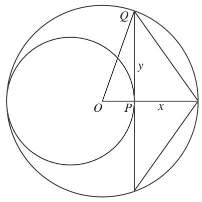
Geometry Solutions
2004 Stanford Mathematics Tournament
February 28, 2004

1. **Answer: $4\sqrt{2}$**

   Consider the following diagram, where O is the center of the larger circle: Since the smaller circle has

   radius 2, we must have $x = 2$ and $OP = 1$. And OQ is a radius of the larger circle, so $OQ = 3$. Applying the Pythagorean theorem to triangle OPQ, we find that

   $$1^2 + y^2 = 3^2.$$

   Solving this for y gives us $y = 2\sqrt{2}$.

   The shaded region is a triangle with base $2y$ and height x. Therefore its area is

   $$\frac{1}{2}(x)(2y) = xy = 4\sqrt{2}.$$

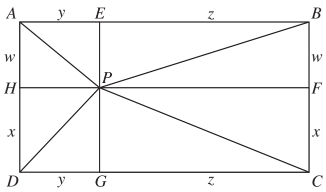
2. **Answer: 8**

   There are 3 possible locations for the 4th vertex. Let (a, b) be its coordinates. If it is opposite to vertex $(1, 2)$, then since the midpoints of the diagonals of a parallelogram coincide, we get $(\frac{a+1}{2}, \frac{b+2}{2}) = (\frac{3+4}{2}, \frac{8+1}{2})$. Thus $(a, b) = (6, 7)$. By similar reasoning for the other possible choices of opposite vertex, the other possible positions for the fourth vertex are $(0, 9)$ and $(2, -5)$, and all of these choices do give parallelograms. So the answer is $6 + 0 + 2 = 8$.

3. **Answer: 4**

   Let $AC = d$, $BC = c$, $AD = a$, $BD = b$ so $\frac{a}{d} = \frac{b}{c}$. Now $\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c} = \frac{a}{c} + \frac{a}{d} = a(\frac{1}{c} + \frac{1}{d})$, $\frac{a+b}{d} = \frac{a}{d} + \frac{b}{d} = \frac{b}{c} + \frac{b}{d} = b(\frac{1}{c} + \frac{1}{d})$, $\frac{c+d}{a+b} = \frac{1}{\frac{1}{c}+\frac{1}{d}} \times (\frac{1}{a} + \frac{1}{b}) = \frac{cd}{c+d} \times \frac{a+b}{ab}$. This implies that $(\frac{c+d}{a+b})^2 = \frac{cd}{ab}$. Now $a + b = 2004$, $c + d = 6012 - 2004 = 4008$ so $\frac{cd}{ab} = (\frac{4008}{2004})^2 = 4$.

4. **Answer: $\sqrt{95}$**

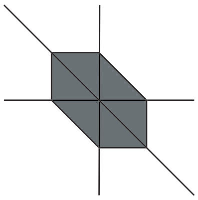
   Draw perpendiculars from P to E on AB, F on BC, G on CD, and H on DA, and let $AH = BF = w$, $HD = FC = x$, $AE = DG = y$, and $EB = GC = z$. Then $PA^2 = w^2 + y^2$, $PB^2 = w^2 + z^2$, $PC^2 = x^2 + z^2$, and $PD^2 = x^2 + y^2$. Adding and subtracting, we see that $PD^2 = PA^2 - PB^2 + PC^2 = 95$, so $PD = \sqrt{95}$.

Sector	Inequality	Simplified inequality
$x \geq 0, y \geq 0, x+y \geq 0$	$x+y+(x+y) \leq 1$	$x+y \leq \frac{1}{2}$
$x \geq 0, y \leq 0, x+y \geq 0$	$x-y+(x+y) \leq 1$	$x \leq \frac{1}{2}$
$x \geq 0, y \leq 0, x+y \leq 0$	$x-y-(x+y) \leq 1$	$y \geq -\frac{1}{2}$
$x \leq 0, y \geq 0, x+y \geq 0$	$-x+y+(x+y) \leq 1$	$y \leq \frac{1}{2}$
$x \leq 0, y \geq 0, x+y \leq 0$	$-x+y-(x+y) \leq 1$	$x \geq -\frac{1}{2}$
$x \leq 0, y \leq 0, x+y \leq 0$	$-x-y-(x+y) \leq 1$	$x+y \geq -\frac{1}{2}$

5. **Answer: $\frac{3}{4}$**

   To graph this region we divide the xy-plane into six sectors depending on which of $x, y, x+y$ are ≥ 0, or ≤ 0. The inequality simplifies in each case:

   We then draw the region; we get a hexagon as shown. The hexagon intersects each region in an isosceles right triangle of area $\frac{1}{8}$, so the total area is $6 \cdot \frac{1}{8} = \frac{3}{4}$.



