Team Test Solutions
2004 Stanford Mathematics Tournament
February 28, 2004

1. **Answer:** $\frac{-1}{\sqrt{2003}}$

   Since $(\sin x - \cos x)^2 = \sin^2 x + \cos^2 x - 2\sin x \cos x = 1 - \sin 2x = \frac{1}{2003}$, we are looking for when $(\sin x - \cos x)^2 = \frac{1}{2003}$. For the interval given, $\sin x < \cos x$ thus our answer is $\frac{-1}{\sqrt{2003}}$.

2. **Answer:** $\frac{30}{61}$

   There is a $\frac{5}{36}$ chance that Dave's roll will be a 6 and a $\frac{1}{6}$ chance that Daly's roll will be a 7. So we can write the probability that Dave wins as: $\frac{5}{36} + \frac{31}{36}\frac{5}{6}\frac{5}{36} + \frac{31}{36}\frac{5}{6}\frac{31}{36}\frac{5}{6}\frac{5}{36} + ...$ So, this is a geometric series whose sum is $\frac{\frac{5}{36}}{1-\frac{5}{6}\frac{31}{36}} = \frac{30}{61}$.

3. **Answer:** $\frac{2}{3}$

   Notice that the figure is a hexagon in union with six congruent triangles. It's easy to see that triangles ACE and BDF are equilateral, so each of these six congruent triangles has a $60°$ angle opposite the side it shares with the smaller hexagon. The other two sides are obviously equivalent, so the triangle is equilateral. "Folding" these six triangles across their common side with the hexagon shows that their area of their union is the same as that of the hexagon. If V is the intersection of AC and BF, then AVB is $120°$ and $BV = AV = JV$ where J is the intersection of AE and BF. This proves that FJ, JV and VB are all equivalent, so JV is $\frac{1}{3}$ the length of BF. If s is the side length of $ABCDEF$, then it's easy to see that BF is $s\sqrt{3}$, so the smaller hexagon has length $\frac{s}{\sqrt{3}}$. The ratio of the area of the smaller hexagon to that of the larger is thus $\frac{1}{3}$, and the answer to the question is $\frac{2}{3}$.

4. **Answer: 772**

   Suppose n is a two-digit integer and that s is its square. Writing n as $(10a + b)$ where a and b are digits, we need the rightmost two digits of $(20ab + b^2)$ and $(30ab^2 + b^2)$ to be the same. We have four cases: $b = 0, 1, 5, 6$. The first case gives all two-digit multiples of ten. The second case yields nothing. The third case yields $a \in \{2, 4, 6, 8\}$. The final case gives $a \in \{2, 7\}$. The answer is $450 + 25 + 45 + 65 + 85 + 26 + 76 = 772$.

5. **Answer:** $p(x) = x^3 - x^2 - 2x - 3$

   The coefficients of $p(x) = x^3 - lx^2 + mx - n$ are $l = a + b + c$ $m = ab + ac + bc$ $n = abc$ $l = 1$ is given. $l^2 = (a+b+c)^2 = a^2+b^2+c^2+2ab+2ac+2bc = a^2+b^2+c^2+2m$. Since $l^2 = 1$ and $a^2+b^2+c^2 = 5$, $m = -2$. $l^3 = (a+b+c)^3 = a^3+b^3+c^3+3a^2b+3ab^2+3a^2c+3ac^2+3b^2c+3bc^2+6abc$. Note that $lm = a^2b+ab^2+a^2c+ac^2+b^2c+bc^2+3abc$, so $l^3 = a^3+b^c+c^3+3lm-3n$. We know the values of everything except n, so we can solve to show $n = 3$.

6. **Answer:** $\frac{80}{143}$

   The students can be grouped into the cars in a total of $\binom{13}{4}\binom{9}{4}\binom{5}{5} = 13 \cdot 11 \cdot 7 \cdot 5 \cdot 3^2 \cdot 2$ ways. We must arrange the Bobs so that two are seated in one car and the other two are in the other cars. If the car holding the two Bobs is the 5-seater, then there are $\binom{9}{3}\binom{4}{2}\binom{6}{3}\binom{2}{1}\binom{3}{3}\binom{1}{1} = 7 \cdot 5 \cdot 3^2 \cdot 2^6$ ways. This is the number of ways of choosing the non-Bob people in the 5-seater times the number of ways of choosing the two Bobs times the number of ways of choosing non-Bob people in one of the four seaters times the number of ways of choosing Bob times the number of ways of choosing non-Bob people in the other four seater times the number of ways of choosing the last Bob. If the two Bobs double up in a 4-seater, there are $\binom{9}{2}\binom{4}{2}\binom{7}{3}\binom{2}{1}\binom{4}{4}\binom{1}{1} = 7 \cdot 5 \cdot 3^3 \cdot 2^4$ possible groupings. There are two different 4-seaters. Adding all these up yields $7 \cdot 5^2 \cdot 3^2 \cdot 2^5$. Thus the probability is $\frac{7 \cdot 5^2 \cdot 3^2 \cdot 2^5}{13 \cdot 11 \cdot 7 \cdot 5 \cdot 3^2 \cdot 2} = \frac{80}{143}$.

