Team Test
2007 Stanford Math Tournament
March 4, 2007

1. How many rational solutions for x are there to the equation $x^4+(2-p)x^3+(2-2p)x^2+(1-2p)x-p=0$ if p is a prime number?

2. If a and b are each randomly and independently chosen in the interval $[-1,1]$, what is the probability that $|a|+|b|<1$?

3. A clock currently shows the time 10:10. The obtuse angle between the hands measures x degrees. What is the next time that the angle between the hands will be x degrees? Round your answer to the nearest minute.

4. What is the area of the smallest triangle with all side lengths rational and all vertices lattice points?

5. How many five-letter "words" can you spell using the letters S, I, and T, if a "word" is defined as any sequence of letters that does not contain three consecutive consonants?

6. $x \equiv \left(\sum_{k=1}^{2007} k\right) \mod 2016$, where $0 \le x \le 2015$. Solve for x.

7. Daniel counts the number of ways he can form a positive integer using the digits 1, 2, 2, 3, and 4 (each digit at most once). Edward counts the number of ways you can use the letters in the word "BANANAS" to form a six-letter word (it doesn't have to be a real English word). Fernando counts the number of ways you can distribute nine identical pieces of candy to three children. By their powers combined, they add each of their values to form the number that represents the meaning of life. What is the meaning of life? (Hint: it's not 42.)

8. A 13-foot-tall extraterrestrial is standing on a very small spherical planet with radius 156 feet. It sees an ant crawling along the horizon. If the ant circles the extraterrestrial once, always staying on the horizon, how far will it travel (in feet)?

9. Let d_n denote the number of derangements of the integers $1,2,\ldots,n$ so that no integer i is in the i^{th} position. It is possible to write a recurrence relation $d_n = f(n)d_{n-1} + g(n)d_{n-2}$; what is $f(n)+g(n)$?

10. A nondegenerate rhombus has side length l, and its area is twice that of its inscribed circle. Find the radius of the inscribed circle.

11. The polynomial $R(x)$ is the remainder upon dividing x^{2007} by x^2-5x+6. $R(0)$ can be expressed as $ab(a^c-b^c)$. Find $a+c-b$.

12. Brownian motion (for example, pollen grains in water randomly pushed by collisions from water molecules) simplified to one dimension and beginning at the origin has several interesting properties. If $B(t)$ denotes the position of the particle at time t, the average of $B(t)$ is $x=0$, but the average of $B(t)^2$ is t, and these properties of course still hold if we move the space and time origins ($x=0$ and $t=0$) to a later position and time of the particle (past and future are independent). What is the average of the product $B(t)B(s)$?

13. Mary Jane and Rachel are playing ping pong. Rachel has a 7/8 chance of returning any shot, and Mary Jane has a 5/8 chance. Mary Jane serves to Rachel (and doesn't mess up the serve). What is the average number of returns made?

14. Let p,q be positive integers and let $x_0=0$. Suppose $x_{n+1}=x_n+p+\sqrt{q^2+4px_n}$. Find an explicit formula for x_n.

15. $\int_0^\infty \frac{\tan^{-1}(\pi x)-\tan^{-1}x}{x}dx$