Algebra Test
2006 Stanford Math Tournament
February 25, 2006

1. A finite sequence of positive integers m_i for $i = 1, 2, \ldots, 2006$ are defined so that $m_1 = 1$ and $m_i = 10m_{i-1} + 1$ for $i > 1$. How many of these integers are divisible by 37?

2. Find the minimum value of $2x^2 + 2y^2 + 5z^2 - 2xy - 4yz - 4x - 2z + 15$ for real numbers x, y, z.

3. A Gaussian prime is a Gaussian integer $z = a + bi$ (where a and b are integers) with no Gaussian integer factors of smaller absolute value. Factor $-4 + 7i$ into Gaussian primes with positive real parts. i is a symbol with the property that $i^2 = -1$.

4. Simplify: $\frac{a^3}{(a-b)(a-c)} + \frac{b^3}{(b-a)(b-c)} + \frac{c^3}{(c-a)(c-b)}$

5. Jerry is bored one day, so he makes an array of Cocoa pebbles. He makes 8 equal rows with the pebbles remaining in a box. When Kramer drops by and eats one, Jerry yells at him until Kramer realizes he can make 9 equal rows with the remaining pebbles. After Kramer eats another, he finds he can make 10 equal rows with the remaining pebbles. Find the smallest number of pebbles that were in the box in the beginning.

6. Let a, b, c be real numbers satisfying:

$$ab - a = b + 119$$
$$bc - b = c + 59$$
$$ca - c = a + 71$$

   Determine all possible values of $a + b + c$.

7. Find all solutions to $aabb = n^4 - 6n^3$, where a and b are non-zero digits, and n is an integer (a and b are not necessarily distinct).

8. Evaluate:

$$\sum_{x=2}^{10} \frac{2}{x(x^2 - 1)}$$

   .

9. Principal Skinner is thinking of two integers m and n and bets Superintendent Chalmers that he will not be able to determine these integers with a single piece of information. Chalmers asks Skinner the numerical value of $mn + 13m + 13n - m^2 - n^2$. From the value of this expression alone, he miraculously determines both m and n. What is the value of the above expression?

10. Evaluate: $\sum\limits_{k=1}^{\infty} \frac{k}{a^{k-1}}$ for $|a| < 1$.