7. **Answer:** $\frac{4}{5}$

Use the identity for geometric series: $\frac{1}{1-a} = 1+a+a^2+...$ Notice that $(a+a^2+a^3+...)(1+a+a^2+...) = a+2a^2+3a^3+...$ Thus $(\frac{1}{1-a}-1)(\frac{1}{1-a}) = \sum_{k=1}^{\infty} kx^k = 20$. Set $y = \frac{1}{1-a}$ and we want to solve $(y-1)(y) = 20$ The two solutions are 5 and -4. Then $a = \frac{4}{5}$ or $\frac{5}{4}$. The second solution gives a series which diverges, hence the only answer is $\frac{4}{5}$.

8. **Answer: 0**

   Fix P a set distance from BC. Draw the line through P which is parallel to BC. It intersects AB and BC at B' and C' respectively. The perpendiculars from P to AB' and AC' are PX and PY respectively. Now $PB = \frac{\sqrt{3}}{2}(PX)$ and $PC' = \frac{\sqrt{3}}{2}(PY)$, so $PX+PY = \frac{2}{\sqrt{3}}(PB'+PC')$ which is constant. So, $d(P)$ does not change when we slide the point in a direction parallel to a side. We can use this tranlstion twice to get to any other point, so we know that $d(P)$ is a constant.

9. **Answer: 60**

   Writing out what happens in the round one will show us the pattern. Owls in positions 2, 8, 11, 12, 14, 18, 20, 24 and 30 stay in the same spot each round. The owls in sports 16 and 19 just trade spots. Thus every even numbered round they are back in the right places. The four owls in spots 21, 26, 28 and 23 just rotate, in that order. Thus every fourth round they are back in the original order. Lastly, the remaining 15 owls form one huge rotation. The order is 1, 6, 9, 7, 4, 29, 27, 25, 22, 17, 15, 13, 10, 5, 3, and 1 again. Thus every 15 rounds these owls return to their original order. One can see each of these three groups that rotate will be back in their original order at the gcd of 2, 4 and 15. The answer is 60.

10. **Answer: $\frac{1}{2}$**

    Recall that (1) $\lfloor -x \rfloor = -\lceil x \rceil$ (2) If x is not an integer $\lceil x \rceil = \lfloor x \rfloor + 1$. (3) $\lfloor x+n \rfloor = \lfloor x \rfloor + n$ for integer n. Let $F(x) = \lfloor 2x^3 - 2\lfloor x^3 \rfloor \rfloor$. Then as long as x^3 is not an integer or half-integer we have:

    $F(-x) = \lfloor -2x^3 - 2\lfloor -x^3 \rfloor \rfloor = \lfloor -2x^3 + 2\lceil x^3 \rceil \rfloor = \lfloor -2x^3 + 2\lfloor x^3 \rfloor + 2 \rfloor = 2 + \lfloor -2x^3 + 2\lfloor x^3 \rfloor \rfloor = 2 - \lceil 2x^3 - 2\lfloor x^3 \rfloor \rceil = 1 - \lfloor 2x^3 - 2\lfloor x^3 \rfloor \rfloor = 1 - F(x)$.

    Thus $F(x) + F(-x) = 1$ except at isolated points. Hence the average is $\frac{1}{2}$.

11. **Answer: $\frac{110592}{676039}$**

    The people lose $\frac{1}{2}$ of the grains on the first step. Then $\frac{1}{3}$ of the remaining numbers are multiples of 3, and so on. So we just need to compute $(1-\frac{1}{2})(1-\frac{1}{3})(1-\frac{1}{5})...(1-\frac{1}{23}) = \frac{2^{12}*3^3}{7*13*17*19*23} = \frac{110592}{676039}$.

12. **Answer: 6621**

    The equation implies that $x + \frac{1}{x} = 3$. The expression to be solved can be factored as $(x^8 + x^{-8})(x + \frac{1}{x})$. The second term is 3 while the first one can be found by squaring $x + \frac{1}{x}$, solving for $x^2 + x^{-2}$ and repeating the procedure twice. The first term turns out to be 2207. The answer is 6621.

13. **Answer: 6**

    Because -5 is the second coefficient, the sum of the roots is 5. 5 can be written as the sum of 4 nonnegative integers 6 distinct ways: (0,0,0,5), (0,0,1,4), (0,0,2,3), (0,1,1,3), (0,1,2,2), (1,1,1,2) Since 5 is the sum of the roots, and the roots must be integers by the rational roots theorem, these are the only roots the polynomial can have. But since the polynomial is monic, fixing the roots fixes the polynomial.

14. **Answer: 2003** Notice that this can be written with partial fractions as $\frac{A}{7} + \frac{B}{5^{2003}}$. $\frac{A}{7}$ is a repeating decimal while $\frac{B}{5^{2003}}$ is a string of 2003 digits which do not repeat. So the length of the leading non-repeating block is 2003.

15. **Answer: $(1-(\frac{5}{6})^k)^5 - (1-(\frac{5}{6})^{k-1})^5$**

    Put all 5 dice in independent cups. Roll all of them k times. Then the probability that one of them is 6 somewhere in k rolls is $1-(\frac{5}{6})^k$. Then the probability that all 5 of them have a 6 somewhere in k rolls is $(1-(\frac{5}{6})^k)^5$. To get the probability of getting all sixes in exactly k rolls, we subtract the probability that all 5 dice have a 6 in $k-1$ rolls. This gives $(1-(\frac{5}{6})^k)^5 - (1-(\frac{5}{6})^{k-1})^5$.