Advanced Topics Solutions
2006 Stanford Math Tournament
February 25, 2006

1. **Answer:** $\pm\frac{\sqrt{2}}{2}(1+i)$

   For an answer in the form $z = a + bi$ note that $z^2 = a^2 - b^2 + 2abi$. The real part is zero, so $a = b$. $2ab = 2a^2 = 1$ so $a = b = \pm\frac{\sqrt{2}}{2}$. Thus $z = \pm\frac{\sqrt{2}}{2}(1+i)$. One can use polar coordinates and De Moivre's theorem to arrive at the same result.

2. **Answer:** $\begin{pmatrix} \mathbf{0} \\ \mathbf{\frac{1}{11}} \end{pmatrix}$

   $A^2 = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = 3I$ Thus $A^4 = 9I^2 = 9I$. We can see $A^6 = 27I$ and $A^8 = 81I$. Thus $A^8 + A^6 + A^4 + A^2 + I = 121I = \begin{pmatrix} 121 & 0 \\ 0 & 121 \end{pmatrix}$. Let $v = \begin{pmatrix} a \\ b \end{pmatrix}$ and then $\begin{pmatrix} 121 & 0 \\ 0 & 121 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 121a \\ 121b \end{pmatrix}$. Setting $121a = 0$ and $121b = 11$. This means $a = 0$ and $b = \frac{1}{11}$. Thus $v = \begin{pmatrix} 0 \\ \frac{1}{11} \end{pmatrix}$.

3. **Answer:** $\binom{2007}{11}$

   Imagine a sequence of n numbers, $\{1, 2, 3 \ldots, n+1\}$. A combination of $k+1$ elements may be chosen by first choosing k from the set {1,...,k} and attaching the $(k+1)$th number. Then another combination can be formed by choosing k from the set $\{1, \ldots, k+1\}$ and attaching the $(k+2)$th number. You may continue in this fashion until choosing k from $\{1, \ldots, n\}$. Therefore the summation that we ask for is equal to $\binom{n+1}{k+1} = \binom{2007}{11}$. To check, you may examine a smaller sum such as $\binom{10}{10} + \binom{11}{10} + \binom{12}{10} = \binom{13}{11}$

4. **Answer:** $\frac{1}{25}$

   C=correct problem; W=wrong problem;
   C*=Smartie thinks a problem is correct; W*=Smartie thinks a problem is wrong;
   S=problem from Stanford; R=problem from Rice
   We are given $P(W|W*) = \frac{3}{4}, P(W*|R) = \frac{1}{5}$, and $P(W*|S) = \frac{1}{10}$. We can solve for $P(R) = \frac{1}{3}, P(S) = \frac{2}{3}$, and $P(C) = \frac{\#\text{correct problems}}{\text{total problems}} = \frac{9\cdot 10+16\cdot 10}{10\cdot 10+20\cdot 10} = \frac{5}{6}$.
   We want to find $P(W*|C)$:

$$
\begin{aligned}
P(W*|C) &= \frac{P(C|W*)\cdot P(W*)}{P(C)}, \text{ where} \\
P(W*) &= P(W*|R)\cdot P(R) + P(W*|S)\cdot P(S) \\
&= \frac{1}{5}\cdot\frac{1}{3} + \frac{1}{10}\cdot\frac{2}{3}, \text{ and} \\
P(C|W*) &= 1 - P(W|W*) \\
&= 1 - \frac{3}{4} = \frac{1}{4}
\end{aligned}
$$

   So $P(W*|C) = \frac{\frac{1}{4}\cdot\frac{2}{15}}{\frac{5}{6}} = \frac{1}{25}$.

5. **Answer:** $\frac{2-\sqrt{2}}{4}$

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{1}{k\sqrt{k+2}+(k+2)\sqrt{k}} &= \sum_{k=1}^{\infty} \frac{1}{\sqrt{k(k+2)}} \frac{1}{\sqrt{k}+\sqrt{k+2}} \\ &= \sum_{k=1}^{\infty} \frac{1}{\sqrt{k(k+2)}} \left(\frac{k+2}{2} - \frac{k}{2}\right) \\ &= \frac{1}{2}\sum_{k=1}^{\infty} \left(\frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+2}}\right) \\ &= \frac{1}{2}\left(\frac{1}{1} + \frac{1}{\sqrt{2}}\right) \\ &= \frac{2-\sqrt{2}}{4}\end{aligned}$$

The infinite series in its final form is a telescoping sum.

6. **Answer: 112**
The teams' scores must sum to $1+2+\ldots+50 = \frac{1}{2}\cdot 50\cdot 51 = 1275$. The winning score must be no larger than $\frac{1}{10}\cdot 1275 = 127.5$ and is at least $1+2+3+4+5 = 15$. However, not all scores between 15 and 127 inclusively are possible because all teams must have integer scores and no team can tie the winning team. If the winning score is s, the sum of all teams' scores is at least $s+9(s+1) = 10s+9$, so solving gives $s \leq 126$. Hence, $126-15+1 = 112$ winning scores are possible.

