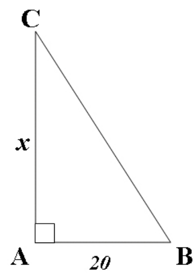
Team Solutions
2006 Stanford Math Tournament
February 25, 2006

1. **Answer: 29**



$140\pi =$ volume of cone M - volume of cone $N = \frac{1}{3} \cdot x^2 \cdot 20\pi - \frac{1}{3} \cdot x \cdot (20)^2\pi = \frac{20x^2\pi}{3} - \frac{400x\pi}{3}$
$20x^2 - 400x = 420 \Rightarrow x^2 - 20x - 21 = 0 \Rightarrow (x-21)(x+1) = 0 \Rightarrow x = 21, -1$
But x must be positive, so $x = 21$.
$\overline{BC} = \sqrt{\overline{AB}^2 + \overline{AC}^2} = \sqrt{20^2 + 21^2} = 29$

2. **Answer: -2**
Let the first element be x, and the second,y. Writing out each element in terms of x and y gives $\{x, y, 2x+y, 5x+3y, 13x+8y, \ldots\}$, which is apparently the fibonacci sequence with every other element as the coefficient of x or y. So the 6th element is $34x+21y$ and the seventh, $89x+55y$. Solving $89 \cdot 2 + 55 \cdot y = 68$ gives $y = -2$.

3. **Answer: 17.5**
Form $\triangle ABC$, and set $a = \overline{BC}, b = \overline{AC}$,and $c = \overline{AB}$. Let 5 be the altitude from A, 7 be the altitude from B, and call the third altitude h.
$5a = 7b = h \cdot c$, so $\frac{a}{c} < \frac{h}{5}$ and $\frac{b}{c} = \frac{h}{7}$.
Since $a < b + c$,

$$\frac{a}{c} = \frac{b}{c} + 1 \Rightarrow \frac{h}{5} < \frac{h}{7} + 1$$

$$h \cdot \left(\frac{1}{5} - \frac{1}{7}\right) < 1$$

so $h < \frac{7 \cdot 5}{7-5} = 17.5$

4. **Answer: $a^6 - 6a^4b + 9a^2b^2 - 2b^3$**
Note: $(x^{n-1} + y^{n-1})(x+y) = x^n + y^n + yx^{n-1} + xy^{n-1} = x^n + y^n + xy(x^{n-2} + y^{n-2})$.
Thus, let $f(n) = x^n + y^n$. We see $f(n) = af(a-1) - bf(n-2)$.
$x^0 + y^0 = 2$, so $f(0) = 2$
$x^1 + y^1 = x + y = a$, so $f(1) = a$
$f(2) = a^2 - 2b$
$f(3) = a^3 - 3ab$
$f(4) = a^4 - 3a^2b - a^2b + 2b^2 = a^4 - 4a^2b + 2b^2$
$f(5) = a^5 - 4a^3b + 2ab^2 - a^3b + 3ab^2 = a^6 - 6a^3b + 5ab^2$
$f(6) = a^6 - 5a^4b + 5a^2b^2 - a^4b + 4a^2b^2 - 2b^3 = a^6 - 6a^4b + 9a^2b^2 - 2b^3$

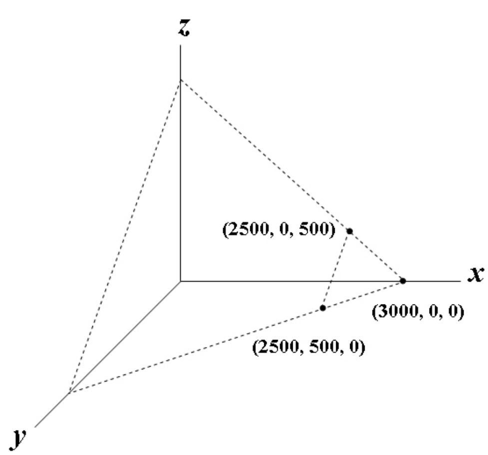
5. **Answer: 1**

$$\begin{aligned}
\sin(\arccos(\tan(\arcsin x))) &= x \\
\sin\left(\arccos\left(\frac{x}{\sqrt{1-x^2}}\right)\right) &= x \\
\sqrt{1-\left(\frac{x}{\sqrt{1-x^2}}\right)^2} &= x \\
\sqrt{\frac{1-2x^2}{1-x^2}} &= x \\
1-2x^2 &= x^2 - x^4 \\
x^4 - 3x^2 + 1 &= 0
\end{aligned}$$

