$$

differentiate w.r.t. u:

$$G'(u) = A'(e)\frac{de}{du}, \qquad \frac{de}{du} = 1 - FG'(u)$$

eliminate de/du to get

$$G'(u) = \frac{A'(e)}{1 + A'(e)F}$$

conclusions: for u s.t. $|A'F| \gg 1$,

- $G' \approx 1/F$ (independent of u) $i.e.$, G is nearly linear!
- slope of G is smaller than slope of A (by factor $1 + A'F$)

A measure of nonlinear distortion

let $w = H(v)$ be a nonlinear I/O characteristic

assume $H(0) = 0$ and look at Taylor series

$$H(v) = H'(0)v + \frac{1}{2}H''(0)v^2 + \cdots$$

ratio of quadratic term to first order term is

$$\frac{H''(0)}{2H'(0)}v,$$

so $H''(0)/H'(0)$ gives a measure of distortion
(for a given input v, or a given output w)

now consider feedback system, with $A(0) = 0$

distortion measure for open-loop system is $A''(0)/A'(0)$

differentiate $G' = A'/(1 + A'F)$ w.r.t. u to get

$$G''(u) = \frac{A''(e)}{(1 + A'(e)F)^2}$$

distortion measure for closed-loop system is

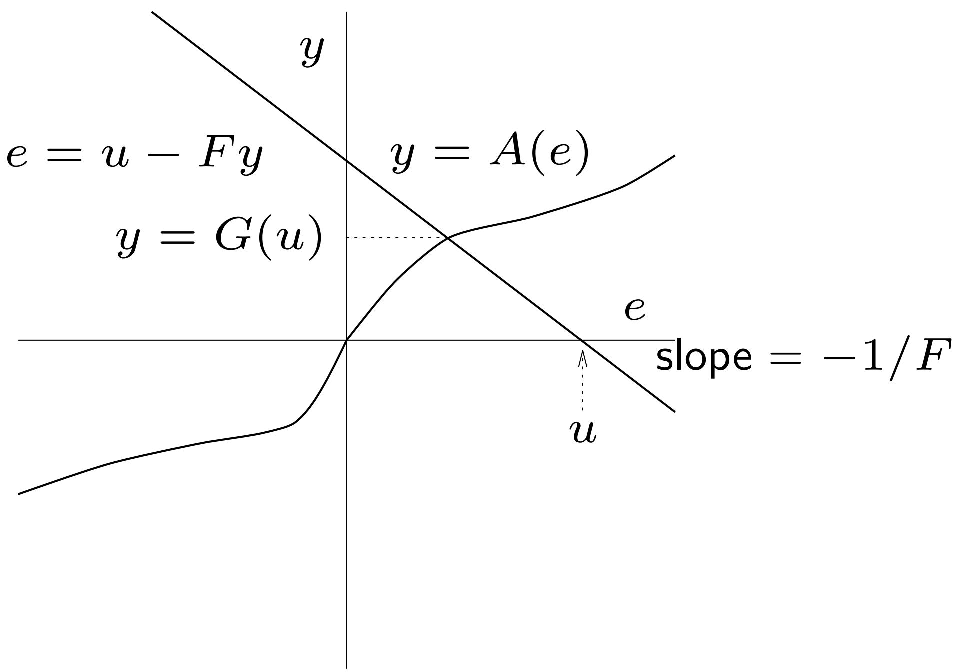
$$G''(0)/G'(0) = \frac{1}{1 + A'(0)F} A''(0)/A'(0)$$

thus, nonlinear distortion measure is reduced by the sensitivity S of the linearized system!

Finding the closed-loop characteristic

Graphical method (load line): write feedback equations as $y = A(e)$, $e = u - Fy$

