Calculus Solutions
2007 Stanford Math Tournament
March 4, 2007

1. **Answer:** $-\frac{1}{6}$

   Use l'Hopital's rule:
   $$\lim_{x\to 0} \frac{-1+\cos x}{3x^2+4x^3} = \lim_{x\to 0} \frac{-\sin x}{6x+12x^2} = \lim_{x\to 0} \frac{-\cos x}{6+12x}$$

2. **Answer:** $\frac{\sqrt[3]{4}}{2}$
   $$y' = 3a^2+3 = \frac{a^3+3a+1}{a} = \frac{y}{x}$$
   $$\frac{2a^3-1}{a} = 0$$
   $$a^3 = \frac{1}{2}$$

3. **Answer:** $\frac{\sqrt{5}-1}{2}$

   The speed will cancel out so assume it is 1. We then have:
   $$\int_{\tau}^{t+1} \frac{1}{t}dt = 2\int_{\tau+1}^{\tau+2} \frac{1}{t}dt$$
   $$\ln\frac{\tau+1}{\tau} = 2\ln\frac{\tau+2}{\tau+1}$$
   $$\frac{\tau+1}{\tau} = \left(\frac{\tau+2}{\tau+1}\right)^2$$
   $$\tau = \frac{-1\pm\sqrt{5}}{2}$$

4. **Answer:** $\frac{5}{2}$

   For odd n, $I(n) = -\left.\frac{\cos(nx)}{n}\right|_0^{\pi} = 2/n$, so $\sum_{n=0}^{\infty} I(5^n) = \sum_{n=0}^{\infty} 2/5^n = 5/2$

5. **Answer: 7**

   We have $f'(x) = \int(\delta_1(x)+\delta_2(x))\,dx = \Theta_1(x)+\Theta_2(x)+C$, and $f'(0)=0$ so $C=0$. Integrating up to f is most easily accomplished graphically; the region under the curve from 0 to 5 is a 1×4 rectangle from $x=1$ to $x=5$ with a 1×3 rectangle from $x=2$ to $x=5$ on top.

6. **Answer:** $\frac{4}{\pi^2}$

   Suppose A lies at polar coordinate $0<\theta<\pi/2$. For the rectangle to lie within the circle, B must lie in the rectangle with vertices at A, A reflected over the x-axis, A reflected over the y-axis, and A reflected over both axes. Thus for this fixed A, the probability is $(2\sin\theta)(2\cos\theta)/\pi = 2\sin(2\theta)/\pi$. The total probability is then $\frac{2}{\pi}\int_0^{\pi/2}\frac{2}{\pi}\sin(2\theta)d\theta$. (Integrating over the circle requires taking the absolute value of the expression for area, which then splits up into four sections identical to the one considered here.)

7. **Answer:** $\boldsymbol{\frac{4\pi}{e^2}}$

$$V = \pi \int_0^2 \left(\sqrt{2x - x^2}e^{-x/2}\right)^2 dx = \pi \int_0^2 (2x - x^2)e^{-x}dx = \pi x^2 e^{-x}\Big|_0^2$$

8. **Answer: 12 cups of coffee**

   The number of theorems proven is $(s + \ln c)(24 - s - c/12)$. Differentiating with respect to s gives $24 - \frac{c}{12} - 2s - \ln c = 0$, so $s = 12 - \frac{c}{24} - \frac{1}{2}\ln c$. This is a maximum in s since the second derivative is -2. Plugging this back in and simplifying gives $(12 - \frac{c}{24} + \frac{\ln c}{2})^2 = f(c)^2$ theorems proven. This differentiates to $2f'(c)f(c)$, so the derivative will be zero when either $f(c)$ or $f'(c)$ is zero. $f(c) = 0$ is difficult to solve, involving both a logarithm and a binomial, but $f'(c) = \frac{1}{2c} - \frac{1}{24}$, so $c = 12$ is a solution. It is a maximum in c since the second derivative is $2f'(c)^2 + 2f(c)f''(c)$, with $f''(12) < 0$, $f(12) > 0$, and $f'(12) = 0$.

9. **Answer: ln 2**

$$\sum_{k=n+1}^{2n} \frac{1}{k} = \frac{n}{n}\sum_{k=1}^{n} \frac{1}{k+n} = \sum_{k=1}^{n} \frac{1}{n}\frac{1}{1+\frac{k}{n}}$$

   This is a Riemann sum: $\int_1^2 \frac{1}{x}dx = \ln 2$.

10. **Answer:** $\boldsymbol{10x^{19}}$

    Note that $\int f(x)dx = \frac{1}{2(1-x^2)} = \frac{1}{4}\left(\frac{1}{1+x} + \frac{1}{1-x}\right)$. These are geometric sums, so we have

$$\begin{aligned} \int f(x)dx &= \frac{1}{4}\left(\sum_{k=0}^{\infty} x^k + \sum_{k=0}^{\infty}(-x)^k\right) \\ &= \frac{1}{2}\sum_{k=0}^{\infty} x^{2k} \\ f(x) &= \sum_{k=0}^{\infty} kx^{2k-1} \end{aligned}$$