General Solutions
2006 Stanford Math Tournament
February 25, 2006

1. **Answer: 4**
   He walks $\frac{1}{3}$ of the way or half as far as he rides, but it takes him twice as long. Therefore, he rides four times as fast as he walks.

2. **Answer: .80 *or* 4/5**
   Let B = event that the customer buys bread, M = the event that the customer buys milk. Then, according to the rule of addition, we have
   $P(B \cup M) = P(B) + P(M) - P(B \cap M) = .60 + .50 - .30 = .80$

3. **Answer: 0**
   If nine letters are in the correct envelopes, the tenth must be also, so the probability is zero.

4. **Answer: 1535**
   511 players must be defeated thrice, the winner might be beaten twice.
   $511 \cdot 3 + 2 = 1535$

5. **Answer: 20**
   The ratio between consecutive terms is $6!/5! = 6$. So the fourth term is 1/6 of the fifth term. We get $\frac{1}{6}(5!) = 20$.

6. **Answer: 15**
   In $3\frac{1}{2}$ hours the alarm clock has become 14 minutes slow. When the alarm clock shows noon, it will fall behind approximately an additonal minute. Its hands will show noon in 15 minutes.

7. **Answer: .999999**
   Let P(S) = the probability of a successful flight.
   Let P(S') = the probability of an unsucessful flight.
   Let P(F) = the probability of an engine failure.
   Since the flight is unsuccessful only when all three engines fail, then the probability of unsuccessful flight is
   $P(S') = P(F \cap F \cap F) = (.01)(.01)(.01) = (.01)^3$
   But $P(S) = 1 - P(S') = 1 - (.01)^3 = 1 - .000001 = .999999$.

8. **Answer: $\mathbf{1, 1, 1, 2, 5}$**
   $1+1+1+2+5 = 1 \cdot 1 \cdot 1 \cdot 2 \cdot 5, 1+1+1+3+3 = 1 \cdot 1 \cdot 1 \cdot 3 \cdot 3, 1+1+2+2+2 = 1 \cdot 1 \cdot 2 \cdot 2 \cdot 2$

9. **Answer: $\mathbf{\frac{1}{5}; \frac{1}{7}}$** .
   Any fraction with 1 for numerator and any odd number (2n-1) for the denominator increases to n times its value when its denominator is added to its numerator and to its denominator. All other answers simplify to one of these fractions.

10. **Answer: 2006**
    sum of the first odd positive integer: $1 = 1 = 1^1$
    sum of the first 2 odd positive integers: $1 + 3 = 4 = 2^2$
    sum of the first 3 odd positive integers: $1 + 3 + 5 = 9 = 3^2$
    sum of the first 4 odd positive integers: $1 + 3 + 5 + 7 = 16 = 4^2$
    sum of the first 5 odd positive integers: $1 + 3 + 5 + 7 + 9 = 25 = 5^2$
    sum of the first 6 odd positive integers: $1 + 3 + 5 + 7 + 9 + 11 = 36 = 6^2$ and so on...
    therefore $\sqrt{2006^2} = 2006$

11. **Answer: 0.26**
Let A denote the event that the policyholder has an accident within a year.
Let B denote the event that the policyholder is accident prone.
$P(A) = P(A|B)P(B) + P(A|B^C)P(B^C) = 0.4 \cdot 0.3 + 0.2 \cdot 0.7 = 0.26$

12. **Answer: 31**
Write 8091 as $8100 - 9 = 90^2 - 3^2 = (90-3)(90+3) = (87)(93) = 3^2 \cdot 29 \cdot 31$. So 31 is the answer.

13. **Answer: $\mathbf{123 - 45 - 67 + 89}$**

14. **Answer: $\boldsymbol{\frac{\pi^3}{2}}$**
By symmetry, the area is simply half the area of the circle.

15. **Answer: 62**
The first digit of 15,951 could not change in 5 hours. Therefore, 1 is the first and last digit of the new number. The second and fourth digits changed to 6. If the middle digit is 0, 1, 2,..., then the car traveled 110, 210, 310, ..., miles in 5 hours. Therefore the car was traveling 62 miles per hour.

16. **Answer: 37** The key is to note that the minor arcs between A_n and A_{n+1} are all congruent with measure 110^o. Then the solution is the smallest n for which $110(n-1)^o$ is a multiple of 360^o, which is clearly 37.

17. **Answer: 4**
The fly will travel for a tenth of an hour at 40 miles per hour, so it will travel a total of 4 miles.

18. **Answer: $\mathbf{\frac{508}{625}}$**
The probability that Brian makes more free throws than Alex at any point is the probability that Brian makes the first and Alex misses the first plus the probability that they made the same number of free throws after each taking one and then Brian makes the second and Alex misses the second.
So $(\frac{3}{5})(\frac{1}{5}) + (\frac{3}{5})(\frac{1}{5})[(\frac{3}{5})(\frac{4}{5}) + (\frac{2}{5})(\frac{1}{5})] = \frac{3}{25} + (\frac{3}{25})(\frac{14}{25}) = \frac{117}{625}$
So the probability that Alex has made at least as many free throws as Brian after each one taken is $1 - \frac{117}{625} = \frac{508}{625}$.