for given u sketch both equations on e-y plane; intersection gives solution



easy to visualize what happens as u or F changes

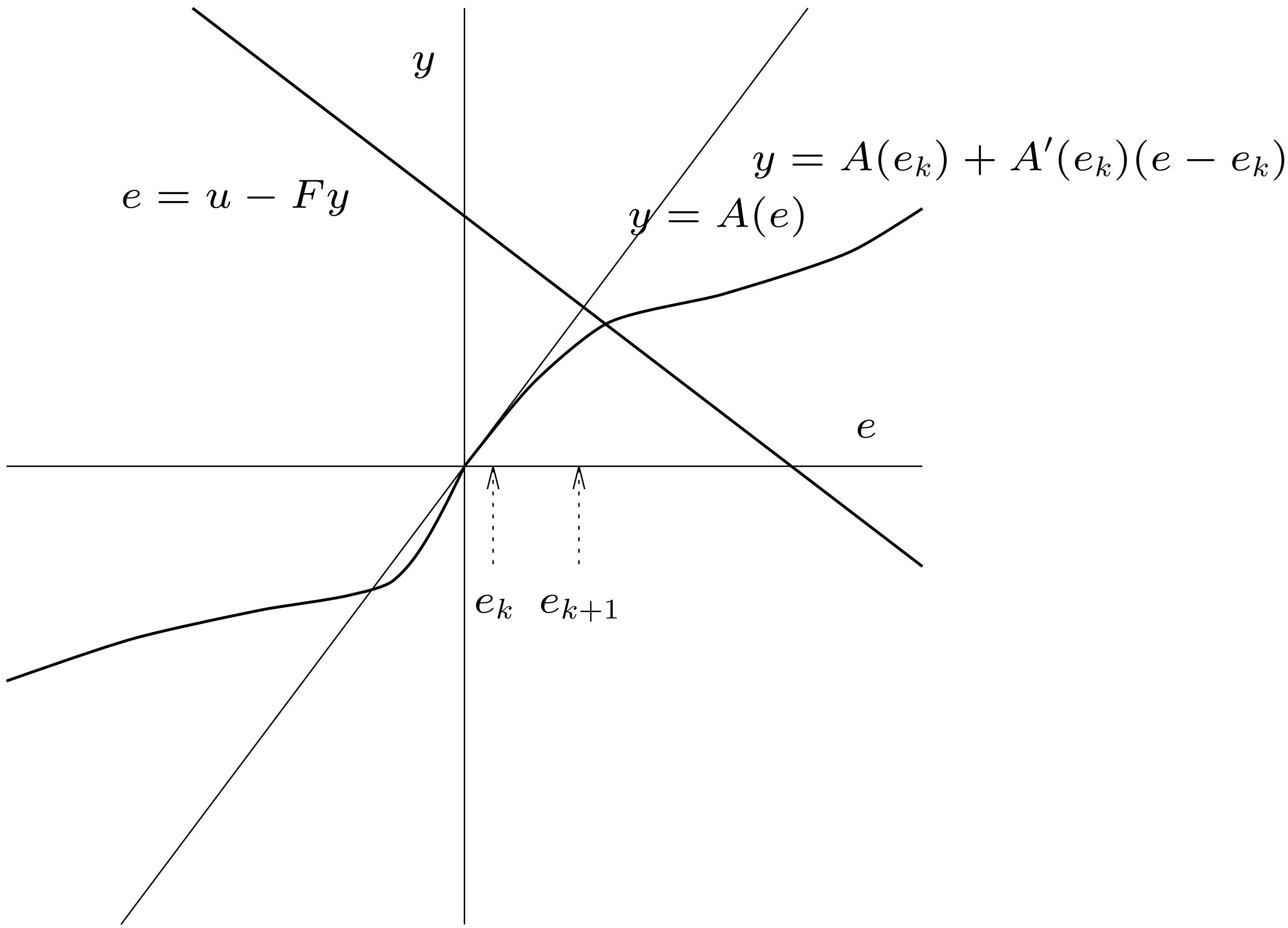
Newton's method to solve $y = A(e)$, $e = u - Fy$ (given A, u, and F)

1. guess a value e_0 for e; set $k = 0$
2. set $y_k := A(e_k)$
3. if $e_k = u - Fy_k$, quit
4. replace nonlinear equation $y = A(e)$ with first-order Taylor expansion near e_k,
$$y \approx A(e_k) + A'(e_k)(e - e_k)$$
Then solve the *linear equations*
$$\begin{aligned} \hat{y} &= A(e_k) + A'(e_k)(\hat{e} - e_k), \\ \hat{e} &= u - F\hat{y} \end{aligned}$$
for $\hat{e}$ and $\hat{y}$; set $e_{k+1} := \hat{e}$

I.e., set $e_{k+1} := \dfrac{u - Fy_k + FA'(e_k)e_k}{1 + FA'(e_k)}$
5. $k := k + 1$; go to 2

works very well when initial guess is good; may not converge for bad initial guess

