1. **Answer: 2**

   Let the set of coins be $\{A, B, C, D, E, F, G, H\}$. First, weigh $\{A, B, C\}$ vs. $\{D, E, F\}$. If one group is lighter (say $\{A, B, C\}$), then use the second weighing to measure A vs B. If we find that one is lighter than the other, than it must be the counterfeit. If A vs B weigh the same, then we know that C must be the counterfeit. If both groups weighed the same, then measure G vs H. If G vs H weigh the same, then no counterfeit exists.

2. **Answer: 29**

   Just try a bunch.

3. **Answer: 3**

   We note that to get a zero at the end of a number, we must multiply by 10. Since $5 \times 2 = 10$, and there are more factors of 2 in 200! than 5, it suffices to count how many 5's appear in the prime factorization of 200. Each of $5, 10, 15, \ldots, 200$ has a factor of 5 in it, which gives 40 factors. In addition, $25, 50, \ldots, 200$ gives a second factor of 5, so that is 8 additional factors of 5. 125 has a third factor of 5. So 200! has 49 factors of 5; it ends with 49 zeros.

   Similarly, 124! ends with 28 zeros and 76! ends with 18 zeros. Therefore,

   $$\binom{200}{124} = \frac{200!}{124!\,76!}$$

   ends in $49 - 28 - 18 = 3$ zeros.

4. **Answer: $\frac{1+\sqrt{5}}{2}$**

   Let $x = \sqrt{1 + \sqrt{1 + \sqrt{1 + \ldots}}}$. Then $x^2 = 1 + \sqrt{1 + \sqrt{1 + \ldots}}$. Thus $x^2 = x + 1$. The positive root of $x^2 - x - 1 = 0$ is $\frac{1+\sqrt{5}}{2}$.

5. **Answer: 19801 and 20201**

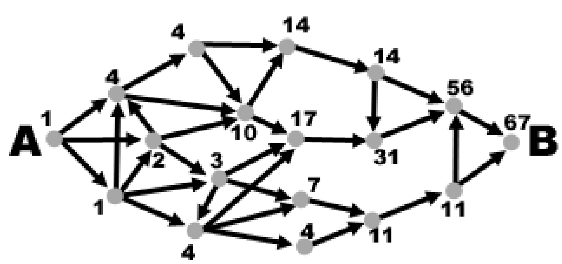
   Notice that $4x^4 + 1 = 4x^4 + 4x^2 + 1 - (2x)^2 = (2x^2 + 2x + 1)(2x^2 - 2x + 1)$. Setting $x = 100$, we have that $400000001 = 19801 \cdot 20201$.

6. **Answer: $14\sqrt{5}$**

   We can use Heron's formula to calculate the area of the triangle. The semiperimeter equals $\frac{7+9+12}{2} = 14$. Applying Heron's formula

   Area $= \sqrt{14 \cdot (14-7) \cdot (14-9) \cdot (14-12)} = 14\sqrt{5}$

7. **Answer: 67**



   Observe that if vertices $V_1, V_2, \ldots, V_n$ is a complete list of vertices that contain an edge leading into a vertex X and it is known that there are $p_1, p_2, \ldots p_n$ different paths into the vertices $V_1, V_2, \ldots, V_n$, respectively, then the number of paths into vertex X is $p_1 + p_2 + \ldots p_n$. Using this rule, we can start from the vertex A which trivially has 1 [empty] path into it and proceed inductively along the grid to get the following diagram, which ends with 67 paths on vertex B.

8. **Answer:** $x = \frac{a \pm a\sqrt{5}}{2}$

$$\begin{aligned} a^2 &= x^2 - ax \\ 0 &= x^2 - ax - a^2 \\ x &= \frac{a \pm \sqrt{a^2 - 4\left(-a^2\right)}}{2} \\ x &= \frac{a \pm |a|\sqrt{5}}{2} = \frac{a \pm a\sqrt{5}}{2} \end{aligned}$$

The $\pm$ takes care of the fact that we do not know the sign of a.

9. **Answer: 180 meters**

The regular trains will intersect at $(150, 0)$ meters at $t = 3$ seconds. Regardless of the location of the fly at any time, we know that the fly will be moving at 60 meters/second. Therefore, at $t = 3$ seconds the fly will have traveled a total of $3 \cdot 60 = 180$ meters.