Solving and restricting x to positive numbers: $x^2 = \frac{3\pm\sqrt{9-4}}{2}$
$x = \sqrt{\frac{3+\sqrt{5}}{2}}$ or $x = \sqrt{\frac{3-\sqrt{5}}{2}}$. Multiplying these together, the answer is $\sqrt{\frac{9-5}{4}}$.

6. **Answer: 8024**
Write the expression as $x^4 + x^2 + 1$ where $x = 2^n$. This is equivalent to $(x^2+1)^2 - x^2$ (by adding and subtracting x^2). This expression can be written as $(x^2+x+1)(x^2-x+1) = \frac{x^3-1}{x-1} \cdot \frac{x^3+1}{x+1} = \frac{x^6-1}{x^2-1} = \frac{2^{6n}-1}{2^{2n}-1}$. Hence $p(n) = 6n$ and $q(n) = 2n$. It's not hard to see that this is the only solution by considering the limit of each expression as n approaches infinity. The highest-order terms predominate: 2^{4n} and $2^{q(n)(p(n)/q(n)-1)}$. This implies that p and q are linear functions. Exact functions can be determined by evaluating the expressions at $n = 1$ and $n = 2$ and solving for two variables. The answer is 8,024.

7. **Answer: $\frac{1}{12}$**



This is a geometric probability problem. The set of 3-tuples above fits an equilateral triangle on the plane $x + y + z = 3000$. We're going to look at the sections of this triangle where $x \geq 2500$. This is a triangle with vertices $(2500, 500, 0)$, $(2500, 0, 500)$, and $(3000, 0, 0)$. This is an equilateral triangle with length $500\sqrt{2}$. The area of this triangle is $\frac{side^2\sqrt{3}}{4} = \frac{(500\sqrt{2})^2\sqrt{3}}{4} = 125000\sqrt{3}$. Since x, y, or z can be larger than 2500, we need to multiply this by 3 to get the total area that works: $125000\sqrt{3} \cdot 3 = 375000\sqrt{3}$. The total possible area is the whole triangle of side length $3000\sqrt{2}$: $\frac{side^2\sqrt{3}}{4} = \frac{(3000\sqrt{2})^2\sqrt{3}}{4} = 4500000\sqrt{3}$. So the overall probability is $\frac{375000\sqrt{3}}{4500000\sqrt{3}} = \frac{1}{12}$.

8. **Answer: 2**

Let $S_n = \sum_{k=n^2}^{(n+1)^2} \frac{1}{\sqrt{k}}$

$$\sum_{k=n^2}^{(n+1)^2} \frac{1}{\sqrt{(n+1)^2}} < S_n < \sum_{k=n^2}^{(n+1)^2} \frac{1}{\sqrt{n^2}}$$
$$\left((n+1)^2 - n^2 + 1\right)\frac{1}{n+1} < S_n < \left((n+1)^2 - n^2 + 1\right)\frac{1}{n}$$
$$\frac{2(n+1)}{n+1} < S_n < \frac{2(n+1)}{n}$$
$$2 < S_n < 2 + \frac{1}{n}$$

Thus $\lim_{n\to\infty} \sum_{k=n^2}^{(n+1)^2} \frac{1}{\sqrt{k}} = 2$

9. **Answer: $\mathbf{\frac{5}{2}}$**

Suppose the medians intersect at P. If $BC = x$, $BP = CP = \frac{x}{\sqrt{2}}$. By a well-known property of centroids, $\frac{MP}{MC} = \frac{1}{3}$, so $MP = \frac{x}{2\sqrt{2}}$. Using the Pythagorean Theorem, we find that $MB = \frac{x\sqrt{\frac{5}{2}}}{2}$ and so $AB = x\sqrt{\frac{5}{2}}$. So $\left(\frac{AB}{BC}\right)^2 = \frac{5}{2}$.

