Stochastic Control for Optimal Market-Making

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Overview

1. Trading Order Book Dynamics
2. Definition of Optimal Market-Making Problem
3. Derivation of Avellaneda-Stoikov Analytical Solution
4. Real-world Optimal Market-Making and Reinforcement Learning
Trading Order Book (TOB)

- Market depth
  - Large sell order
  - Mid-price
- Volume
- Best buy
- Spread
- Best sell
- Price p/share
- Limit buy orders
- Limit sell orders
Basics of Trading Order Book (TOB)

- Buyers/Sellers express their intent to trade by submitting bids/asks
- These are Limit Orders (LO) with a price $P$ and size $N$
- Buy LO $(P, N)$ states willingness to buy $N$ shares at a price $\leq P$
- Sell LO $(P, N)$ states willingness to sell $N$ shares at a price $\geq P$
- Trading Order Book aggregates order sizes for each unique price
- So we can represent with two sorted lists of $(Price, Size)$ pairs

Bids: $[(P_i^{(b)}, N_i^{(b)}) \mid 1 \leq i \leq m], P_i^{(b)} > P_j^{(b)}$ for $i < j$

Asks: $[(P_i^{(a)}, N_i^{(a)}) \mid 1 \leq i \leq n], P_i^{(a)} < P_j^{(a)}$ for $i < j$

- We call $P_1^{(b)}$ as simply Bid, $P_1^{(a)}$ as Ask, $\frac{P_1^{(a)} + P_1^{(b)}}{2}$ as Mid
- We call $P_1^{(a)} - P_1^{(b)}$ as Spread, $P_n^{(a)} - P_m^{(b)}$ as Market Depth
- A Market Order (MO) states intent to buy/sell $N$ shares at the best possible price(s) available on the TOB at the time of MO submission
A new Sell LO \((P, N)\) potentially removes best bid prices on the TOB

Removal: \[
((P_i^{(b)}, \min(N_i^{(b)}, \max(0, N - \sum_{j=1}^{i-1} N_j^{(b)})))) \mid (i : P_i^{(b)} \geq P)\]

After this removal, it adds the following to the asks side of the TOB

\[
(P, \max(0, N - \sum_{i : P_i^{(b)} \geq P} N_i^{(b)}))
\]

A new Buy MO operates analogously (on the other side of the TOB)

A Sell Market Order \(N\) will remove the best bid prices on the TOB

Removal: \[
((P_i^{(b)}, \min(N_i^{(b)}, \max(0, N - \sum_{j=1}^{i-1} N_j^{(b)})))) \mid 1 \leq i \leq m\]

A Buy Market Order \(N\) will remove the best ask prices on the TOB

Removal: \[
((P_i^{(a)}, \min(N_i^{(a)}, \max(0, N - \sum_{j=1}^{i-1} N_j^{(a)})))) \mid 1 \leq i \leq n\]
TOB Dynamics and Market-Making

- Modeling TOB Dynamics involves predicting arrival of MOs and LOs
- Market-makers are liquidity providers (providers of Buy and Sell LOs)
- Other market participants are typically liquidity takers (MOs)
- But there are also other market participants that trade with LOs
- Complex interplay between market-makers & other mkt participants
- Hence, TOB Dynamics tend to be quite complex
- We view the TOB from the perspective of a single market-maker who aims to gain with Buy/Sell LOs of appropriate width/size
- By anticipating TOB Dynamics & dynamically adjusting Buy/Sell LOs
- Goal is to maximize *Utility of Gains* at the end of a suitable horizon
- If Buy/Sell LOs are too narrow, more frequent but small gains
- If Buy/Sell LOs are too wide, less frequent but large gains
- Market-maker also needs to manage potential unfavorable inventory (long or short) buildup and consequent unfavorable liquidation
Notation for Optimal Market-Making Problem

- We simplify the setting for ease of exposition
- Assume finite time steps indexed by \( t = 0, 1, \ldots, T \)
- Denote \( W_t \in \mathbb{R} \) as Market-maker’s trading PnL at time \( t \)
- Denote \( I_t \in \mathbb{Z} \) as Market-maker’s inventory of shares at time \( t \) (\( I_0 = 0 \))
- \( S_t \in \mathbb{R}^+ \) is the TOB Mid Price at time \( t \) (assume stochastic process)
- \( P_t^{(b)} \in \mathbb{R}^+, N_t^{(b)} \in \mathbb{Z}^+ \) are market maker’s Bid Price, Bid Size at time \( t \)
- \( P_t^{(a)} \in \mathbb{R}^+, N_t^{(a)} \in \mathbb{Z}^+ \) are market-maker’s Ask Price, Ask Size at time \( t \)
- Assume market-maker can add or remove bids/asks costlessly
- Denote \( \delta_t^{(b)} = S_t - P_t^{(b)} \) as Bid Spread, \( \delta_t^{(a)} = P_t^{(a)} - S_t \) as Ask Spread
- Random var \( X_t^{(b)} \in \mathbb{Z}_{\geq 0} \) denotes bid-shares “hit” up to time \( t \)
- Random var \( X_t^{(a)} \in \mathbb{Z}_{\geq 0} \) denotes ask-shares “lifted” up to time \( t \)