10. **Answer: 100000000**

Note that the digits of 14641_n in base n are the binomial coefficients $\binom{4}{i}$ so that $14641_n = \sum_{i=0}^{4} \binom{4}{i} n^i = (n+1)^4$. When $n = 99$, this is simply 100000000.

11. **Answer:** $\mathbf{150\sqrt{3}}$

This regular hexagon has side length 10. Notice that a regular hexagon can be split up into 6 equilateral triangles, each with a side length of 10. The area of an equilateral triangle with side length 10 is $\frac{10^2\sqrt{3}}{4}$ and hence the area of the hexagon will be $\frac{10^2\sqrt{3}}{4} \cdot 6 = 150\sqrt{3}$

12. **Answer: 1027**

If $S(n)$ is the nth partial sum, note that if m is the kth triangular number, $S(m) = k^2$. Since $44^2 = 1936$ and $45^2 = 2025$, we want to begin our search at $44(44+1)/2 = 990$. Because $(2010 - 1936)/2 = 37$, 37 more 2s are needed, so the needed term is $n = 990 + 37 = 1027$.

13. **Answer: 21,26,31,36,41,46**

$$\begin{aligned} 6x + 5 &\equiv -19 \mod 10 \\ 6x &\equiv -24 \mod 10 \\ x &\equiv -4 \mod \frac{10}{\gcd(10, 6)} \\ x &\equiv -4 \mod 5 \\ x &\equiv 1 \mod 5 \end{aligned}$$

That is, x is in the form $5k + 1$ where k is an integer.

14. **Answer: 17**

The open lockers will be the ones with an odd number of odd divisors. These numbers are of the form $2^k \cdot n^2$, where n is odd. We can simply check that the open lockers are numbered

$$1, 2, 4, 8, 9, 16, 18, 25, 32, 36, 49, 50, 64, 72, 81, 98, 100.$$

15. **Answer:** $\mathbf{\frac{26}{15}}$

Since $\frac{p}{q} - \sqrt{3} = \frac{p-\sqrt{3}q}{q} = \frac{p^2-3q^2}{q(p+\sqrt{3}q)}$, we should look for (p, q) which minimizes $|p^2 - 3q^2|$. There are no solutions for $p^2 - 3q^2 = -1$ (consider mod 3) and for $p^2 - 3q^2 = 1$, we have $(2, 1)$, $(7, 4)$, $(26, 15)$ as solutions where $q \leq 15$. Since $q = 15$ also maximizes the denominator, $(p, q) = (26, 15)$ is the best choice.

16. **Answer: 21 revolutions**

The wheel goes through three laps of $2\pi \cdot 7 = 14\pi$ distance, for a total distance of $15 \cdot 14\pi = 210\pi$. In one full turn, the wheel goes through a distance of $2\pi\dot{5} = 10\pi$ and thus after all three laps the wheel undergoes $\frac{210\pi}{10\pi} = 21$ revolutions.

17. **Answer:** $\boldsymbol{R\sqrt{3}}$

Label the center of the circle O and the vertices of the triangle A, B, and C. Then we can find the length of segment $\overline{AB}$ by using the law of cosines on triangle AOB. By symmetry, we can see that the angle AOB equals $\frac{360}{3} = 120$ degrees. Applying the law of cosines

(Length of $\overline{AB})^2 = R^2 + R^2 - 2 \cdot R \cdot R \cdot \cos(120°) = 2R^2 - 2R^2 \cdot \left(-\frac{1}{2}\right) = 3R^2$

18. **Answer: 96**

The number of blue cells is $n + m - 1$; the number of total cells is nm. So $2010(m + n - 1) = nm$, or $nm - 2010n - 2010m + 2010 = 0$. This factors as $(n - 2010)(m - 2010) - 2010^2 + 2010 = 0$, or $(n-2010)(m-2010) = 2010\cdot2009$. Thus each of $n-2010$ and $m-2010$ must be one of the positive factors of $2010 \cdot 2009$; for each positive factor, there is one ordered pair. Since $2010 \cdot 2009 = 2 \cdot 3 \cdot 5 \cdot 7^2 \cdot 41 \cdot 67$, there are $2 \cdot 2 \cdot 2 \cdot 3 \cdot 2 \cdot 2 = 2^5 \cdot 3 = 96$ solutions.

