Power Solutions
2007 Stanford Math Tournament
March 4, 2007

1. $n \leq x < n+1$ if and only if n is the greatest integer less than or equal to x. The second condition is equivalent to the first since $x-1 < n \Rightarrow x < n+1$. The corresponding statements are $\lceil x \rceil = n \iff n-1 < x \leq n \iff x \leq n < x+1$.

2. From the first problem, we have $\lfloor -x \rfloor = n \Rightarrow n \leq -x < n+1 \Rightarrow -n-1 < x \leq -n \Rightarrow -n = \lceil x \rceil$.

3. Assume first that $x < n$. Then by problem 1, $\lfloor x \rfloor \leq x < n$. Now assume $\lfloor x \rfloor < n$; by problem 1 we know $x < \lfloor x \rfloor + 1$, and since both are integers, $\lfloor x \rfloor \leq n$. Similarly, $n < x \iff n < \lceil x \rceil$, $x \leq n \iff \lceil x \rceil \leq n$, and $n \leq x \iff n \leq \lfloor x \rfloor$.

4. Let $m = \lfloor n + x \rfloor$. Then $m \leq n + x < m+1$, so $m - n \leq x < m - n + 1$, so $\lfloor x \rfloor = m - n$ and thus $m = n + \lfloor x \rfloor$. Similarly $\lceil n + x \rceil = n + \lceil x \rceil$.

5. We split x into floor and fractional part: $\lfloor nx \rfloor = \lfloor n \lfloor x \rfloor + n \{x\} \rfloor = n \lfloor x \rfloor + \lfloor n \{x\} \rfloor$. Thus for the two to be equal, $\lfloor n \{x\} \rfloor = 0$ so $0 \leq n \{x\} < 1$, so $\{x\} < 1/n$.

6. To round up, take $\left\lfloor x + \frac{1}{2} \right\rfloor$. We see this works by splitting the inside into a floor and a fractional part; if $\{x\} < 1/2$, adding $1/2$ doesn't change the floor, but if $\{x\} \geq 1/2$, adding $1/2$ increases the floor by 1. A similar argument gives $\left\lceil x - \frac{1}{2} \right\rceil$ for rounding down.

7. $\frac{2x+1}{2} = x + \frac{1}{2}$, so the first term rounds looks like our rounding formula, except the result is always one too high except when $x + 1/2$ is an integer, in which case it correcly rounds up. Now notice that $\lceil \alpha \rceil - \lfloor \alpha \rfloor$ is 0 if α is an integer and 1 otherwise, so the next two terms subtract 1 if $\frac{2x+1}{4} = \frac{x+1/2}{2}$ is not an integer. Thus the other terms correct the first term to the correctly rounded value when $x+1/2$ is not an integer. When $x + 1/2$ is an integer, the other terms leave the first term alone if it's an even one, but subtract one if it's odd. Thus the formula always rounds x to the nearest integer, rounding halves up or down when $x + 1/2$ is even or odd.

8. Let $k = \left\lceil \frac{n}{m} \right\rceil$. We have $k - 1 < \frac{n}{m} \leq k$. Since $\frac{m-1}{m} < 1$, $\frac{n+m-1}{m} < k+1$. Since n, m are integers, and $\frac{n}{m} > k-1$, we know that $\frac{n}{m} \geq k - 1 + \frac{1}{m}$, so $\frac{n+m-1}{m} > k$. Thus $k = \left\lfloor \frac{n+m-1}{m} \right\rfloor$.

9. First note that if α and β are integers, the answer in both cases is $\beta - \alpha$. Let n be an integer in $[\alpha, \beta)$; by problem 3 we have that $\lceil \alpha \rceil \leq n < \lceil \beta \rceil$, so the number of integers in the interval is $\lceil \beta \rceil - \lceil \alpha \rceil$. Similarly, $n \in (\alpha, \beta]$ implies $\lfloor \alpha \rfloor < n \leq \lfloor \beta \rfloor$, giving $\lfloor \beta \rfloor - \lfloor \alpha \rfloor$.

10. Since α is irrational, we know $0 < \{m\alpha\} < 1$, and also $n/\alpha < 1$. Plugging in $\lfloor m\alpha \rfloor = m\alpha - \{m\alpha\}$, we obtain $\lfloor m\alpha n/\alpha - \{m\alpha\} n/\alpha \rfloor = \lfloor mn - \{m\alpha\} n/\alpha \rfloor = mn - 1$.