\[
W_{t+1} = W_t + P_t^{(a)} \cdot (X_{t+1}^{(a)} - X_t^{(a)}) - P_t^{(b)} \cdot (X_{t+1}^{(b)} - X_t^{(b)}) , I_t = X_t^{(b)} - X_t^{(a)}
\]

- Goal to maximize \( \mathbb{E}[U(W_T + I_T \cdot S_T)] \) for appropriate concave \( U(\cdot) \)
Markov Decision Process (MDP) Formulation

- Order of MDP activity in each time step $0 \leq t \leq T - 1$:
  - Observe $State := (t, S_t, W_t, I_t)$
  - Perform $Action := (P_t(b), N_t(b), P_t(a), N_t(a))$
  - Experience TOB Dynamics resulting in:
    - random bid-shares hit $= X_{t+1}^{(b)} - X_t^{(b)}$ and ask-shares lifted $= X_{t+1}^{(a)} - X_t^{(a)}$
    - update of $W_t$ to $W_{t+1}$, update of $I_t$ to $I_{t+1}$
    - stochastic evolution of $S_t$ to $S_{t+1}$
  - Receive next-step $(t + 1)$ Reward $R_{t+1}$

\[
R_{t+1} := \begin{cases} 
0 & \text{for } 1 \leq t + 1 \leq T - 1 \\
U(W_{t+1} + I_{t+1} \cdot S_{t+1}) & \text{for } t + 1 = T
\end{cases}
\]

- Goal is to find an *Optimal Policy* $\pi^*$:

\[
\pi^*(t, S_t, W_t, I_t) = (P_t^{(b)}, N_t^{(b)}, P_t^{(a)}, N_t^{(a)}) \text{ that maximizes } \mathbb{E}\left[\sum_{t=1}^{T} R_t\right]
\]

- Note: Discount Factor when aggregating Rewards in the MDP is 1
We go over the landmark paper by Avellaneda and Stoikov in 2006
They derive a simple, clean and intuitive solution
We adapt our discrete-time notation to their continuous-time setting
$X_t^{(b)}, X_t^{(a)}$ are Poisson processes with hit/lift-rate means $\lambda_t^{(b)}, \lambda_t^{(a)}$

\[
dX_t^{(b)} \sim \text{Poisson}(\lambda_t^{(b)} \cdot dt) , \ dX_t^{(a)} \sim \text{Poisson}(\lambda_t^{(a)} \cdot dt)
\]

$\lambda_t^{(b)} = f^{(b)}(\delta_t^{(b)}), \ \lambda_t^{(a)} = f^{(a)}(\delta_t^{(a)})$ for decreasing functions $f^{(b)}, f^{(a)}$

\[
dW_t = P_t^{(a)} \cdot dX_t^{(a)} - P_t^{(b)} \cdot dX_t^{(b)} , \ I_t = X_t^{(b)} - X_t^{(a)} \quad \text{(note: } I_0 = 0)\]

Since infinitesimal Poisson random variables $dX_t^{(b)}$ (shares hit in time $dt$) and $dX_t^{(a)}$ (shares lifted in time $dt$) are Bernoulli (shares hit/lifted in time $dt$ are 0 or 1), $N_t^{(b)}$ and $N_t^{(a)}$ can be assumed to be 1
This simplifies the Action at time $t$ to be just the pair: $(\delta_t^{(b)}, \delta_t^{(a)})$

TOB Mid Price Dynamics: $dS_t = \sigma \cdot dz_t$ (scaled brownian motion)
Utility function $U(x) = -e^{-\gamma x}$ where $\gamma > 0$ is coeff. of risk-aversion
Hamilton-Jacobi-Bellman (HJB) Equation

- We denote the Optimal Value function as $V^*(t, S_t, W_t, I_t)$

$$V^*(t, S_t, W_t, I_t) = \max_{\delta_t^{(b)}, \delta_t^{(a)}} \mathbb{E}[-e^{-\gamma(W_T+I_T \cdot S_T)}]$$

- $V^*(t, S_t, W_t, I_t)$ satisfies a recursive formulation for $0 \leq t < t_1 < T$

$$V^*(t, S_t, W_t, I_t) = \max_{\delta_t^{(b)}, \delta_t^{(a)}} \mathbb{E}[V^*(t_1, S_{t_1}, W_{t_1}, I_{t_1})]$$