19. **Answer:** $\boldsymbol{x = -2, -1, -\frac{1}{2}, \frac{2}{3}}$

Simple trial and error of the first few integers yields the roots $x = -2, -1$. From there, the remaining polynomial can be solved using the quadratic formula. The final factored form is

$$(2 + x)(1 + x)\left(6x^2 - x - 2\right) = (2 + x)(1 + x)(1 + 2x)(-2 + 3x)$$

20. **Answer: 20**

Each circle can intersect with any other circle in at most two distinct points. As long as the intersection points between every two circles are all unique, then we can calculate the greatest number of intersections possible starting with the case of two circles and working up to five circles:
2 circles $\rightarrow$ 2 points max
3 circles $\rightarrow 2 + 2 \cdot 2 = 6$ points max
4 circles $\rightarrow 6 + 2 \cdot 3 = 12$ points max
5 circles $\rightarrow 12 + 2 \cdot 4 = 20$ points max

21. **Answer: 1**

Factor it as $(x^2 - 1)(x^2 + 1) = 2y^2$. Since $x^2 - 1$ and $x^2 + 1$ differ by 2, their gcd is 1 or 2. In either case both $x^2 - 1$ and $x^2 + 1$ has even powers of all odd primes, so $x^2 - 1 = m^2$, $x^2 + 1 = 2n^2$ or $x^2 - 1 = 2m^2$, $x^2 + 1 = n^2$. In the first case $x^2 - m^2 = 1$, so we only have $(x, m) = (1, 0)$ possible, giving the answer $(x, y) = (1, 0)$. In the second case $n^2 - x^2 = 1$, so $x = 0$. But this is impossible since then $x^4 - 2y^2 = -2y^2$ would not be 1. Thus we only have one solution $(x, y) = (1, 0)$.

22. **Answer:** $\mathbf{12\sqrt{5}}$

Note that $1010100_\phi - .010101_\phi = \phi^6 + \phi^4 + \phi^2 - \phi^{-2} - \phi^{-4} - \phi^{-6} = \phi^6 + \phi^4 + \phi^2 - (-\phi)^{-2} - (-\phi)^{-4} - (-\phi)^{-6}$. Thus, $1010100_\phi - .010101_\phi = \sqrt{5}(F(6) + F(4) + F(2)) = \sqrt{5}(8 + 3 + 1) = 12\sqrt{5}$.

23. **Answer: 3**

Note that the function is antisymmetric, so all three terms in the numerator are equal.

24. **Answer: 6029**

If we consider a single 1×1 square, and find two regions within it on which the center of the coin of radius $\frac{1}{4}$ can land — the center $\frac{1}{2} \times \frac{1}{2}$ square, which has area $\frac{1}{4}$, and the outside edge, where an overlap will occur, of area $\frac{3}{4}$.

The total area that the center of the coin can land on is thus

$$\left(2010 - \frac{1}{2}\right)\left(2010 - \frac{1}{2}\right) = \frac{4019^2}{4}.$$

Thus, the probability is $\frac{2010^2}{4019^2}$, so $a + b = 6029$.

25. **Answer: 7**

Let 1, 2, 3, 4, 5 be five balls.
Compare 1, 2, Without Loss of Generality (WLOG) $1 < 2$
Compare 3, 4, WLOG $3 < 4$
Compare 1, 3, WLOG $1 < 3$
Compare 3, 5.

(a) If $3 < 5$, there are eight remaining cases: compare 4, 5 (WLOG $4 < 5$), compare 2, 4
   i. If $2 < 4$, compare 2, 3; done
   ii. If $4 < 2$, compare 2, 5; done

(b) If $3 > 5$, there are seven remaining cases: compare 2, 3.
   i. If $2 < 3$, compare 1, 5. If $5 < 1$ we are done, if $1 < 5$, compare 2, 5; done.
   ii. If $2 > 3$, compare 1, 5, and then compare 2, 4; done.