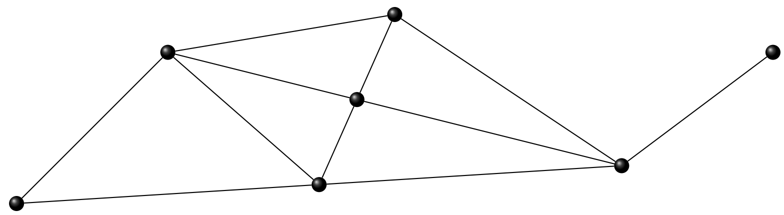
2010 SMT Power Round

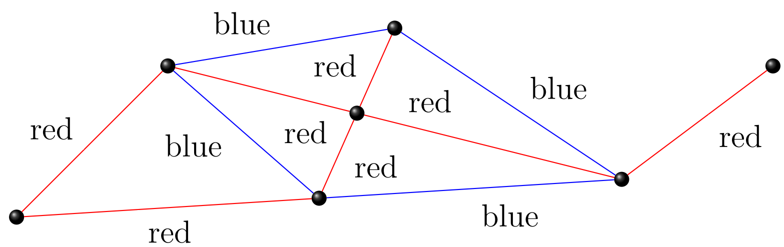
Definitions

A *graph* is a collection of points (vertices) connected by line segments (edges). In this test, all graphs will be *simple* – any two vertices will be connected by at most one edge – and *connected* – you can get from any vertex to any other by following edges.



A simple connected graph with 7 vertices and 11 edges.

An *edge n-coloring* of a graph G is an assignment of one of n colors to each edge of G.



A 2-coloring of the earlier graph.

A *complete* graph is one in which any two vertices are connected by an edge.

1. a. (5 points) Draw a simple connected graph with 8 vertices and 7 edges, and 3-color its edges.
   Solution: Many graphs work. The only thing to note is that the graph must be a tree, that is, it should have no cycles.

   b. (5 points) Draw a complete graph on 5 vertices, and 2-color its edges so that it does *not* contain a red triangle or a blue triangle (3 vertices, the edges between which are all red or all blue).
   Solution: The simplest depiction is a pentagon, with blue sides and red diagonals.

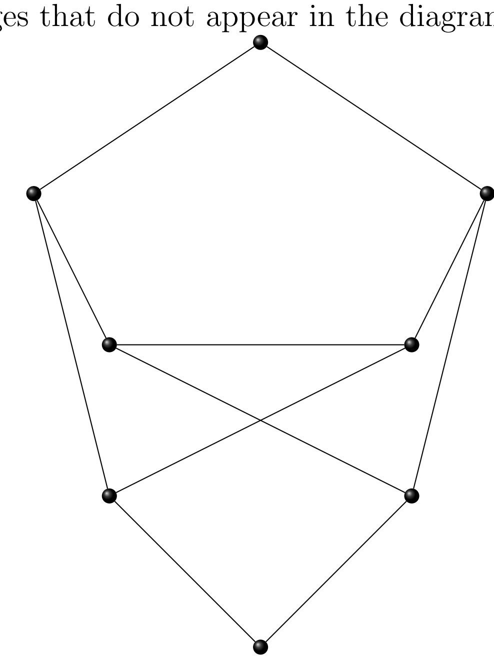
We will use K_n to denote a complete graph on n vertices. A *monochromatic* K_n is one in which every edge has the same color. Hence, problem 1(b) could have been phrased "Color K_5 so that it has no monochromatic K_3".

2. (10 points) Show that no matter how you 2-color K_6, it will contain a monochromatic K_3. (Hint: Think about all the edges coming from one vertex).
   Solution: Pick any vertex v of K_6. Then there must be three edges of one color coming from v, WLOG they are all blue. Now consider the three vertices on the other side of these edges from v, and the edges between them. If any of the edges are blue, they form a blue K_3 with the blue edges from its endpoints going to v. However, if none of them are blue, then we have a red K_3.

The *Ramsey number* $R(k)$ is the least number n such that no matter how you 2-color the edges of K_n, there will be a monochromatic K_k. In problems 1(b) and 2, you have shown that $R(3) = 6$.

Interestingly, $R(4)$ is a difficult quantity to calculate, and $R(5)$ is still unknown! Since we cannot go much further in this vein, let us try looking at generalizations of Ramsey numbers. Define $R(k, j)$ as the least n such that every red, blue edge 2-coloring of K_n contains either a red K_k or a blue K_j. Then $R(n)$ is just $R(n, n)$ under this new definition.