- Rewriting in stochastic differential form, we have the HJB Equation

$$\max_{\delta_t^{(b)}, \delta_t^{(a)}} \mathbb{E}[dV^*(t, S_t, W_t, I_t)] = 0 \text{ for } t < T$$

$$V^*(T, S_T, W_T, I_T) = -e^{-\gamma(W_T+I_T \cdot S_T)}$$
Converting HJB to a Partial Differential Equation

- Change to $V^*(t, S_t, W_t, I_t)$ is comprised of 3 components:
  - Due to pure movement in time $t$
  - Due to randomness in TOB Mid-Price $S_t$
  - Due to randomness in hitting/lifting the Bid/Ask

- With this, we can expand $dV^*(t, S_t, W_t, I_t)$ and rewrite HJB as:

$$\max_{\delta_t^{(b)}, \delta_t^{(a)}} \left\{ \frac{\partial V^*}{\partial t} dt + \mathbb{E}\left[ \sigma \frac{\partial V^*}{\partial S_t} dz_t + \frac{\sigma^2}{2} \frac{\partial^2 V^*}{\partial S_t^2} (dz_t)^2 \right] ight. $$

$$+ \lambda_t^{(b)} \cdot dt \cdot V^*(t, S_t, W_t - S_t + \delta_t^{(b)}, I_t + 1)$$

$$+ \lambda_t^{(a)} \cdot dt \cdot V^*(t, S_t, W_t + S_t + \delta_t^{(a)}, I_t - 1)$$

$$+ (1 - \lambda_t^{(b)} \cdot dt - \lambda_t^{(a)} \cdot dt) \cdot V^*(t, S_t, W_t, I_t)$$

$$- V^*(t, S_t, W_t, I_t) \right\} = 0$$
Converting HJB to a Partial Differential Equation

We can simplify this equation with a few observations:

- \( \mathbb{E}[dz_t] = 0 \)
- \( \mathbb{E}[(dz_t)^2] = dt \)
- Organize the terms involving \( \lambda_t^{(b)} \) and \( \lambda_t^{(a)} \) better with some algebra
- Divide throughout by \( dt \)

\[
\max_{\delta_t^{(b)}, \delta_t^{(a)}} \left\{ \frac{\partial V^*}{\partial t} + \frac{\sigma^2}{2} \frac{\partial^2 V^*}{\partial S^2} + \lambda_t^{(b)} \cdot (V^*(t, S_t, W_t - S_t + \delta_t^{(b)}, I_t + 1) - V^*(t, S_t, W_t, I_t)) + \lambda_t^{(a)} \cdot (V^*(t, S_t, W_t + S_t + \delta_t^{(a)}, I_t - 1) - V^*(t, S_t, W_t, I_t)) \right\} = 0
\]
Next, note that $\lambda_t^{(b)} = f^{(b)}(\delta_t^{(b)})$ and $\lambda_t^{(a)} = f^{(a)}(\delta_t^{(a)})$, and apply the max only on the relevant terms

$$
\frac{\partial V^*}{\partial t} + \frac{\sigma^2}{2} \frac{\partial^2 V^*}{\partial S_t^2} + \max_{\delta_t^{(b)}} \{ f^{(b)}(\delta_t^{(b)}) \cdot (V^*(t, S_t, W_t - S_t + \delta_t^{(b)}, l_t + 1) - V^*(t, S_t, W_t, l_t)) \}
+ \max_{\delta_t^{(a)}} \{ f^{(a)}(\delta_t^{(a)}) \cdot (V^*(t, S_t, W_t + S_t + \delta_t^{(a)}, l_t - 1) - V^*(t, S_t, W_t, l_t)) \} = 0
$$

This combines with the boundary condition:

$$
V^*(T, S_T, W_T, l_T) = -e^{-\gamma(W_T + l_T S_T)}
$$
Converting HJB to a Partial Differential Equation

- We make an “educated guess” for the structure of $V^*(t, S_t, W_t, I_t)$:
  \[ V^*(t, S_t, W_t, I_t) = -e^{-\gamma(W_t + \theta(t, S_t, I_t))} \]  
  (1)

and reduce the problem to a PDE in terms of $\theta(t, S_t, I_t)$
- Substituting this into the above PDE for $V^*(t, S_t, W_t, I_t)$ gives:
  \[
  \frac{\partial \theta}{\partial t} + \frac{\sigma^2}{2} \left( \frac{\partial^2 \theta}{\partial S_t^2} - \gamma \left( \frac{\partial \theta}{\partial S_t} \right)^2 \right) \\
  + \max_{\delta^{(b)}_t} \left\{ \frac{f^{(b)}(\delta^{(b)}_t)}{\gamma} \cdot (1 - e^{-\gamma(\delta^{(b)}_t - S_t + \theta(t, S_t, I_t + 1) - \theta(t, S_t, I_t))}) \right\} \\
  + \max_{\delta^{(a)}_t} \left\{ \frac{f^{(a)}(\delta^{(a)}_t)}{\gamma} \cdot (1 - e^{-\gamma(\delta^{(a)}_t + S_t + \theta(t, S_t, I_t - 1) - \theta(t, S_t, I_t))}) \right\} = 0
  \]