Graphical interpretation of Newton's method



Tracing the closed-loop characteristic curve

write feedback equations as

$$y = A(e), \qquad u = e + Fy$$

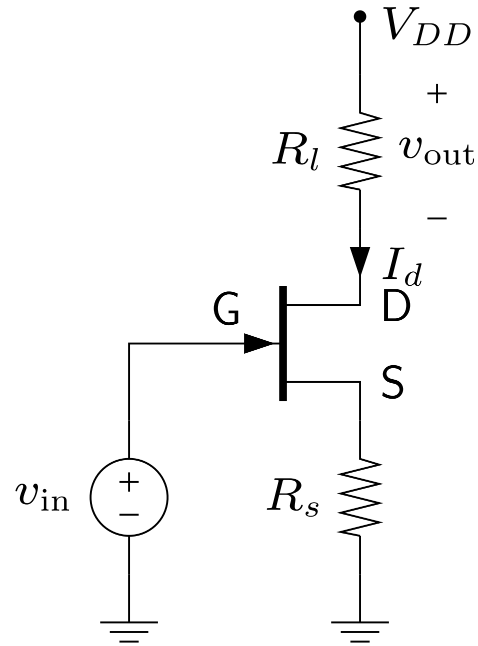
given error e, we can easily find associated y and u!

can use this to trace the curve, parametrized by e:

1. choose $e_1, e_2, \ldots, e_n$ that cover an appropriate range for e
2. for $i = 1$ to n, set $y_i := A(e_i)$, $u_i := e_i + Fy_i$
3. plot $(u_1, y_1), \ldots, (u_n, y_n)$

note that here we don't specify the u values (as in Newton's method)

Example: JFET amplifier (we assume $v_{\rm GS} \leq 0$)



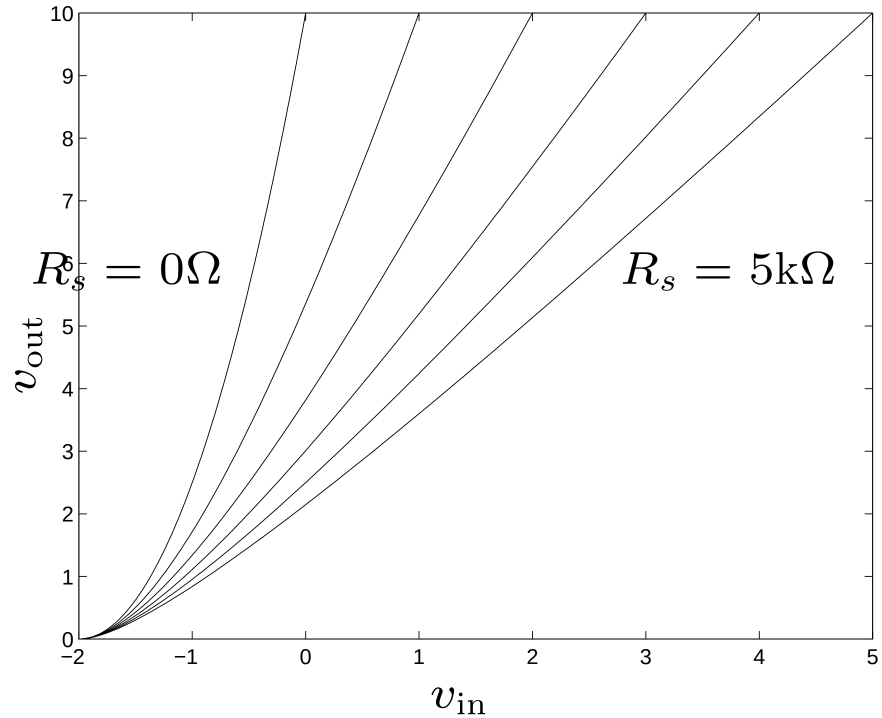
can express as static nonlinear feedback system:

$$v_{\rm out} = A(v_{GS}), \qquad v_{GS} = v_{\rm in} - F v_{\rm out},$$

with $F = R_s/R_l$ and

$$A(v_{GS}) = \begin{cases} R_l I_{DSS}(1 - v_{GS}/V_P)^2 & V_P \leq v_{GS} \leq 0 \\ 0 & v_{GS} < V_P \end{cases}$$

we'll take $R_l = 10\text{k}\Omega$, $I_{DSS} = 1\text{mA}$, $V_P = -2\text{V}$



plot shows v_{out} vs. v_{in} for $R_s = 0, 1, \ldots, 5\text{k}\Omega$
(corresponds to $F = 0, 0.1, \ldots, 0.5$)

as feedback increases, closed-loop 'gain' is smaller; closed-loop characteristic is more linear

Summary

- using feedback we can trade raw gain for lower sensitivity, greater linearity
- benefits determined by $S = 1/(1 + AF)$:
  sensitivity and nonlinearity are both reduced by S
- large loop gain $L = AF$ (positive *or* negative) yields small S hence benefits of feedback