Calculus Test
2006 Stanford Math Tournament
February 25, 2006

1. Evaluate:

$$\lim_{x \to 0} \frac{\frac{d}{dx}\frac{\sin x}{x}}{x}$$

2. Given the equation $4y'' + 3y' - y = 0$ and its solution $y = e^{\lambda t}$, what are the values of λ?

3. Find the volume of an hourglass constructed by revolving the graph of $y = \sin^2(x) + \frac{1}{10}$ from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$ about the x-axis.

4. Evaluate

$$\lim_{x \to 0} \frac{\ln(x+1)}{x \cdot ((1+x)^{\frac{1}{2}} - e)}$$

5. Evaluate: $\int (x \tan^{-1} x) dx$

6. Evaluate

$$\int_0^{\pi/2} \frac{\sin^3 x}{\sin^3 x + \cos^3 x} dx$$

.

7. Find $H_{n+1}(x)$ in terms of $H_n(x), H_n'(x), H_n''(x), \ldots$ for

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} \left(e^{-x^2}\right)$$

8. A unicorn is tied to a cylindrical wizard's magic tower with an elven rope stretching from the unicorn to the top of the tower. The tower has radius 2 and height 8; the rope is of length 10. The unicorn begins as far away from the center of the tower as possible. The unicorn is startled and begins to run as close to counterclockwise as possible; as it does so the rope winds around the tower. Find the area swept out by the shadow of the rope, assuming the sun is directly overhead. Also, you may assume that the unicorn is a point on the ground, and that the elven rope is so light it makes a straight line from the unicorn to the tower.

9. Define the function $\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$ Let $\tanh^{-1}$ denote the inverse function of tanh. Evaluate and simplify:

$$\frac{d}{dx} \tanh^{-1} \tan x$$

10. Four ants Alan, Bill, Carl, and Diane begin at the points $(0,0)$, $(1,0)$, $(1,1)$, and $(0,1)$, respectively. Beginning at the same time they begin to walk at constant speed so that Alan is always moving directly toward Bill, Bill toward Carl, Carl toward Diane, and Diane toward Alan. An approximate solution finds that after some time, Alan is at the point $(0.6, 0.4)$. Assuming for the moment that this approximation is correct (it is, to better than 1%) and so the pont lies on Alan's path, what is the radius of curvature at that point. In standard Cartesian coordinates, the radius of curvature of a function $y(x)$ is given by:

$$R = \frac{\left(1 + \left(\frac{dy}{dx}\right)^2\right)^{3/2}}{\left|\frac{d^2y}{dx^2}\right|}$$