- The boundary condition is:
  \[ \theta(T, S_T, I_T) = I_T \cdot S_T \]
Indifference Bid/Ask Price

- It turns out that $\theta(t, S_t, I_t + 1) - \theta(t, S_t, I_t)$ and $\theta(t, S_t, I_t) - \theta(t, S_t, I_t - 1)$ are equal to financially meaningful quantities known as Indifference Bid and Ask Prices.

- Indifference Bid Price $Q^{(b)}(t, S_t, I_t)$ is defined as:

\[
V^*(t, S_t, W_t - Q^{(b)}(t, S_t, I_t), I_t + 1) = V^*(t, S_t, W_t, I_t) \tag{2}
\]

- $Q^{(b)}(t, S_t, I_t)$ is the price to buy a share with guarantee of immediate purchase that results in Optimum Expected Utility being unchanged.

- Likewise, Indifference Ask Price $Q^{(a)}(t, S_t, I_t)$ is defined as:

\[
V^*(t, S_t, W_t + Q^{(a)}(t, S_t, I_t), I_t - 1) = V^*(t, S_t, W_t, I_t) \tag{3}
\]

- $Q^{(a)}(t, S_t, I_t)$ is the price to sell a share with guarantee of immediate sale that results in Optimum Expected Utility being unchanged.

- We abbreviate $Q^{(b)}(t, S_t, I_t)$ as $Q_t^{(b)}$ and $Q^{(a)}(t, S_t, I_t)$ as $Q_t^{(a)}$. 
Indifference Bid/Ask Price in the PDE for $\theta$

- Express $V^*(t, S_t, W_t - Q_t^{(b)}, l_t + 1) = V^*(t, S_t, W_t, l_t)$ in terms of $\theta$:

$$-e^{-\gamma(W_t - Q_t^{(b)} + \theta(t, S_t, l_t + 1))} = -e^{-\gamma(W_t + \theta(t, S_t, l_t))}$$

$$\Rightarrow Q_t^{(b)} = \theta(t, S_t, l_t + 1) - \theta(t, S_t, l_t)$$ (4)

- Likewise for $Q_t^{(a)}$, we get:

$$Q_t^{(a)} = \theta(t, S_t, l_t) - \theta(t, S_t, l_t - 1)$$ (5)

- Using equations (4) and (5), bring $Q_t^{(b)}$ and $Q_t^{(a)}$ in the PDE for $\theta$

$$\frac{\partial\theta}{\partial t} + \frac{\sigma^2}{2} \left( \frac{\partial^2\theta}{\partial S_t^2} - \gamma \left( \frac{\partial\theta}{\partial S_t} \right)^2 \right) + \max_{\delta_t^{(b)}} g(\delta_t^{(b)}) + \max_{\delta_t^{(a)}} h(\delta_t^{(a)}) = 0$$

where $g(\delta_t^{(b)}) = \frac{f(b)(\delta_t^{(b)})}{\gamma} \cdot (1 - e^{-\gamma(\delta_t^{(b)} - S_t + Q_t^{(b)})})$

and $h(\delta_t^{(a)}) = \frac{f(a)(\delta_t^{(a)})}{\gamma} \cdot (1 - e^{-\gamma(\delta_t^{(a)} + S_t - Q_t^{(a)})})$
To maximize $g(\delta_t^{(b)})$, differentiate $g$ with respect to $\delta_t^{(b)}$ and set to 0

$$e^{-\gamma(\delta_t^{(b)} - S_t + Q_t^{(b)})} \cdot (\gamma \cdot f(b)(\delta_t^{(b)})) - \frac{\partial f(b)}{\partial \delta_t^{(b)}}(\delta_t^{(b)}) + \frac{\partial f(b)}{\partial \delta_t^{(b)}}(\delta_t^{(b)}) = 0$$

$$\Rightarrow \delta_t^{(b)} = S_t - Q_t^{(b)} + \frac{1}{\gamma} \cdot \ln \left(1 - \gamma \cdot \frac{f(b)(\delta_t^{(b)})}{\partial f(b)/\partial \delta_t^{(b)}(\delta_t^{(b)})}\right) \quad (6)$$

