CALCULUS SOLUTIONS
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1. **Answer:** $-\frac{1}{3}$

$$\begin{aligned}\lim_{x\to 0}\frac{\frac{d}{dx}\frac{\sin x}{x}}{x} &= \lim_{x\to 0}\frac{x\cos x-\sin x}{x^3}\\ &= \lim_{x\to 0}\frac{\cos x - x\sin x-\cos x}{3x^2}\\ &= \lim_{x\to 0}-\frac{\sin x}{3x}=\lim_{x\to 0}-\frac{\cos x}{3}=-\frac{1}{3}\end{aligned}$$

2. **Answer:** $\frac{1}{4}, -1$
You substitute the solution of $y=e^{\lambda t}$ into the differential equation. You then get $4\lambda^2e^{\lambda t}+3\lambda e^{\lambda t}-e^{\lambda t}=0$ where you can divide through by $e^{\lambda t}$ and end up with $4\lambda^2+3\lambda-\lambda=0$. You get $(4\lambda-1)(\lambda+1)=0$ where λ must be equal to $\frac{1}{4}$, -1.

3. **Answer:** $\frac{97\pi^2}{200}$
The volume is

$$\begin{aligned}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\pi\left(\sin^2 x+\frac{1}{10}\right)dx &= \pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\sin^4 x+\frac{1}{5}\sin^2 x+\frac{1}{100}\right)dx\\ &= \frac{\pi^2}{100}+\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\sin^2 x(1-\cos^2 x)+\frac{1}{5}\sin^2 x\right)dx\\ &= \frac{\pi^2}{100}+\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\frac{6}{5}\sin^2 x-\frac{1}{4}\sin^2 2x\right)dx\\ &= \frac{\pi^2}{100}+\pi\left(\frac{6}{5}\frac{\pi}{2}-\frac{1}{4}\frac{\pi}{2}\right)=\frac{97\pi^2}{200}\end{aligned}$$

4. **Answer:** $\frac{1}{e}$
Rearrange to get $\lim_{x\to 0}\frac{\ln(x+1)^{\frac{1}{x}}}{(1+x)^{\frac{1}{x}}-e}$. Let $y=\ln(x+1)^{\frac{1}{x}}$. Using L'Hopital's rule, $\lim_{x\to 0}\frac{\ln(y)}{y-e}=\lim_{x\to 0}\frac{\frac{y'}{y}}{y'}=\lim_{x\to 0}\frac{1}{y}$. To evaluate the limit in the denominator,

$$\lim_{x\to 0}(1+x)^{\frac{1}{x}}=\lim_{x\to 0}e^{\ln\left((1+x)^{\frac{1}{x}}\right)}=e^{\lim_{x\to 0}\frac{\ln(1+x)}{x}}=e^{\lim_{x\to 0}\frac{\frac{1}{1+x}}{1}}=e$$

so the answer is $\frac{1}{e}$

5. **Answer:** $-\frac{x}{2}+\frac{1+x^2}{2}\tan^{-1}x$

$$\begin{aligned}\int(x\tan^{-1}x)dx &= -\frac{x}{2}+\frac{1}{2}\int\left(1+2x\tan^{-1}x\right)dx\\ &= -\frac{x}{2}+\frac{1}{2}\int\left(\frac{1+x^2}{1+x^2}+2x\tan^{-1}x\right)dx\\ &= -\frac{x}{2}+\frac{1}{2}\int\left((1+x^2)\frac{d}{dx}(\tan^{-1}x)+\frac{d}{dx}(1+x^2)\tan^{-1}x\right)dx\\ &= -\frac{x}{2}+\frac{1+x^2}{2}\tan^{-1}x\end{aligned}$$

Integration by parts can also lead to the solution.