11. If $\lfloor x \rfloor = x$, we are done; otherwise, $\lfloor x \rfloor < x$. Thus $f(\lfloor x \rfloor) < f(x)$ since f is increasing, and so $\lfloor f(\lfloor x \rfloor) \rfloor \leq \lfloor f(x) \rfloor$. If $\lfloor f(\lfloor x \rfloor) \rfloor < \lfloor f(x) \rfloor$, since f is continuous there must be a number y such that $\lfloor x \rfloor \leq y < x$ and $f(y) = \lfloor f(x) \rfloor$. By the special property of f, this means y is an integer, but there can be no integer between x and its floor! Thus we must have $\lfloor f(\lfloor x \rfloor) \rfloor = \lfloor f(x) \rfloor$. Similarly, for decreasing f, $\lfloor f(x) \rfloor = \lfloor f(\lceil x \rceil) \rfloor$.

12. (Proof by contrapositive) Suppose $\alpha \neq \beta$, and assume without loss of generality that $\alpha < \beta$. Then there must be a positive integer m such that $m(\beta - \alpha) \geq 1$. Thus $m\beta - m\alpha \geq 1$ so $\lfloor m\beta \rfloor > \lfloor m\alpha \rfloor$, so the m^{th} elements of the spectra are different.

13. Suppose n is a winner; let $k = \lfloor \sqrt[3]{n} \rfloor$. Then $k^3 \leq n < (k+1)^3$ and $n = km$ for some m. Note that N^3 is a winner; let's assume $n < N^3$, so that $1 \leq k < N$. Now substituting km for n, $k^3 \leq km < (k+1)^3$ so $k^2 \leq m < (k+1)^3/m$. Using our formula for the number of integers in a half-open interval,

there are $\lceil (k+1)^3/k \rceil - \lceil k^2 \rceil = \lceil k^2 + 3k + 3 + 1/k \rceil - k^2 = 3k + 4$ of these. We then simply sum this for the possible values of k (it's an arithmetic series), and add back in the $n = 1000$ case to get $1 + 4(N-1) + \frac{3}{2}(N-1)N = \frac{1}{2}(3N^2 + 5N - 6)$.

14. A proof by induction is quickest (though not the most general or elegant). The statement is true for $n = 0$, and starting from n and moving up to $n+1$:

$$\begin{aligned}\frac{1}{6}n(n+1)(2n+1) + (n+1)^2 &= (n+1)\left(\frac{n^2}{3} + \frac{n}{6} + n + 1\right) \\ &= \frac{1}{6}(n+1)(2n^2 + 7n + 6) \\ &= \frac{1}{6}(n+1)(n+2)(2n+3)\end{aligned}$$

15. Note that the terms for $a^2 \leq k < n$ are all equal to a, so they contribute $(n - a^2)a$ to the sum. We now consider the rest of the sum, $0 \leq k < a^2$. Let $m = \lfloor \sqrt{k} \rfloor$; then $m \leq \sqrt{k} < m+1$ so $m^2 \leq k < (m+1)^2 \leq a^2$. We sum over k first instead of m; there are $(m+1)^2 - m^2$ possible values of k, so our new sum is:

$$\sum_{m=0}^{a-1} m((m+1)^2 - m^2 = \sum_{m=0}^{a-1} m(2m+1) = 2\frac{1}{6}(a-1)a(2a-1) + \frac{1}{2}a(a-1)$$

Expanding, we have $\frac{2a^3}{3} - \frac{a^2}{2} - \frac{a}{6}$; adding in the $k \geq a^2$ terms, we obtain the desired result.

16. There are $2n - 1$ each of horizontal lines vertical lines between cells of the grid, and the circle crosses each one twice. Since r^2 is not an integer, the circle cannot pass through the corner of any cell, by the Pythagorean theorem. Thus the circle passes through a cell for each time it crosses a line, giving $4(2n-1) = 8n - 4 = 8r$ cells. $f(n,k) = 4\left\lfloor r^2 - k^2 \right\rfloor$: consider $f(n,k)/4$; placing the x, y axes along the grid with origin at the center we can easily see from the equation of a circle that this is the number of cells above $x = k$ within the circle.