Geometry Test
2006 Stanford Math Tournament
February 25, 2006

1. Given a cube, determine the ratio of the volume of the octahedron formed by connecting the centers of each face of the cube to the volume of the cube.

2. Given square $ABCD$ of side length 1, with E on $\overline{CD}$ and F in the interior of the square so that $\overline{EF} \perp \overline{DC}$ and $\overline{AF} \cong \overline{BF} \cong \overline{EF}$, find the area of the quadrilateral $ADEF$.

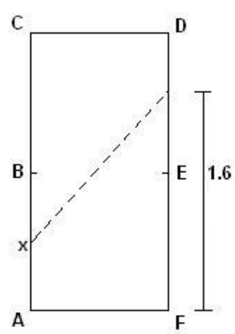
3. Circle γ is centered at $(0, 3)$ with radius 1. Circle δ is externally tangent to circle γ and tangent to the x-axis. Find an equation, solved for y if possible, for the locus of possible centers (x, y) of circle δ.

4. The distance AB is l. Find the area of the locus of points X such that $15^o \leq \angle AXB \leq 30^o$ and X is on the same side of line AB as a given point C.

5. Let S denote a set of points (x, y, z). We define the *shadow* of S to be the set of points (x, y) for which there exists a real number z such that (x, y, z) is in S. For example, the shadow of a sphere with radius r centered on the z axis is a circle in the xy plane centered at the origin with radius r. Suppose a cube has a shadow consisting of a regular hexagon witih area $147\sqrt{3}$. What is the side length of the cube?

6. A circle of radius R is placed tangent to two perpendicular lines. Another circle is placed tangent to the same two lines and the first circle. In terms of R, what is the radius of a third circle that is tangent to one line and tangent to both other circles?

7. A certain 2' by 1' pool table has pockets, denoted $[A, \dots, F]$ as shown. A pool player strikes a ball at point x, $\frac{1}{4}$ of the way up side $\overline{AC}$, aiming for a point 1.6' up the opposite side of the table. He makes his mark, and the ball ricochets around the edges of the table until it finally lands in one of the pockets. How many times does it ricochet before it falls into a pocket, and which pocket? Write your answer in the form $\{C, 2006\}$.



8. In triangle $\triangle PQR$, the altitudes from P, Q and R measure 5, 4 and 4, respectively. Find $\overline{QR}^2$.

9. Poles A, B, and $P_1, P_2, P_3, \dots$ are vertical line segments with bases on the x-axis. The tops of poles A and B are (0,1) and (200,5), respectively. A string S connects (0,1) and (200,0) and intersects another string connecting (0,0) and (200,5) at point T. Pole P_1 is constructed with T as its top point. For each integer $i > 1$, pole P_i is constructed so that its top point is the intersection of S and the line segment connecting the base of P_{i-1} (on the x-axis) and the top of pole B. Find the height of pole P_{100}.

10. In triangle $\triangle ABC$, points P, Q and R lie on sides $\overline{AB}, \overline{BC}$ and $\overline{AC}$, respectively, so that $\frac{AP}{PB} = \frac{BQ}{QC} = \frac{CR}{RA} = \frac{1}{3}$. If the area of $\triangle ABC$ is 1, determine the area of the triangle formed by the points of intersection of lines $\overline{AQ}, \overline{BR}$ and $\overline{CP}$.