To maximize $g(\delta_t^{(a)})$, differentiate $h$ with respect to $\delta_t^{(a)}$ and set to 0

$$e^{-\gamma(\delta_t^{(a)} + S_t - Q_t^{(a)})} \cdot (\gamma \cdot f(a)(\delta_t^{(a)})) - \frac{\partial f(a)}{\partial \delta_t^{(a)}}(\delta_t^{(a)}) + \frac{\partial f(a)}{\partial \delta_t^{(a)}}(\delta_t^{(a)}) = 0$$

$$\Rightarrow \delta_t^{(a)} = Q_t^{(a)} - S_t + \frac{1}{\gamma} \cdot \ln \left(1 - \gamma \cdot \frac{f(a)(\delta_t^{(a)})}{\partial f(a)/\partial \delta_t^{(a)}(\delta_t^{(a)})}\right) \quad (7)$$

(6) and (7) are implicit equations for $\delta_t^{(b)}$ and $\delta_t^{(a)}$ respectively
Solving for $\theta$ and for Optimal Bid/Ask Spreads

Let us write the PDE in terms of the Optimal Bid and Ask Spreads

\[
\frac{\partial \theta}{\partial t} + \frac{\sigma^2}{2} \left( \frac{\partial^2 \theta}{\partial S_t^2} - \gamma \left( \frac{\partial \theta}{\partial S_t} \right)^2 \right) + f^{(b)}(\delta_c^{(b)})^* \cdot \left(1 - e^{-\gamma(\delta_c^{(b)} - S_t + \theta(t, S_t, I_t + 1) - \theta(t, S_t, I_t))} \right) \\
+ f^{(a)}(\delta_c^{(a)})^* \cdot \left(1 - e^{-\gamma(\delta_c^{(a)} + S_t + \theta(t, S_t, I_t - 1) - \theta(t, S_t, I_t))} \right) = 0
\]

with boundary condition $\theta(T, S_T, I_T) = I_T \cdot S_T$

- First we solve PDE (8) for $\theta$ in terms of $\delta_c^{(b)}$ and $\delta_c^{(a)}$
- In general, this would be a numerical PDE solution
- Using (4) and (5), we have $Q_t^{(b)}$ and $Q_t^{(a)}$ in terms of $\delta_c^{(b)}$ and $\delta_c^{(a)}$
- Substitute above-obtained $Q_t^{(b)}$ and $Q_t^{(a)}$ in equations (6) and (7)
- Solve implicit equations for $\delta_c^{(b)}$ and $\delta_c^{(a)}$ (in general, numerically)
Building Intuition

- Define *Indifference Mid Price* $Q_t^{(m)} = \frac{Q_t^{(b)} + Q_t^{(a)}}{2}$
- To develop intuition for Indifference Prices, consider a simple case where the market-maker doesn’t supply any bids or asks

$$V^*(t, S_t, W_t, I_t) = \mathbb{E}[-e^{-\gamma(W_t + I_t \cdot S_T)}]$$

- Combining this with the diffusion $dS_t = \sigma \cdot dz_t$, we get:

$$V^*(t, S_t, W_t, I_t) = -e^{-\gamma(W_t + I_t \cdot S_t - \frac{\gamma I_t^2 \cdot \sigma^2 (T-t)}{2})}$$

- Combining this with equations (2) and (3), we get:

$$Q_t^{(b)} = S_t - (2I_t + 1)\frac{\gamma \sigma^2 (T-t)}{2}, \quad Q_t^{(a)} = S_t - (2I_t - 1)\frac{\gamma \sigma^2 (T-t)}{2}$$

$$Q_t^{(m)} = S_t - I_t \gamma \sigma^2 (T-t), \quad Q_t^{(a)} - Q_t^{(b)} = \gamma \sigma^2 (T-t)$$

- These results for the simple case of no-market-making serve as approximations for our problem of optimal market-making
Building Intuition

- Think of $Q_t^{(m)}$ as *inventory-risk-adjusted* mid-price (adjustment to $S_t$)
- If market-maker is long inventory ($I_t > 0$), $Q_t^{(m)} < S_t$ indicating inclination to sell than buy, and if market-maker is short inventory, $Q_t^{(m)} > S_t$ indicating inclination to buy than sell
- Armed with this intuition, we come back to optimal market-making, observing from eqns (6) and (7): $P_t^{(b)*} < Q_t^{(b)} < Q_t^{(m)} < Q_t^{(a)} < P_t^{(a)*}$
- Think of $[P_t^{(b)*}, P_t^{(a)*}]$ as “centered” at $Q_t^{(m)}$ (rather than at $S_t$), i.e., $[P_t^{(b)*}, P_t^{(a)*}]$ will (together) move up/down in tandem with $Q_t^{(m)}$ moving up/down (as a function of inventory position $I_t$)

$$Q_t^{(m)} - P_t^{(b)*} = \frac{Q_t^{(a)} - Q_t^{(b)}}{2} + \frac{1}{\gamma} \cdot \ln \left(1 - \gamma \cdot \frac{f(b)(\delta_t^{(b)*})}{\partial f(b)/\partial \delta_t^{(b)}(\delta_t^{(b)*})}\right) \quad (9)$$

$$P_t^{(a)*} - Q_t^{(m)} = \frac{Q_t^{(a)} - Q_t^{(b)}}{2} + \frac{1}{\gamma} \cdot \ln \left(1 - \gamma \cdot \frac{f(a)(\delta_t^{(a)*})}{\partial f(a)/\partial \delta_t^{(a)}(\delta_t^{(a)*})}\right) \quad (10)$$
Simple Functional Form for Hitting/Lifting Rate Means