7. **Answer: 1200**
The midpoint of the segment connecting (x,y) and (x',y') is $\left(\frac{x+x'}{2}, \frac{y+y'}{2}\right)$. Therefore a and a' must have the same parity, as must b and b' for the midpoint to be a lattice point. We therefore divide the set into four groups: (even,even), (even,odd), (odd,even), (odd,odd), with the number of points in each group a,b,c,d. The number of such segments is then

$$\begin{aligned}\binom{a}{2}+\binom{b}{2}+\binom{c}{2}+\binom{d}{2} &= \frac{a(a-1)}{2}+\frac{b(b-1)}{2}+\frac{c(c-1)}{2}+\frac{d(d-1)}{2} \\ &= \frac{1}{2}\left(a^2-a+b^2-b+c^2-c+d^2-d\right) \\ &= \frac{1}{2}\left(a^2+b^2+c^2+d^2-100\right)\end{aligned}$$

This is minimized when $a=b=c=d$, giving a value of $\frac{1}{2}(4\cdot 25^2-100) = 1200$.

8. **Answer: 1**
Expanding $k_i(j)$ we have

$$k_i(j) = \frac{(n+1)n!n!(i+j)!(2n-i-j)!}{(2n+1)i!(n-i)!j!(n-j)!(2n)!} = \frac{\binom{i+j}{i}}{\binom{2n-i-j}{n-i}}\binom{2n+1}{n+1}.$$

We claim that

$$\sum_{j=0}^{n}\binom{i+j}{i}\binom{2n-i-j}{n-i} = \binom{2n+1}{n+1}.$$

We show this by bijection. If we pick $n+1$ items from among $2n+1$ we must choose the $i+1$st element at position $i+1$, $i+2$, ... , or $2n+1-(n-i)$. For each such choice, we can pick the first i objects from among the first $i+j$ and the last $n-i$ from among the last $2n-i-j$, $0\leq j\leq n$. Thus

$$\sum_{j=0}^{n}\binom{i+j}{i}\binom{2n-i-j}{n-i} = \binom{2n+1}{n+1}.$$

9. **Answer: 88**
Let $f(n) = \frac{2006}{n}$. For sufficiently small n, $\lfloor f(n) \rfloor$ takes a different value. Consequently, for all sufficiently small m, there exists at least one value of n for which $\lfloor f(n) \rfloor = m$. Note that if a and b are positive real numbers for which $a = \lfloor a \rfloor + a'$ and $b = \lfloor b \rfloor + b'$, then $\lfloor a \rfloor - \lfloor b \rfloor = a - b + (b' - a')$. Note also that $|b' - a'| < 1$. Hence, if $f(n) - f(n+1) > 1$, then $\lfloor f(n) \rfloor > \lfloor f(n+1) \rfloor$. Also, if $f(n) - f(n+1) < 1$, then $\lfloor f(n) \rfloor - \lfloor f(n+1) \rfloor < 2$ (i.e. equals 0 or 1). The equation $\frac{2006}{x} - \frac{2006}{x+1} = 1$ implies $x^2 + x - 2006 = 0$, or $x = \frac{1}{2}(5\sqrt{321} - 1) < \frac{1}{2}(5(18) - 1) = 44.5$. Note also that $x > \frac{1}{2}(5(17) - 1) = 42$. So $42 < x < 45$, implying that if $n \geq 45$, $f(n) - f(n+1) < 1$ and that if $n \leq 42$, $f(n) - f(n+1) > 1$. Evaluating $\lfloor f(n) \rfloor$ for $n = 42, 43, 44$, and 45, we see that each are unique. We conclude that the first 44 terms are unique integers. The rest of the terms take on the values $1, 2, \ldots, \lfloor \frac{2006}{45} \rfloor$, or 44 additional terms.

10. **Answer: $\boldsymbol{\frac{\pi}{2}}$**
We show by induction that

$$\sum_{n=1}^{m} \arctan\left(\frac{1}{n^2 - n + 1}\right) = \arctan(m)$$

.
Clearly $\arctan\left(\frac{1}{1-1+1}\right) = \arctan 1$.
If $\sum_{n=1}^{m} \arctan\left(\frac{1}{n^2-n+1}\right) = \arctan(m)$, then

$$\begin{aligned}
\tan\left(\sum_{n=1}^{m+1} \arctan\left(\frac{1}{n^2 - n + 1}\right)\right) &= \tan\left(\arctan(m) + \arctan\left(\frac{1}{(m+1)^2 - (m+1) + 1}\right)\right) \\
&= \frac{m + \frac{1}{m^2+m+1}}{1 - m\frac{1}{m^2+m+1}} \\
&= \frac{m(m^2 + m + 1) + 1}{m^2 + m + 1 - m} \\
&= \frac{m^3 + m^2 + m + 1}{m^2 + 1} \\
&= \frac{(m+1)(m^2 + 1)}{m^2 + 1} \\
\tan\left(\sum_{n=1}^{m+1} \arctan\left(\frac{1}{n^2 - n + 1}\right)\right) &= m + 1
\end{aligned}$$

$$\sum_{n=1}^{m+1} \arctan\left(\frac{1}{n^2 - n + 1}\right) = \arctan(m+1)$$

Thus as $m \to \infty$ the sum goes to $\arctan(+\infty) = \frac{\pi}{2}$.