10. **Answer: 638**

Notice that $n^3 + 8$ is divisible by $n + 2$. Therefore, $m - 8$ must be divisible by $n + 2$ for the expression to be an integer. If f is a factor of $m - 8$, $n = f - 2$ is a corresponding suitable n; we then need $f \geq 3$ to make $n > 0$. Thus $m - 8$ must have twelve each odd and even factors including 1 and 2. To make the number of odd and even factors equal in order to minimize m, the power of 2 in the prime factorization of $m - 8$ must be 1. Suppose the prime factorization of $m - 8$ is then $2^1 \cdot 3^a \cdot 5^b \cdot 7^c \cdot 11^d$ (larger prime factors will clearly not minimize m). Then $(a+1)(b+1)(c+1)(d+1) \geq 12$. To minimize m, $a \geq b \geq c \geq d$. We then examine values of $\frac{m-8}{2}$ to determine the best (a, b, c, d). $3 \cdot 5 \cdot 7 \cdot 11 = 1155$, $3^2 \cdot 5 \cdot 7 = 315$. Moving any more factors into smaller primes involves multiplying by $\frac{3^2}{7}$ or $\frac{3^2}{5}$ (or subsequent larger powers of 3), which increases the value. Therefore $m - 8 = 2 \cdot 3^2 \cdot 5 \cdot 7$, so $m = 638$.

11. **Answer: 64**

Using the first condition with $j = 1003$ we get $c_i = 2(1003 - i)c_{2006-i}$. Replace the coefficients of P in this manner and notice that $x^{2006}\frac{P(\frac{2}{x})}{2006} = P(x)$. Therefore if r is a solution of $P(x) = 0$ then $P(2/r) = 0$. Then:

$$\sum_{i\neq j, i=1, j=1}^{2006} \frac{r_i}{r_j} = \sum_{i=1}^{2006} r_i \sum_{i=1}^{2006} \frac{1}{r_i} - 2006 = \frac{1}{2}\left(\sum_{i=1}^{2006} r_i\right)^2 - 2006 = 42$$

Solving for the desired sum gives 64.

12. **Answer: 17**

$\sum_{i=1}^{k}\left(180 - \frac{360}{n_i}\right) = 0$, so $k/2 - 1 = \sum_{i=1}^{k} \frac{1}{n_i}$. Clearly, $3 \leq k \leq 6$, since the interior angles are less than $180°$, and six equilateral triangles maximize k. For each k, bounds can be established on the smallest or largest n_i. From then, we can fix all but two of the n_i, solve algebraically, then use reasonable guesswork to find all integer solutions. For $k = 3$, fix n_1 at 3, 4, 5, or 6 and then solve $\frac{3}{2} - 1 = \frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3}$. This yields 10 solutions. For $k = 4$, $n_4 = 3$ or 4; there are 4 solutions. For $k = 5$, $n_5 = n_4 = n_3 = 3$, giving two solutions. Finally there is of course only one solution for $k = 6$. $10 + 4 + 2 + 1 = 17$

13. **Answer: $\frac{2\sqrt{7}}{7}$**
It is clear from drawing the graph that we want to find the cosine of the smallest angle $\theta (0 < \theta < \frac{\pi}{2})$ such that a ray leaving the origin at angle θ will hit the graph of the hyperbola in the first quadrant. Since $\cos\theta$ is a decreasing function on this interval, we want the largest possible value of $\cos\theta$.
We begin by writing the hyperbola in polar coordinates: $r^2\sin^2\theta = r^2\cos^2\theta - r\cos\theta + 1$.
Using $\sin^2\theta = 1 - \cos^2\theta$ and collecting like terms, we get: $(2\cos^2\theta - 1)r^2 - (\cos\theta)r + 1 = 0$.
Now we can use the quadratic formula to solve for r:

$$r = \frac{\cos\theta \pm \sqrt{\cos^2\theta - 4(2\cos^2\theta - 1)}}{4\cos^2\theta - 2}$$