- The PDE for $\theta$ and the implicit equations for $\delta_t^{(b)^*}, \delta_t^{(a)^*}$ are messy
- We make some assumptions, simplify, derive analytical approximations
- First we assume a fairly standard functional form for $f(b)$ and $f(a)$
  \[ f(b)(\delta) = f(a)(\delta) = c \cdot e^{-k \cdot \delta} \]
- This reduces equations (6) and (7) to:
  \[ \delta_t^{(b)^*} = S_t - Q_t^{(b)} + \frac{1}{\gamma} \ln \left( 1 + \frac{\gamma}{k} \right) \quad (11) \]
  \[ \delta_t^{(a)^*} = Q_t^{(a)} - S_t + \frac{1}{\gamma} \ln \left( 1 + \frac{\gamma}{k} \right) \quad (12) \]
- $\Rightarrow P_t^{(b)^*}$ and $P_t^{(a)^*}$ are equidistant from $Q_t^{(m)}$
- Substituting these simplified $\delta_t^{(b)^*}, \delta_t^{(a)^*}$ in (8) reduces the PDE to:
  \[ \frac{\partial \theta}{\partial t} + \frac{\sigma^2}{2} \left( \frac{\partial^2 \theta}{\partial S_t^2} - \gamma \left( \frac{\partial \theta}{\partial S_t} \right)^2 \right) + \frac{c}{k + \gamma} (e^{-k \cdot \delta_t^{(b)^*}} + e^{-k \cdot \delta_t^{(a)^*}}) = 0 \quad (13) \]
  with boundary condition $\theta(T, S_T, I_T) = I_T \cdot S_T$
Simplifying the PDE with Approximations

- Note that this PDE (13) involves $\delta_t^{(b)^*}$ and $\delta_t^{(a)^*}$
- However, equations (11), (12), (4), (5) enable expressing $\delta_t^{(b)^*}$ and $\delta_t^{(a)^*}$ in terms of $\theta(t, S_t, I_t - 1), \theta(t, S_t, I_t), \theta(t, S_t, I_t + 1)$
- This would give us a PDE just in terms of $\theta$
- Solving that PDE for $\theta$ would not only give us $V^*(t, S_t, W_t, I_t)$ but also $\delta_t^{(b)^*}$ and $\delta_t^{(a)^*}$ (using equations (11), (12), (4), (5))
- To solve the PDE, we need to make a couple of approximations
- First we make a linear approx for $e^{-k \cdot \delta_t^{(b)^*}}$ and $e^{-k \cdot \delta_t^{(a)^*}}$ in PDE (13):
  \[
  \frac{\partial \theta}{\partial t} + \frac{\sigma^2}{2} \left( \frac{\partial^2 \theta}{\partial S_t^2} - \gamma \left( \frac{\partial \theta}{\partial S_t} \right)^2 \right) + \frac{c}{k + \gamma} \left( 1 - k \cdot \delta_t^{(b)^*} + 1 - k \cdot \delta_t^{(a)^*} \right) = 0 \quad (14)
  \]
- Equations (11), (12), (4), (5) tell us that:
  \[
  \delta_t^{(b)^*} + \delta_t^{(a)^*} = \frac{2}{\gamma} \ln \left( 1 + \frac{\gamma}{k} \right) + 2\theta(t, S_t, I_t) - \theta(t, S_t, I_t + 1) - \theta(t, S_t, I_t - 1)
  \]
Asymptotic Expansion of $\theta$ in $I_t$

- With this expression for $\delta_t^{(b)*} + \delta_t^{(a)*}$, PDE (14) takes the form:

$$
\frac{\partial \theta}{\partial t} + \frac{\sigma^2}{2} \left( \frac{\partial^2 \theta}{\partial S^2_t} - \gamma \left( \frac{\partial \theta}{\partial S_t} \right)^2 \right) + \frac{c}{k + \gamma} \left( 2 - \frac{2k}{\gamma} \ln \left( 1 + \frac{\gamma}{k} \right) \right)

- k \left( 2\theta(t, S_t, I_t) - \theta(t, S_t, I_t + 1) - \theta(t, S_t, I_t - 1) \right) = 0

