1. **Answer:** $\frac{\sqrt{5}-1}{4}$**.**

   Consider:

$$\begin{aligned}\sin 18^\circ &= \cos 72^\circ = 2\cos^2 36^\circ - 1 = 2(1-2\sin^2 18^\circ)^2 - 1\\ &= 1 - 8\sin^2 18^\circ + 8\sin^4 18^\circ\\ 0 &= 8x^4 - 8x^2 - x + 1 = (x-1)(2x-1)(4x^2+2x+1).\end{aligned}$$

   Clearly, $x \neq 1, \frac{-1}{2}$, because $0 < \sin 18^\circ < \sin 90^\circ = 1$. We solve the remaining term:

$$0 = 4x^2 + 2x + 1 \implies x = \frac{-2 \pm \sqrt{4+4(4)}}{2(4)} = \frac{-1 \pm \sqrt{5}}{4}.$$

   The only root that is within our bounds is $\frac{\sqrt{5}-1}{4}$.

2. **Answer:** $\frac{44}{15} + \frac{4}{15}i$**.**

   Consider:

$$\begin{aligned}3.\overline{0123} &= 3 + \sum_{n=0}^{\infty} 0(2i)^{-(4n+1)} + \sum_{n=0}^{\infty} 1(2i)^{-(4n+2)} + \sum_{n=0}^{\infty} 2(2i)^{-(4n+3)} + \sum_{n=0}^{\infty} 3(2i)^{-(4n+4)}\\ &= 3 + \sum_{n=0}^{\infty}\left[-\frac{1}{4}\left(\frac{1}{16}\right)^n + \frac{i}{4}\left(\frac{1}{16}^n\right)^n + \frac{3}{16}\left(\frac{1}{16}\right)^n\right]\\ &= 3 + \sum_{n=0}^{\infty}\left(-\frac{1}{16} + \frac{1}{4}i\right)\left(\frac{1}{16}\right)^n\\ &= 3 + \left(-\frac{1}{16} + \frac{1}{4}i\right)\frac{1}{1-\frac{1}{16}}\\ &= 3 + \frac{16}{15}\left(-\frac{1}{16} + \frac{1}{4}i\right)\\ &= \frac{44}{15} + \frac{4}{15}i.\end{aligned}$$

3. **Answer: 6.**

   This is a trivial application of Ramsey Theory. It is isomorphic to Problem 2 of the Power Round. Consider one of the people, $P$, in the group, and that he or she may or may not have met last year. Assume without loss of generality that $P$ met at least three of them last year: $A$, $B$, and $C$. If any two of these met each other last year, then those two and $P$ all met each other last year. Alternatively, none of $A$, $B$, and $C$ met each other last year.

4. **Answer:** $\frac{1}{16}\cos 5\theta + \frac{5}{16}\cos 3\theta + \frac{5}{8}\cos\theta$

   Notice that

$$\cos(n\theta) + i\sin(n\theta) = (\cos\theta + i\sin\theta)^n = z^n$$

   and

$$\cos(-n\theta) + i\sin(-n\theta) = (\cos\theta + i\sin\theta)^{-n} = z^{-n}.$$

   Adding these two equations, we get that $\cos(n\theta) = (z^n + z^{-n})/2$. Then $(\cos(\theta))^5 = (z + z^{-1})^5/32$. Expanding yields the binomial coefficients:

$$(z+z^{-1})^5 = z^5 + 5z^4(z^{-1}) + 10z^3(z^{-2}) + 10z^2(z^{-3}) + 5z(z^{-4}) + z^{-5}.$$

   Then

$$(z+z^{-1})^5/32 = \frac{1}{16}(z^5+z^{-5})/2 + \frac{5}{16}(z^3+z^{-3})/2 + \frac{5}{8}(z+z^{-1})/2 = \frac{1}{16}\cos 5\theta + \frac{5}{16}\cos 3\theta + \frac{5}{8}\cos\theta.$$

5. **Answer: $\mathbf{2^{2011}-1}$**

We evaluate the inner sum by the Hockey Stick Identity. This identity is

$$\sum_{i=r}^{n}\binom{i}{r}=\binom{n+1}{r+1}\implies\sum_{i=j}^{2010}\binom{i}{j}=\binom{2011}{j+1},$$

so that

$$\sum_{j=0}^{2010}\sum_{i=j}^{2010}\binom{i}{j}=\sum_{j=0}^{2010}\binom{2011}{j+1}.$$

Now, using the fact that

$$\sum_{i=0}^{n}\binom{n}{i}=2^n,$$

we obtain

$$\sum_{j=0}^{2010}\binom{2011}{j+1}=\sum_{j=1}^{2011}\binom{2011}{j}=\sum_{j=0}^{2011}\binom{2011}{j}-\binom{2011}{0}=2^{2011}-1.$$

6. **Answer: 96**

The number of blue cells is $n+m-1$; the number of total cells is $nm$. So $2010(m+n-1)=nm$, or $nm-2010n-2010m+2010=0$. This factors as $(n-2010)(m-2010)-2010^2+2010=0$, or $(n-2010)(m-2010)=2010\cdot 2009$. Thus each of $n-2010$ and $m-2010$ must be one of the positive factors of $2010*2009$; for each positive factor, there is one ordered pair. Since $2010\cdot 2009=2\cdot 3\cdot 5\cdot 7^2\cdot 41\cdot 67$, there are $2\cdot 2\cdot 2\cdot 3\cdot 2\cdot 2=2^5\cdot 3=96$ solutions.

