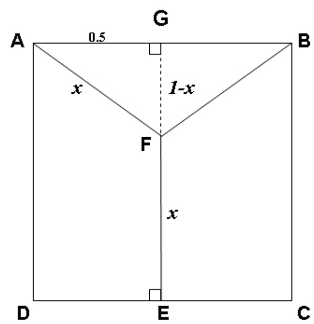
Geometry Solutions
2006 Stanford Math Tournament
February 25, 2006

1. **Answer:** $\frac{1}{6}$
   Let s represent the side length of the cube. The octahedron has a volume equivalent to the volume of two pyramid with height $\frac{s}{2}$ and a square base with side length $\frac{s}{2}\sqrt{2}$. The volume is therefore $2 \cdot \frac{1}{3} \cdot \left(\frac{s}{2}\sqrt{2}\right)^2 \cdot \frac{s}{2} = \frac{1}{6} \cdot s^3$, or 1/6 of the cube volume.

2. **Answer:** $\frac{13}{32}$
   Let $\overline{EF} = x$.



   From pythagorean theorem:
   $(\frac{1}{2})^2 + (1-x)^2 = x^2$
   $1 + 4x^2 - 8x + 4 = 4x^2$
   $8x = 5$
   $x = \frac{5}{8}$
   area of $ADEF =$ area of $ADEG-$ area of $AFG = \frac{1}{2} - \frac{(\frac{1}{2})(\frac{3}{8})}{2} = \frac{1}{2} - \frac{3}{32} = \frac{13}{32}$

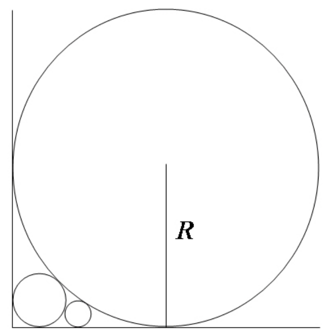
3. **Answer:** $y = \frac{x^2}{8} + 1$
   Since circle δ is tangent to the x-axis, its radius is y. Thus from the Pythagorean Theorem:
$$(3-y)^2 + x^2 = (y+1)^2$$
$$9 - 6y + x^2 = 2y + 1$$
$$8 + x^2 = 8y$$
$$1 + \frac{x^2}{8} = y$$

4. **Answer:** $\frac{1}{2} + \pi\left(1 + \frac{11\sqrt{3}}{12}\right)$
   Since inscribed angles intercept arcs of measure twice that of the inscribed angle, this is the area above line AB between circles centered at P and Q, with $\angle AQB = 60^\circ$ and $\angle APB = 30^\circ$, A, B on both circles, and P, Q on the perpendicular bisector of $\overline{AB}$. Let M be the midpoint of $\overline{AB}$. $\triangle AQB$ is then equilateral, so $QM = \frac{\sqrt{3}}{2}$, so the radius of circle Q is 1. We see that since $\angle APB = 30^\circ$, P is on circle Q, so $PM = 1 + \frac{\sqrt{3}}{2}$, and by the Pythagorean theorem, $(PA)^2 = \left(1 + \frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = 2 + \sqrt{3}$. We find the area in circles P and Q above line AB by taking the major sector AB of each the circles above the line and adding in the areas of $\triangle APB$ and $\triangle AQB$ respectively:
$$A = \pi\left(2+\sqrt{3}\right)\frac{330}{360} + \frac{1}{2}(1)\left(1 + \frac{\sqrt{3}}{2}\right) - \left(\pi \cdot 1^2 \frac{300}{360} + \frac{1}{2}(1)\left(\frac{\sqrt{3}}{2}\right)\right)$$

This simplifies to the given answer.

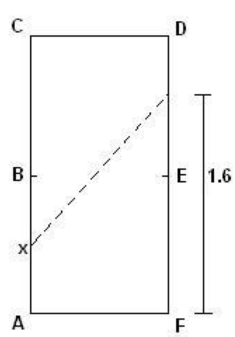
5. **Answer: $\mathbf{7\sqrt{3}}$**
Let the side length of the cube be s. It is apparent that in order for the shadow to be a regular hexagon, the cube must have two vertices with the same x and y coordinates; call these vertices A and B. Let T be another vertex of the cube. Clearly, $\triangle ABT$ is a right triangle with hypotenuse $AB = s\sqrt{3}$ the space diagonal of the cube, and legs s and $s\sqrt{2}$. Notice that a segment from T to AB has for its shadow a segment between the center of the hexagon and one of its vertices; thus the distance from T to AB is the same as the center to vertex distance. Using similar triangles, this length can be found to be $\frac{s\sqrt{6}}{3}$. Thus the area of the hexagon is $s^2\sqrt{3} = 147\sqrt{3}$ and therefore $s = 7\sqrt{3}$.