6. **Answer: $\frac{\pi}{4}$**
Let $u = \frac{\pi}{2} - x$. Substituting then changing u to x gives

$$\int_0^{\pi/2} \frac{\sin^3 x}{\sin^3 x + \cos^3 x} dx = \int_0^{\pi/2} \frac{\cos^3 x}{\cos^3 x + \sin^3 x} dx$$

Adding the two integrals

$$2 \cdot \int_0^{\pi/2} \frac{\sin^3 x}{\sin^3 x + \cos^3 x} dx = \int_0^{\pi/2} dx$$

so

$$\int_0^{\pi/2} \frac{\sin^3 x}{\sin^3 x + \cos^3 x} dx = \frac{\pi}{2} \cdot \frac{1}{2} = \frac{\pi}{4}$$

7. **Answer: $2xH_n(x) - H_n'(x)$**

$$\begin{aligned} H_{n+1}(x) &= -e^{x^2} \frac{d}{dx} \left(e^{-x^2} H_n(x)\right) \\ &= -e^{x^2} \left(-2xe^{-x^2} H_n(x) + e^{-x^2} H_n'(x)\right) \\ &= 2xH_n(x) - H_n'(x) \end{aligned}$$

8. **Answer: $18 + 9\pi$**

The shadow of the rope has length 6 by the Pythagorean theorem; we can work the rest of the problem pretending we have a horizontal rope of length 6 from the unicorn to the tower. The unicorn travels a quarter-circle before the rope begins to wrap around the tower; from then on, if the rope has wrapped around an angle θ of the tower, the rope remaining is $6 - 2\theta$ long. Using the Pythagorean theorem, we find that if r is the unicorn's distance from the center of the tower, $r^2 = 2^2 + (6 - 2\theta)^2$. The area swept out is the initial quarter-circle travelled by the unicorn plus the area swept out by the line from the unicorn to the center of the tower minus the area of the tower covered in the angle traversed:

$$\frac{\pi 6^2}{4} + \frac{1}{2} \int_0^{6/2} (2^2 + (6 - 2\theta)^2) d\theta - \pi 2^2 \frac{6/2}{2\pi}$$

9. **Answer: $\sec 2x$**

First we solve for the inverse function; let $x = \tanh^{-1} y$.

$$\begin{aligned}
y = \tanh x &= \frac{e^x - e^{-x}}{e^x + e^{-x}} \\
y &= \frac{e^{2x} - 1}{e^{2x} + 1} \\
\left(e^{2x} + 1\right) y &= e^{2x} - 1 \\
e^{2x}(y - 1) &= y + 1 \\
x &= \frac{1}{2} \ln \frac{y+1}{y-1}
\end{aligned}$$

Now we return to the given expression:

$$\begin{aligned}
\frac{d}{dx} \tanh^{-1} \tan x &= \frac{d}{dx} \frac{1}{2} \ln \frac{\tan x + 1}{\tan x - 1} \\
&= \frac{1}{2} \frac{d}{dx} \left(\ln(\tan x + 1) - \ln(\tan x - 1)\right) \\
&= \frac{1}{2} \left(\frac{\sec^2 x}{\tan x + 1} - \frac{\sec^2 x}{\tan x - 1} \right) \\
&= \frac{1}{2} \sec^2 x \frac{(\tan x - 1) - (\tan x + 1)}{\tan^2 x - 1} \\
&= \frac{\sec^2 x}{1 - \tan^2 x} = \frac{\frac{1}{\cos^2 x}}{1 - \frac{\sin^2 x}{\cos^2 x}} \\
&= \frac{1}{\cos^2 x - \sin^2 x} = \sec 2x
\end{aligned}$$

10. **Answer: $\mathbf{\frac{1}{5}}$**

By symmetry, Bill's path must be the same as Alan's, rotated 90° counter-clockwise and shifted to start at $(1, 0)$. If Alan's position is given by (x, y), Bill's is then given by $(1-y, x)$. The slope of Alan's path at a point (x, y) is then $y' = \frac{x-y}{1-y-x}$. This is undefined at the given point, so noting that the sign of the derivatives in the radius of curvature does not matter, we switch to the function $x(y)$, with $x' = \frac{1-y-x}{x-y} = \frac{1-0.6-0.4}{0.6-0.4} = 0$ at the given point. Differentiating, $x'' = \frac{(-1-x')(x-y)-(x'-1)(1-y-x)}{(x-y)^2} = \frac{(-1)(.6-.4)-(-1)(1-.6-.4)}{(.6-.4)^2} = -5$. The radius of curvature is then

$$R = \frac{\left(1 + \left(\frac{dx}{dy}\right)^2\right)^{3/2}}{\left|\frac{d^2x}{dy^2}\right|} = \frac{(1+0)^{3/2}}{|-5|} = \frac{1}{5}$$

The answer can also found using the parametric form of the radius curvature with $(x'(t), y'(t))$