Algebra Solutions
2006 Stanford Math Tournament
February 25, 2006

1. **Answer: 668**
   Note that $111 = 3 \cdot 37$. It follows that m_i is divisible by 37 for all $i = 3, 6, 9, \ldots, 2004$. The others will clearly leave remainders of 1 or 11.

2. **Answer: 10**
   The expression can be written as $(x-2)^2 + (x-y)^2 + (y-2z)^2 + (z-1)^2 + 10$. This clearly must be at least 10. Indeed, if $x = 2$, $y = 2$, $z = 1$, this value is achieved.

3. **Answer: $\boldsymbol{(1+2i)(2+3i)}$**
   We write $-4 + 7i = (a + bi)(c + di)$. The solution can be intuitive after the first line of expansion, in the same way as factoring of polynomials. However, we can assume $a = 1$ and then move factors from $(c + di)$ back to $(a + bi)$ if we don't end up with integers (fortunately, in this case we're lucky).

$$\begin{aligned}
-4 + 7i &= (a + bi)(c + di) \\
&= ac - bd + (ad + bc)i \\
&= c - bd + (d + bc)i
\end{aligned}$$

   We then know c should be positive (and not too large), so we can try $c = 1$, giving $1 - bd = -4$ and $b + d$=7, which clearly has no rational solution. We then try $c = 2$, giving $6 = bd$ and $2b + d = 7$, which is easily solved giving the final solution.

4. **Answer: $\boldsymbol{a+b+c}$**

$$\begin{aligned}
\frac{a^3}{(a-b)(a-c)} + \frac{b^3}{(b-a)(b-c)} + \frac{c^3}{(c-a)(c-b)} &= \frac{a^3(c-b) + b^3(a-c) + c^3(b-a)}{(a-b)(b-c)(c-a)} \\
&= \frac{a^3(c-b) + a(b^3-c^3) + bc^3 - cb^3}{(a-b)(b-c)(c-a)} \\
&= \frac{(c-b)(a^3 - a(b^2+bc+c^2) + b^2c + c^2b}{(a-b)(b-c)(c-a)} \\
&= -\frac{b^2(c-a) + b(c^2-ac) + a^3 - ac^2}{(a-b)(c-a)} \\
&= -\frac{(c-a)(b^2+bc-ac-a^2)}{(a-b)(c-a)} \\
&= \frac{(a+b)(a-b) + c(a-b)}{a-b} \\
&= a + b + c
\end{aligned}$$

5. **Answer: 352**

   Let N represent the number of remaining pebbles after Kramer eats the second. Then N is divisible by 10, and $N + 1$, which must end in 1, is divisible by 9. Put $N + 1 = 100a + 10b + 1$, where a and b are digits summing to 8 or 17 (so the sum of the digits will be divisible by 9 - hence the number will be divisible by 9). Now we need $N + 2$ to be divisible by 8. Try 82, 172, 262, and 352 to get 352 as the answer.

6. **Answer: 31, −25**
From the first equation:

$$\begin{aligned} ab - a &= b + 119 \\ a(b-1) &= (b-1) + 120 \\ (a-1)(b-1) &= 120 \end{aligned}$$

Similarly, $(b-1)(c-1) = 60$ and $(a-1)(c-1) = 72$. Therefore $\frac{a-1}{c-1} = 2$, and so $2(c-1)^2 = 72$. This gives $c = 7$, and then it is easy to find $a = 13$ and $b = 11$. The other solution is $c = -5$, so $a = -11$, and $b = -9$. The sums are 31 and -25.

7. **Answer: $\boldsymbol{a = 6, b = 5}$**
Since $11|aabb$, $aabb = 11 \cdot a0b$. Factor $n^4 - 6n^3 = (n-6)n^3$, so clearly $n > 6$, as $aabb > 0$. Also, $a0b < 1000$, so unless $n = 11$, $n < 10$. Trying $n = 7, 8, 9$ yields no solutions, so $n = 11$ must be the only solution, if it exists. Indeed we get $6655 = (11-6) \cdot 11^3$.

8. **Answer: $\boldsymbol{\frac{27}{55}}$**
$\frac{2}{x(x^2-1)} = \frac{1}{x}\left(\frac{1}{x-1} - \frac{1}{x+1}\right) = \frac{1}{x(x-1)} - \frac{1}{x(x+1)}$
Let $f(x) = \frac{1}{x(x-1)}$. Then:
$\sum_{x=2}^{10} \frac{2}{x(x^2-1)} = \sum_{x=2}^{10}(f(x) - f(x+1)) = \sum_{x=2}^{10} f(x) - \sum_{x=3}^{11} f(x) = f(2) - f(11) = \frac{1}{2 \cdot 1} - \frac{1}{11 \cdot 10} = \frac{1}{2} - \frac{1}{110} = \frac{27}{55}$

9. **Answer: 169**
Let A be the value of the expression. We have: $m^2 + n^2 - 13m - 13n - mn + A = 0$. Multiplying by 2 yields:

$$\begin{aligned} m^2 - 2mn + n^2 + m^2 - 26m + n2 - 26n + 2A = 0 \\ (m-n)^2 + (m-13)^2 + (n-13)^2 = 2 \cdot 13 \cdot 2 - 2A \end{aligned}$$

In order for there to be a single solution, the sum of the squares must equal zero, yielding $A = 169$. If instead the sum is a positive integer with a solution (m, n), then (n, m) will provide an additional solution unless $m = n$. In that case, $(26-m, 26-n)$ is an additional solution. Hence, it is both sufficient and necessary that the sum of the squares equal zero in order that the solution be unambiguous.

10. **Answer: $\boldsymbol{\left(\frac{a}{a-1}\right)^2}$**

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{k}{n^{k-1}} &= \frac{1}{1} + \frac{2}{k} + \frac{3}{k^2} + \frac{4}{k^3} + \cdots \\ &= \left(\frac{1}{1} + \frac{1}{k} + \frac{1}{k^2} + \frac{1}{k^3} + \cdots\right) + \left(\frac{1}{k} + \frac{1}{k^2} + \frac{1}{k^3} + \cdots\right) + \left(\frac{1}{k^2} + \frac{1}{k^3} + \cdots\right) + \cdots \\ &= \left(\frac{1}{1} + \frac{1}{k} + \frac{1}{k^2} + \frac{1}{k^3} + \cdots\right) + \frac{1}{k}\left(\frac{1}{1} + \frac{1}{k} + \frac{1}{k^2} + \cdots\right) + \frac{1}{k^2}\left(\frac{1}{1} + \frac{1}{k} + \cdots\right) + \cdots \\ &= \left(\frac{1}{1} + \frac{1}{k} + \frac{1}{k^2} + \frac{1}{k^3} + \cdots\right)\left(\frac{1}{1} + \frac{1}{k} + \frac{1}{k^2} + \frac{1}{k^3} + \cdots\right) \\ &= \left(\frac{1}{1 - 1/k}\right)^2 \\ &= \left(\frac{k}{k-1}\right)^2 \end{aligned}$$