If there are any solutions for r, the quantity under the square root must be nonnegative:

$$\cos^2\theta \geq 8\cos^2\theta - 4$$
$$7\cos^2\theta \leq 4$$
$$\cos\theta \leq \frac{2\sqrt{7}}{7}$$

So the angle we are looking for has

$$\cos\theta = \frac{2\sqrt{7}}{7}$$

14. **Answer: 292**
First we find the largest power of an integer d that divides $k!$. Notice that $\lfloor \frac{k}{d} \rfloor$ of the integers $1, 2, \ldots, k$ are divisible by d, $\lfloor \frac{k}{d^2} \rfloor$ are divisible by d^2, and so on. The largest power we are looking for is then $\lfloor \frac{k}{d} \rfloor + \lfloor \frac{k}{d^2} \rfloor + \lfloor \frac{k}{d^3} \rfloor + \ldots$. Now let $m = 2006 - n$, so that $\binom{2006}{n} = \frac{2006!}{n!m!}$; the largest power of 7 divisor is then $\left(\lfloor \frac{2006}{7} \rfloor - \lfloor \frac{n}{7} \rfloor - \lfloor \frac{m}{7} \rfloor\right) + \left(\lfloor \frac{2006}{7^2} \rfloor - \lfloor \frac{n}{7^2} \rfloor - \lfloor \frac{m}{7^2} \rfloor\right) + \ldots$ Note that if $\frac{n}{d} = \lfloor \frac{n}{d} \rfloor + n'$ and $\frac{m}{d} = \lfloor \frac{m}{d} \rfloor + m'$, then $\frac{2006}{d} = \frac{n+m}{d}$ leaves a remainder of $r = n' + m'$ or $n' + m' - d$, whichever satisfies $0 \leq r < d$. Therefore $\lfloor \frac{2006}{d} \rfloor - \lfloor \frac{m}{d} \rfloor - \lfloor \frac{n}{d} \rfloor = 0$ or 1. To make this 1 in order to get large divisors of $\binom{2006}{n}$, we need $m', n' > r$. We therefore find the remainders when 2006 is divided by 7, 7^2, and 7^3: 4, 46, and 291. Therefore n must leave a remainder of at least 292 when divided by 343, so we try $n = 292$, which has remainders of 5 and 47 when divided by 7 and 49.

15. **Answer: $\frac{12}{\pi^2}$**
Write

$$\prod_{p\ prime} \frac{p^2}{p^2 - 1} \prod_{c\ composite} \frac{c^2}{c^2 - 1} = \prod_{n=2}^{\infty} \frac{n^2}{n^2 - 1} = \prod_{n=2}^{\infty} \frac{n}{n-1} \frac{n}{n+1}$$

which telescopes and evaluates to 2. Meanwhile we can write

$$\prod_{p\ prime} \frac{p^2}{p^2 - 1} = \prod_{p\ prime} \frac{1}{1 - \frac{1}{p^2}}.$$

The latter is equivalently rewritten:

$$\prod_{p\ prime} 1 + \frac{1}{p^2} + \frac{1}{p^4} + \ldots = \prod_{p\ prime} \left(\sum_{n=0}^{\infty} \frac{1}{p^{2n}} \right).$$

When we distribute the infinite product over the infinite sum, we get a sum of terms. Each term is of the form $\frac{1}{m^2}$ for integer m. Each m appears exactly once, so the product is equal to $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$. Hence

$$\prod_{c\ composite} \frac{c^2}{c^2 - 1} = \frac{2}{\frac{\pi^2}{6}} = \frac{12}{\pi^2}.$$