6. **Answer: $\mathbf{\frac{R}{2}(3-2\sqrt{2})}$**



Let the radius of the second circle be r.
$R\sqrt{2} - R = r + r\sqrt{2}$
$r = \frac{R(\sqrt{2}-1)}{\sqrt{2}+1} = R(3-2\sqrt{2})$
Let the radius of the third circle be ρ.
$\sqrt{(r+\rho)^2 - (r-\rho)^2} + \sqrt{(R+\rho)^2 - (R-\rho)^2} = \sqrt{(R+r)^2 - (R-r)^2}$
$\sqrt{4r\rho} + \sqrt{4R\rho} = \sqrt{4Rr}$
$\sqrt{r\rho} + \sqrt{R\rho} = \sqrt{Rr}$
$\sqrt{\rho} = \frac{\sqrt{Rr}}{\sqrt{R}+\sqrt{r}}$
$\rho = \frac{Rr}{R+r+2\sqrt{Rr}} = \frac{R^2(3-2\sqrt{2})}{R(4-2\sqrt{2})+2R\sqrt{3-2\sqrt{2}}} = \frac{R^2(3-2\sqrt{2})}{R(4-2\sqrt{2})+2R(\sqrt{2}-1)} = \frac{R^2(3-2\sqrt{2})}{2R} = \frac{R}{2}(3-2\sqrt{2})$

7. **Answer: $\{\boldsymbol{D}, \mathbf{6}\}$**



Note that each time the ball bounces up the wall, it is equivalent to forming a mirror image of the table and extending the path taken. Set up sides $\overline{AF}$ and $\overline{AC}$ as the x and y coordinate axes, respectively.

Since the ball is hit at (0,0.5), it can travel in an imaginary straight line through imaginary images of the table until it hits an integer coordinate (i.e. a pocket). Therefore,

$$0.5 + (1.6 - 1.5)x = y1 + \frac{11}{5} \cdot x = 2y$$

It is clear that the first instance of integer (x, y) occurs when $x = 5$ and $y = 6$. Simply counting, 5 units in the x direction ends up on side $\overline{DF}$, and 6 units in the y direction would be on side $\overline{CD}$. Therefore, the ball must have fallen in at this intersection, into pocket D. Drawing iterations of the pool table to fill the rectangle from $(0, 0)$ to $(5, 6)$, we see that the ball has crossed four vertical boundaries and two horizontal boundaries, making 6 ricochets.

8. **Answer:** $\mathbf{\frac{400}{21}}$
If M is the midpoint of $\overline{QR}$, then $\overline{PM} \cdot \overline{QR} = 2A$, where A is the area of the triangle. So $\overline{QM} = \frac{A}{5}$ and, by the same logic, $\overline{PQ} = \frac{A}{2}$. Use the Pythagorean Theorem on triangle $\triangle PQM$ to get $A = \frac{50}{\sqrt{21}} \Rightarrow \overline{QR} = \frac{20}{\sqrt{21}}$.

9. **Answer:** $\mathbf{\frac{1}{21}}$
Arbitrarily label the heights of poles A and B as a and b, respectively. Suppose poles A and B are p and q units, respectively, from pole P_1 (as measured along the x-axis). Then the height of P_1, call it x, satisfies: $\frac{x}{a} = \frac{q}{p+q}$ and $\frac{x}{b} = \frac{p}{p+q} \Rightarrow x = \frac{ab}{a+b}$. The same procedure yields the height of P_2: just replace a by $\frac{ab}{a+b}$ in the above equation to get $\frac{ab}{2a+b}$. Generalize by replacing a by $\frac{ab}{na+b}$ to get $\frac{ab}{(n+1)a+b}$ as the height of P_{n+1}. Now put $a = 1, b = 5$ and $n = 100$ to get $\frac{1}{21}$.

10. **Answer:** $\mathbf{\frac{4}{13}}$
Let O be the intersection of $\overline{AQ}$ and $\overline{BR}$. Our goal is to find the area of $\triangle ABO = 1 \cdot \frac{BQ}{BC} \cdot \frac{AO}{AQ} = \frac{1}{4} \cdot \frac{AO}{AQ}$. Using mass points, place a mass of 1 at B and therefore a mass of $\frac{1}{3}$ at C since $1\overline{BQ} = \frac{1}{3}\overline{QC}$. Likewise, vertex A bears a mass of $\frac{1}{9}$. Replace the masses at B and C with a mass of $1 + \frac{1}{3} = \frac{4}{3}$ at Q. Thus, $\frac{AO}{OQ} = \frac{4/3}{1/9} = 12$. Hence, $\frac{AO}{AQ} = \frac{12}{13}$. Thus, the area of $\triangle ABO = \frac{1}{4} \cdot \frac{12}{13} = \frac{3}{13}$, which is independent of the side lengths of $\triangle ABC$. There are two additional nonoverlapping triangles like $\triangle ABO$ that must also have an area of $\frac{3}{13}$. The area of the central triangle is $1 - 3 \cdot \frac{3}{13} = \frac{4}{13}$.