(15)

- To solve PDE (15), we consider this asymptotic expansion of $\theta$ in $I_t$:

$$
\theta(t, S_t, I_t) = \sum_{n=0}^{\infty} \frac{I^n_t}{n!} \cdot \theta^{(n)}(t, S_t)
$$

- So we need to determine the functions $\theta^{(n)}(t, S_t)$ for all $n = 0, 1, 2, \ldots$

- For tractability, we approximate this expansion to the first 3 terms:

$$
\theta(t, S_t, I_t) \approx \theta^{(0)}(t, S_t) + I_t \cdot \theta^{(1)}(t, S_t) + \frac{I_t^2}{2} \cdot \theta^{(2)}(t, S_t)
$$
Approximation of the Expansion of $\theta$ in $I_t$

- We note that the Optimal Value Function $V^*$ can depend on $S_t$ only through the current *Value of the Inventory* (i.e., through $I_t \cdot S_t$), i.e., it cannot depend on $S_t$ in any other way.
- This means $V^*(t, S_t, W_t, 0) = -e^{-\gamma(W_t + \theta^{(0)}(t, S_t))}$ is independent of $S_t$.
- This means $\theta^{(0)}(t, S_t)$ is independent of $S_t$.
- So, we can write it as simply $\theta^{(0)}(t)$, meaning $\frac{\partial \theta^{(0)}}{\partial S_t}$ and $\frac{\partial^2 \theta^{(0)}}{\partial S_t^2}$ are 0.
- Therefore, we can write the approximate expansion for $\theta(t, S_t, I_t)$ as:

$$\theta(t, S_t, I_t) = \theta^{(0)}(t) + I_t \cdot \theta^{(1)}(t, S_t) + \frac{I_t^2}{2} \cdot \theta^{(2)}(t, S_t) \quad (16)$$
Solving the PDE

Substitute this approximation (16) for \(\theta(t, S_t, I_t)\) in PDE (15)

\[
\frac{\partial \theta^{(0)}}{\partial t} + I_t \frac{\partial \theta^{(1)}}{\partial t} + \frac{I_t^2}{2} \frac{\partial \theta^{(2)}}{\partial t} + \frac{\sigma^2}{2} \left( I_t \frac{\partial^2 \theta^{(1)}}{\partial S_t^2} + \frac{I_t^2}{2} \frac{\partial^2 \theta^{(2)}}{\partial S_t^2} \right) \\
- \frac{\gamma \sigma^2}{2} \left( I_t \frac{\partial \theta^{(1)}}{\partial S_t} + \frac{I_t^2}{2} \frac{\partial \theta^{(2)}}{\partial S_t} \right)^2 + \frac{c}{k + \gamma} \left( 2 - \frac{2k}{\gamma} \ln \left( 1 + \frac{\gamma}{k} \right) + k \cdot \theta^{(2)} \right) = 0
\]

with boundary condition:

\[
\theta^{(0)}(T) + I_T \cdot \theta^{(1)}(T, S_T) + \frac{I_T^2}{2} \cdot \theta^{(2)}(T, S_T) = I_T \cdot S_T
\]

(17)

We will separately collect terms involving specific powers of \(I_t\), each yielding a separate PDE:

- Terms devoid of \(I_t\) (i.e., \(I_t^0\))
- Terms involving \(I_t\) (i.e., \(I_t^1\))
- Terms involving \(I_t^2\)
Solving the PDE

- We start by collecting terms involving $l_t$

\[
\frac{\partial \theta^{(1)}}{\partial t} + \frac{\sigma^2}{2} \cdot \frac{\partial^2 \theta^{(1)}}{\partial S_t^2} = 0 \text{ with boundary condition } \theta^{(1)}(T, S_T) = S_T
\]

- The solution to this PDE is:

\[
\theta^{(1)}(t, S_t) = S_t \tag{18}
\]

- Next, we collect terms involving $l_t^2$

\[
\frac{\partial \theta^{(2)}}{\partial t} + \frac{\sigma^2}{2} \cdot \frac{\partial^2 \theta^{(2)}}{\partial S_t^2} - \gamma \sigma^2 \cdot \left(\frac{\partial \theta^{(1)}}{\partial S_t}\right)^2 = 0 \text{ with boundary } \theta^{(2)}(T, S_T) = 0
\]

- Noting that $\theta^{(1)}(t, S_t) = S_t$, we solve this PDE as:

\[
\theta^{(2)}(t, S_t) = -\gamma \sigma^2 (T - t) \tag{19}
\]
Solving the PDE

Finally, we collect the terms devoid of $I_t$

$$
\frac{\partial \theta^{(0)}}{\partial t} + \frac{c}{k + \gamma} \left( 2 - \frac{2k}{\gamma} \ln \left( 1 + \frac{\gamma}{k} \right) + k \cdot \theta^{(2)} \right) = 0 \text{ with boundary } \theta^{(0)}(T) = 0
$$