7. **Answer: $\mathbf{\frac{1}{p}-1}$**

Let the probability that a bug's descendant's die out be $x$. There are two ways for the bugs to die out: either the initial bug dies (with probability $1-p$), or the bug successfully splits (probability $p$) and both of its descendants die out (each with probability $x$). Therefore, $x=(1-p)+px^2$. Solving this quadratic equation yields the two solutions $x=1$ and $x=\frac{1}{p}-1$. Which is correct?

Define $p_n$ to be the probability that the bug dies out within $n$ generations. Then, by the same reasoning as before, $p_{n+1}=(1-p)+pp_n^2$. From the definition of $p_n$, we see that the sequence is always increasing. We will show that $p_n<\frac{1}{p}-1$ for every $n$, which would imply that $x=\frac{1}{p}-1$ is the correct solution. This can be done by induction. Notice that $p_0=0<\frac{1}{p}-1$. Now, suppose that $p_k<\frac{1}{p}-1$ for some $k$. Then,

$$p_{k+1}=(1-p)+pp_k^2<(1-p)+p(\frac{1}{p}-1)^2=1-p+p\left(\frac{1}{p^2}-2\frac{1}{p}+1\right)=1-p+\frac{1}{p}-2+p=\frac{1}{p}-1.$$

This completes the induction, so we indeed have $p_n<\frac{1}{p}-1$, and hence the correct answer is indeed $\frac{1}{p}-1$.

8. **Answer: $\mathbf{(1,1)}$ and $\mathbf{(3,2)}$**

From that $3^1\equiv 3$, $3^2\equiv 9$, $3^3\equiv 11, 3^4\equiv 1 \mod 16$, we can see that $3^y\not\equiv 1 \mod 16$. Thus $2^x\not\equiv 0 \mod 16$, so $x$ is at most 3. The solutions are $(1,1)$ and $(3,2)$.

9. **Answer: 4**

If $(x,y)$ is a solution, then $x$ and $y$ are roots of the quadratic equation $t^2+at+b=(t-x)(t-y)=0$. Then $-a=x+y$ and $b=xy$. We find that $x^2+y^2=a^2-2b=9$ and $\frac{1}{x}+\frac{1}{y}=\frac{x+y}{xy}=-\frac{a}{b}=9$. So $a=-9b$. Substituting this in the first equation, we get $81b^2-2b-9=0$. This has two roots for $b$, both of them real. Therefore there are two corresponding values of $a$, both real. In each case, the quadratic leads to two ordered pairs, which gives four total ordered pairs. It is easy to check that they are, indeed, distinct.

10. **Answer: 120**

Note that $n^2 \equiv 0, 1, 4 \mod 5$. We consider three cases.

Case 1: $n^2 \equiv 0 \mod 5$, so that $\lfloor \frac{n^2}{5} \rfloor = \frac{n^2}{5}$. In this case, $n \equiv 0 \mod 5$, so $n = 5a$ for some integer $a$. Then $\frac{n^2}{5} = 5a^2$, which is not prime unless $a = 1$. Therefore, for this case, $n = 5$ is the only value of $n$ for which $\lfloor \frac{n^2}{5} \rfloor$ is prime.

Case 2: $n^2 \equiv 1 \mod 5$, so that $\lfloor \frac{n^2}{5} \rfloor = \frac{n^2-1}{5} = \frac{(n-1)(n+1)}{5}$. In this case, we have either $n = 5a + 1$ or $n = 5a - 1$ for some integer $a$. Then $\frac{n^2}{5} = a(n \pm 1)$, which cannot be prime if $a \neq 1$. Therefore, for this case, $n = 4, 6$ are the only values of $n$ for which $\lfloor \frac{n^2}{5} \rfloor$ might be prime. We can check that these values of $n$ do indeed yield primes 3 and 7.

Case 3: $n^2 \equiv 4 \mod 5$, so that $\lfloor \frac{n^2}{5} \rfloor = \frac{n^2-4}{5} = \frac{(n-2)(n+2)}{5}$. In this case, we have either $n = 5a + 2$ or $n = 5a - 2$ for some integer $a$. Then $\frac{n^2}{5} = a(n \pm 2)$, which cannot be prime if $a \neq 1$. Therefore, for this case, $n = 3, 7$ are the only values of $n$ for which $\lfloor \frac{n^2}{5} \rfloor$ might be prime. None of these values actually yield primes however, as they give $\lfloor \frac{n^2}{5} \rfloor = 1, 9$.

Therefore, the only values of $n$ for which $\lfloor \frac{n^2}{5} \rfloor$ is prime are $n = 4, 5, 6$, and the product of these values of $n$ is $\boxed{120}$.

Generalization: Note that this procedure can be carried out when the denominator 5 is replaced by any other number whose quadratic residues are all perfect squares. Which numbers satisfy this property?