3. a. (5 points) Show that $R(4, 3) > 8$ by exhibiting a 2-coloring.
   Solution: A counterexample is given in the diagram below. Here, the graph shows only blue edges; the edges that do not appear in the diagram are presumed to be red.



   b. (15 points) Show that $R(4, 3) = 9$ (Hint: Use problem 2.)
   Solution: Suppose that we have a 2-coloring of the edges of K_9 in which there are no blue triangles, nor red K_4. Then any vertex v can have no more than 5 red edges coming from it, because if it had six, the points on the other end would form a K_6, and we already know that a K_6 must contain a blue triangle – which we supposed to be impossible – or a red triangle, which would form a red K_4 with the edges from v. Likewise, no vertex can have more than 3 blue edges coming from it, for if four blue edges came from one vertex, the K_4 on the other end would have to be all red, or have a blue edge and complete a blue triangle. Now, since every vertex has 8 edges coming from it, we see that every vertex must have exactly 3 blue and 5 red edges. However, this violates the handshake theorem – basically, there are 27 requests for a blue edge, but these can only be granted in pairs because each edge connects two vertices.

4. a. (15 points) Show that

$$R(n, m) \leq R(n, m - 1) + R(n - 1, m).$$

   (Hint: see hint to problem 2.)
   Solution: Proof by induction on $n + m$. If $n = 2$, then $R(n, m) = m$, and likewise

if $m = 2$. $R(n, m)$ is not defined for smaller arguments. If we know the result for $n + m < k$, let us take a complete graph on $R(n-1, m) + R(n, m-1)$ vertices. Pick any one vertex v, and consider the $R(n-1, m) + R(n, m-1) - 1$ edges coming from it. By the pigeonhole principle, there are either $R(n-1, m)$ blue edges or $R(n, m-1)$ red edges among these. WLOG, there are $R(n-1, m)$ blue ones. Now, the vertices on the other side of those edges form a $K_R(n-1, m)$, and so must contain either a red K_m or a blue K_{n-1}, which with v forms a blue K_n.

b. (5 points) Conclude that $R(n, m)$ is well defined, that is, that it exists for every n and every m.
Solution: In the proof above, we showed that given m and n, every sufficiently large graph must contain either a red K_m or a blue K_n, which is precisely the condition for $R(n, m)$ to exist.

From here on, we will explore some interesting properties and generalizations of Ramsey numbers. Each section is independent.

Bounds on Ramsey Numbers

5. Color a graph of n^2 points, laid out in a $n \times n$ grid, as follows: The edge (u, v) is blue if u and v are in the same row, and red otherwise.
   a. (5 points) Show that any K_{n+1} in that graph contains at least one red edge and at least one blue edge.
   Solution: By pigeonhole, there must be two points of the K_{n+1} which are in the same row, and so have a blue edge between them. Similarly, there must be two not in the same row, and so those two will have a red edge between them.
   b. (5 points) Conclude that $R(n+1, n+1) > n^2$.
   Solution: We just gave an example of a graph that has n^2 vertices, but no monochromatic K_n. Hence $R(n, n)$ is greater than or equal to n^2.

Problem 6 gives us a polynomial lower bound for $R(n, n)$, and it does so constructively – we know exactly which graph will give a counterexample. Erdős has shown that, if we are willing to be nonconstructive, we can get a much better lower bound:

6. a. (5 points) Show that if the edges of K_m are colored red or blue randomly with equal probability (i.e., by flipping a coin for each edge), then the probability that it contains a monochromatic K_n is at most

$$\binom{m}{n} \cdot 2^{1-\binom{n}{2}}.$$

   Solution: The probability that n vertices chosen at random form a monochromatic K_n is $2^{1-\binom{n}{2}}$. There are $\binom{m}{n}$ ways to choose n vertices, so by the union bound the chance of at least one of those choices being monochromatic is that product.
   b. (5 points) Show that if $\binom{m}{n} < 2^{\binom{n}{2}-1}$, that probability is less than 1.
   Solution: Multiply both sides by $2^{\binom{n}{2}-1}$.
   c. (10 points) Using the fact that $\binom{m}{n} < m^n$, show that if $m = 2^{\frac{n}{2}-\frac{1}{n}-\frac{1}{2}}$ then $\binom{m}{n} < 2^{1-\binom{n}{2}}$, and conclude that

$$R(n, n) > 2^{\frac{n}{2}-\frac{1}{n}-\frac{1}{2}}.$$

Solution: $m^n = (2^{\frac{n}{2}-\frac{1}{n}-\frac{1}{2}})^n = 2^{\frac{n^2}{2}-\frac{n}{2}-1} < 2^{\binom{n}{2}-1}$. Now, from the previous parts we have seen that this implies that a randomly colored K_m will have probability less than 1 of containing a monochromatic K_n. But this means that there is some coloring which does not contain a monochromatic K_n, and so $R(n,n) > m$.