6. **Answer: $2\pi - 4$**

   The first square has side length of $\sqrt{2}$. The second circle has half of the area of the first one. Thus the shaded region has half of the area of the first one. The second shaded area is half of the size of the first. But the sum of $\frac{1}{2^n}$ from zero to infinity is 2 so the shaded area is $2(\pi - 2) = 2\pi - 4$.

7. **Answer: $3 - 2\sqrt{2}$**

   A homothety (scaling) about P takes triangle ADP into BCP, since AD, BC are parallel and A, P, C; B, P, D are collinear. The ratio of homothety is thus $\sqrt{2}$. It follows that, if we rescale to put $[ADP] = 1$, then $[ABP] = [CDP] = \sqrt{2}$, just by the ratios of lengths of bases. So $[ABCD] = 3 + 2\sqrt{2}$, so $\frac{[ADP]}{[ABCD]} = \frac{1}{3+2\sqrt{2}}$. Simplifying this, we get $3 - 2\sqrt{2}$.

8. **Answer: 3**

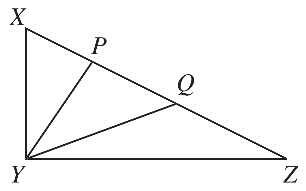
   There are many solutions to this problem, here is one. The given triangle is a right 3-4-5 triangle, so the circumcenter is the midpoint of the hypotenuse. Coordinatizing for convenience, put the vertex at $(0,0)$ and the other vertices at $(0,18)$ and $(24,0)$. Then the circumcenter is $(12,9)$. The centroid is at one-third the sum of the three vertices, which is $(8,6)$. Finally, since the area equals the inradius times half the perimeter, we can see that the inradius is $\frac{\frac{18 \cdot 24}{2}}{\frac{18+24+30}{2}} = 6$. So the incenter of the triangle is $(6,6)$. So the small triangle has a base of length 2 and a height of 3, hence its area is 3.

9. **Answer: $\frac{1}{2}$**

   Imagine placing the tetrahedron $ABCD$ flat on a table with vertex A at the top. By vectors or otherwise, we see that the center is $\frac{3}{4}$ of the way from A to the bottom face, so the reflection of this face lies in a horizontal plane halfway between A and BCD. In particular, it cuts off the smaller tetrahedron obtained by scaling the original tetrahedron by a factor of $\frac{1}{2}$ about A. Similarly, the reflections of the other three faces cut off tetrahedra obtained by scaling $ABCD$ by $\frac{1}{2}$ about B, C, and D. On the other hand, the octahedral piece remaining remaining after we remove these four smaller tetrahedra is in the intersection of $ABCD$ with its reflection, since the reflection sends this piece to itself. So the answer we seek is just the volume of this piece, which is

$$\begin{gathered}\text{(volume of } ABCD) - 4 \cdot \text{(volume of } ABCD \text{ scaled by a factor of } \tfrac{1}{2}) \\ = 1 - 4(\tfrac{1}{2})^3 = \tfrac{1}{2}.\end{gathered}$$

10. **Answer: 1370736**



The triangle's area is $\frac{228 \cdot 2004}{2} = 228456$. All the angles at Y are 30 degrees, so by the sine area formula, the areas of the three small triangles in the diagram are $\frac{QY \cdot YZ}{4}$, $\frac{PY \cdot QY}{4}$, and $\frac{XY \cdot PY}{4}$, which sum to the area of the triangle. So expanding $(PY + YZ)(QY + XY)$, we see that it equals

$$4 \cdot 228456 + XY \cdot YZ = 6 \cdot 228456 = 1370736.$$