19. **Answer: $\mathbf{27,000,001}$**
The numbers can be grouped by pairs:
999,999 and 0; 999,998 and 1; 999,997 and 2; and so on.
There are half a million pairs, and the sum of the digits in each pair is 54. The digits in the unpaired number 1,000,000 add to 1 then
$(500,000 \cdot 54) + 1 = 27,000,001$

20. **Answer: 8**

$$a_1 = 0; a_2 = \frac{1}{4}$$
$$a_n = a_{n-1} + \frac{1}{4} \Rightarrow a_n = \frac{1}{4}(n-1)$$
$$\frac{1}{4}(33-1) = \frac{1}{4}(32) = 8.$$

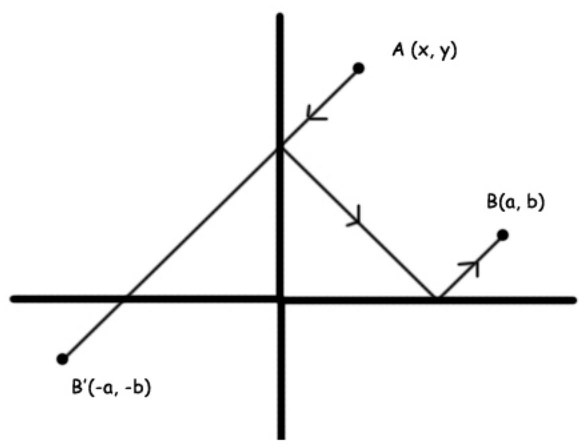
21. **Answer: 1442**

First, factor 1001 into $7 \cdot 11 \cdot 13$. A number relatively prime to 1001 must not be divisible by 7, 11 nor 13. There are 2004 integers less than 2005 and $\lfloor \frac{2004}{7} \rfloor = 286$ of them are multiples of 7. $\lfloor \frac{2004}{11} \rfloor = 182$ of them are multiples of 11 and $\lfloor \frac{2004}{13} \rfloor = 154$ of them are multiples of 13. We've double counted integers divisible by $7 \cdot 11$, $11 \cdot 13$ and $7 \cdot 13$. We must add back in $\lfloor \frac{2004}{77} \rfloor = 26$, $\lfloor \frac{2004}{91} \rfloor = 22$, and $\lfloor \frac{2004}{143} \rfloor = 14$. However, we counted multiples of 7, 11 and 13 three times in the first computation and then we took them out 3 times in the last computation. We need to add them back. There are $\lfloor \frac{2004}{7 \cdot 11 \cdot 13} \rfloor = 2$ of them. Thus, the number of integers not relatively prime to 1001 is $286 + 182 + 154 - 26 - 22 - 14 + 2 = 562$. Thus $2004 - 562 = 1442$ is our answer.

22. **Answer:** $\frac{\mathbf{95}}{\mathbf{159}}$
Let M = the event that he works in the math library.
Let C = the event that he works in the chemistry library.
Let F = he finishes his problem set
$P(M) = 60\%$
$P(C) = 40\%$
$P(F|M) = .95 \cdot .75 = .7125$
$P(F|C) = .90 \cdot .80 = .72$

$$P(M|F) = \frac{P(F|M)P(M)}{P(F|M)P(M) + P(F|C)P(C)}$$
$$= \frac{(0.7125)(0.6)}{(0.7125)(0.6) + (0.72)(0.4)} = \frac{95}{159}$$

23. **Answer:** $\boldsymbol{\sqrt{(x+a)^2 + (y+b)^2}}$



If you consider the mirror image of that light ray's path for both mirrors you get a second path of length equal to the actual path, which travels from (x, y) to $(-a, -b)$ (as seen in the second picture). So the total length the light ray travels is the distance from (x, y) to $(-a, -b)$. This would be $\sqrt{(x+a)^2 + (y+b)^2}$.

24. **Answer: 37, 101, 9901**
$555555555555 = 555 \cdot 1001001001$
$= 555 \cdot 1001 \cdot 1000001$
$= (5 \cdot 3 \cdot 37)(7 \cdot 11 \cdot 13)(100^3 + 1)$
$= (3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 37)(100 + 1)(100^2 - 100 + 1)$
$= 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 37 \cdot 101 \cdot 9901$

25. **Answer:** $\frac{\mathbf{1620}}{\mathbf{1003}}$
If $\Sigma(n)$ is the sum of the positive factors of n then $S(n) = \Sigma(n)/n$.
Since we have the prime factorization $2006 = 2 \cdot 17 \cdot 59$, the sum of its factors is $(1+2)(1+17)(1+59) = 3 \cdot 18 \cdot 60$. Hence $S(2006) = \frac{3240}{2006} = \frac{1620}{1003}$ in lowest terms.