Power Test
2007 Stanford Math Tournament
March 4, 2007

Definitions:

- *Floor:* $\lfloor x \rfloor$ is the greatest integer less than or equal to x.
- *Ceiling:* $\lceil x \rceil$ is the least integer greater than or equal to x.
- *Fractional part:* $\{x\} = x - \lfloor x \rfloor$.
- *Intervals:*
  - *Open:* $(\alpha, \beta) = \{\alpha < x < \beta\}$
  - *Closed:* $[\alpha, \beta] = \{\alpha \leq x \leq \beta\}$
  - *Half-open:* $[\alpha, \beta) = \{\alpha \leq x < \beta\}$ and $(\alpha, \beta] = \{\alpha < x \leq \beta\}$ (called half-closed by pessimists)

In all problems, assume that x, y, α, β are real and m, n are integers. (If you define new variables in your proofs please try to keep to this convention!) Note that for $i < j$ you may use the result of problem i for problem j even if you have not solved it.

1. Show that $\lfloor x \rfloor = n$ if and only if $n \leq x < n+1$ and if and only if $x - 1 < n \leq x$. Write similar statements for ceilings (you needn't prove them separately).
2. Show that $\lfloor -x \rfloor = - \lceil x \rceil$.
3. Show that $x < n$ if and only if $\lfloor x \rfloor < n$. Write similar statements for $n < x$, $x \leq n$, and $n \leq x$ (you needn't prove them separately).
4. Show that $\lfloor n + x \rfloor = n + \lfloor x \rfloor$, and write a similar statement for $\lceil n + x \rceil$ (again, you needn't prove it separately).
5. Determine, with proof, under what conditions $\lfloor nx \rfloor = n \lfloor x \rfloor$.
6. How can we round, that is, find the nearest integer to x? We usually round up ties (when x is halfway between integers), so give two formulas, one which rounds ties up and one which rounds them down.
7. Show that $\left\lceil \frac{2x+1}{2} \right\rceil + \left\lfloor \frac{2x+1}{4} \right\rfloor - \left\lceil \frac{2x+1}{4} \right\rceil$ is either $\lfloor x \rfloor$ or $\lceil x \rceil$, and when each is true.
8. Show that $\left\lceil \frac{n}{m} \right\rceil = \left\lfloor \frac{n+m-1}{m} \right\rfloor$ when $m > 0$.
9. Find, with proof, forumulas for the number of integers contained in the half-open intervals $[\alpha, \beta)$ and $(\alpha, \beta]$, assuming $\alpha \leq \beta$.
10. Show that $\lfloor \lfloor m\alpha \rfloor n/\alpha \rfloor = mn - 1$ where $m, n > 0$ and $\alpha > n$ is irrational.
11. Suppose $f(x)$ is a continuous and increasing function such that if $f(x)$ is an integer, x is an integer. Show that $\lfloor f(\lfloor x \rfloor) \rfloor = \lfloor f(x) \rfloor$. What is a similar statement $\lfloor f(x) \rfloor = ?$ if f is decreasing instead of increasing? (The relevant property of continuous functions is that if $f(x_1) = y_1$ and $f(x_2) = y_2$, then f passes through all y-values between y_1 and y_2 at some point as x goes from x_1 to x_2.)
12. The spectrum of a real number x is the sequence of integers $\text{Spec}(x) = \{\lfloor x \rfloor, \lfloor 2x \rfloor, \lfloor 3x \rfloor, \ldots\}$. Show that spectra are unique, i.e. that $\text{Spec}(\alpha) = \text{Spec}(\beta)$ if and only if $\alpha = \beta$.
13. A casino has a roulette wheel with one thousand slots, numbered 1 to N^3. If the number n that comes up is divisible by the floor of its cube root ($\lfloor \sqrt[3]{n} \rfloor \mid n$), it's a winner. Determine with proof the number of winners.

14. Show that $\sum_{j=0}^{n} j^2 = \frac{1}{6}n(n+1)(2n+1)$

15. Show that, if $a = \lfloor\sqrt{n}\rfloor$, $\sum_{k=0}^{n-1} \left\lfloor \sqrt{k} \right\rfloor = na - \frac{1}{3}a^3 - \frac{1}{2}a^2 - \frac{1}{6}a$.

16. A circle, $2r = 2n-1$ units in diameter, is drawn centered at the center of a $2n \times 2n$ square grid. Show that the circle passes through $8r$ cells of the grid, determine an $f(n,k)$ such that $\sum_{k=1}^{n-1} f(n,k)$ is the number of cells entirely contained inside the circle.