Noting that $\theta^{(2)}(t, S_t) = -\gamma \sigma^2 (T - t)$, we solve as:

$$
\theta^{(0)}(t) = \frac{c}{k + \gamma} \left( \left( 2 - \frac{2k}{\gamma} \ln \left( 1 + \frac{\gamma}{k} \right) \right)(T - t) - \frac{k \gamma \sigma^2}{2} (T - t)^2 \right) \quad (20)
$$

This completes the PDE solution for $\theta(t, S_t, I_t)$ and hence, for $V^*(t, S_t, W_t, I_t)$

Lastly, we derive formulas for $Q_t^{(b)}, Q_t^{(a)}, Q_t^{(m)}, \delta_t^{(b)*}, \delta_t^{(a)*}$
Formulas for Prices and Spreads

- Using equations (4) and (5), we get:

\[ Q_{t}^{(b)} = \theta^{(1)}(t, S_t) + (2I_t+1) \cdot \theta^{(2)}(t, S_t) = S_t - (2I_t+1) \frac{\gamma \sigma^2 (T - t)}{2} \]  
(21)

\[ Q_{t}^{(a)} = \theta^{(1)}(t, S_t) + (2I_t-1) \cdot \theta^{(2)}(t, S_t) = S_t - (2I_t-1) \frac{\gamma \sigma^2 (T - t)}{2} \]  
(22)

- Using equations (11) and (12), we get:

\[ \delta_{t}^{(b)*} = \frac{(2I_t + 1) \gamma \sigma^2 (T - t)}{2} + \frac{1}{\gamma} \ln (1 + \frac{\gamma}{k}) \]  
(23)

\[ \delta_{t}^{(a)*} = \frac{(1 - 2I_t) \gamma \sigma^2 (T - t)}{2} + \frac{1}{\gamma} \ln (1 + \frac{\gamma}{k}) \]  
(24)

Optimal Bid-Ask Spread

\[ \delta_{t}^{(b)*} + \delta_{t}^{(a)*} = \gamma \sigma^2 (T - t) + \frac{2}{\gamma} \ln (1 + \frac{\gamma}{k}) \]  
(25)

Optimal “Mid”

\[ Q_{t}^{(m)} = \frac{Q_{t}^{(b)} + Q_{t}^{(a)}}{2} = \frac{P_{t}^{(b)*} + P_{t}^{(a)*}}{2} = S_t - I_t \gamma \sigma^2 (T - t) \]  
(26)
Think of $Q_t^{(m)}$ as inventory-risk-adjusted mid-price (adjustment to $S_t$)
If market-maker is long inventory ($I_t > 0$), $Q_t^{(m)} < S_t$ indicating inclination to sell than buy, and if market-maker is short inventory, $Q_t^{(m)} > S_t$ indicating inclination to buy than sell
Think of $[P_t^{(b)}^{*}, P_t^{(a)}^{*}]$ as “centered” at $Q_t^{(m)}$ (rather than at $S_t$), i.e., $[P_t^{(b)}^{*}, P_t^{(a)}^{*}]$ will (together) move up/down in tandem with $Q_t^{(m)}$ moving up/down (as a function of inventory position $I_t$)
Note from equation (25) that the Optimal Bid-Ask Spread $P_t^{(a)}^{*} - P_t^{(b)}^{*}$ is independent of inventory $I_t$
Useful view: $P_t^{(b)}^{*} < Q_t^{(b)} < Q_t^{(m)} < Q_t^{(a)} < P_t^{(a)}^{*}$, with these spreads:

**Outer Spreads** $P_t^{(a)}^{*} - Q_t^{(a)} = Q_t^{(b)} - P_t^{(b)}^{*} = \frac{1}{\gamma} \ln (1 + \frac{\gamma}{k})$

**Inner Spreads** $Q_t^{(a)} - Q_t^{(m)} = Q_t^{(m)} - Q_t^{(b)} = \frac{\gamma \sigma^2 (T - t)}{2}$
Real-world TOB dynamics are non-stationary, non-linear, complex
Frictions: Discrete Prices/Sizes, Constraints on Prices/Sizes, Fees
Need to capture various market factors in the State & TOB Dynamics
This leads to Curse of Dimensionality and Curse of Modeling
The practical route is to develop a simulator capturing all of the above
Simulator is a Market-Data-learnt Sampling Model of TOB Dynamics
Using this simulator and neural-networks func approx, we can do RL
References: 2018 Paper from University of Liverpool and 2019 Paper from JP Morgan Research
Exciting area for Future Research as well as Engineering Design