7. (20 points) Prove a complementary upper bound: $R(n,n) \leq 4^n$.
Solution: Take any coloring of K_{4^n}. Pick a vertex v, and consider all $2^{2n}-1$ edges coming from it. Then there are at least 2^{2n-1} edges of one color, so we give v a flag of that color and restrict our attention to the smaller graph, of those vertices on the other ends of the majority-color edges. In this graph we repeat this argument, and we continue doing this until we get to a single point. At the end of this process we have a set of $2n$ vertices, all flagged with different colors, so some color must have at least n flags, by pigeonhole, and we can take the K_n given by the vertices flagged with this color. Now, if v_i and v_j are in this subgraph, with $i < j$, then the edge from v_i to v_j is colored the same color as the flag on v_i. But all of the vertices in this subgraph have the same color flags, so it is monochromatic. Then, since we have shown that every coloring of K_{4^n} must have a monochromatic K_n, $R(n,n) \leq 4^n$.

k-color Ramsey Numbers

Similar to our definition $R(n,m)$, we can define $R(n_1, n_2, n_3, \ldots, n_k)$ to be the least m such that if K_m is colored with k colors, there is some monochromatic K_{n_i} of color c_i.

8. (15 points) Prove that $R(3,3,3) \leq 17$. (In fact, $R(3,3,3) = 17$, but this is difficult to show.)
Solution: Take a 3-coloring of K_{17}, and pick any vertex v. Consider the edges coming from v. There are sixteen of them, and $16/3 > 5$, so there must be some color with 6 edges of that color coming from v. Call this color green, the others red and blue. Consider the K_6 on the other end. Since all the edges going from v to it are green, if it contains a single green edge, we have a green triangle. But if not, that is a 2 colored K_6, so since $R(3,3) = 6$, we must have a red or blue triangle instead.

9. (20 points) Show that

$$R(n_1, \ldots, n_k) \leq R(n_1, n_2, \ldots n_{k-2}, R(n_k, n_{k-1}))$$

Solution: Take a k-coloring of K_m where

$$m = R(n_1, \ldots, n_k) \leq R(n_1, n_2, \ldots n_{k-2}, R(n_k, n_{k-1}))$$

and repaint it so that colors m and $m-1$ look the same. Then by the definition of the Ramsey number, we must either have a K_{n_i} in color $i < k-1$, or we must have a $K_{R(n_{k-1}, n_k)}$ in the last two colors. But even in the last case, by the definition of $R(n,m)$, we must have a $K_{n_{k-1}}$ in color $k-1$ or a K_{n_k} in color k, which completes the proof.

This gives us the existence of $R(n_1, \ldots, n_k)$ for all $\{n_1, \ldots, n_k\}$.

10. Prove that

a. (10 points)

$$R(\underbrace{3,\ldots,3}_{r\text{ 3's}}) \leq 3r!$$

Solution: Induction on r. The base case is done in problems 2 and 8. To induct, let us suppose that

$$R(\underbrace{3,\ldots,3}_{r-1\text{ 3's}}) \leq 3(r-1)!$$

and that we have a coloring of $K_{3(r-1)!}$. Then we can choose a vertex v, and it has many edges of up to r colors coming from it, but in particular we can choose some color c so that there are at least $3(r-1)!$ edges of color c coming from v. Now, if the $K_{3(r-1)!}$ on the other end has any edges colored c, then we have a triangle of color c. But if not, then that graph is colored with only $r-1$ colors, and so by the inductive hypothesis it contains a monochromatic triangle.

b. (15 points)

$$R(\underbrace{3,\ldots,3}_{r\text{ 3's}}) > 2^r$$

Solution: As before, we induct, and the base case is problem 1a. Now, take a $r-1$-coloring of $K_{2^{r-1}}$, which does not have any monochromatic triangles, put two copies side by side, and connect all of their vertices by edges of color r. Then there are no triangles of color r, because two points of the triangle would have to lie in the same copy of the $K_{2^{r-1}}$, and there are no monochromatic triangles of any other color by the induction hypothesis.

Infinite Ramsey Numbers

11. (30 points) Define $K_{\mathbb{N}} = (V, E)$, where $V = \{1, 2, 3, \ldots\}$, and $E = \{(i, j) : i, j \in V,\ i < j\}$. This is in some sense an infinite complete graph. Show that if every edge is colored red or blue, there is some infinite subset V' of V such that all of the edges between points of V' are the same color.

    Solution: The proof is similar to that of problem 7. Take any vertex v_1 and consider the edges coming from it. There must be an infinite number of either red or blue edges, so we put a flag of that color on v_1, and then we restrict our attention to the smaller (but still infinite!) subgraph on the other ends of these edges. Repeat this process (infinitely). When we finish, we have a sequence $v_1, v_2, v_3, \ldots$ of vertices of the graph, each with a red or blue flag, such that if $i < j$, then v_i and v_j are connected by an edge of the same color as the flag on v_i. Well, this sequence must contain an infinite number of vertices all flagged the same color, so take this subset, and the edges between them are all the same color. This is an infinite monochromatic complete subgraph, as desired.