Stanford Mathematics Tournament

February 28, 2004

Individual Round: General Test — Problems

1. Add *this* to *that*, divide by three,
   The square of *this* of course you'll see,
   If *that* to *this* is eight to one,
   Find *that* and you'll be done.

2. In the Championship World Series of the future, Rice and Stanford play three games or until one team wins two games. In each game, both teams have a 1/3 chance of winning, and there is a 1/3 chance that they will tie. What is the probability that Rice wins the championship?

3. Find all natural numbers n that satisfy the equation:
$$(n-1)+(n-1)=n!$$

4. Find the ordered 4-tuple of positive integers (a,b,c,d) so that $37/13=a+1/[b+1/(c+1/d)]$

5. Each vertex of a given square is the center of one of four circles. The circles are all congruent and each one is tangent to two others. What is the probability that a randomly chosen point in the figure is inside both a circle and the square?

6. There are 1000 rooms in a row along a long corridor. Initially the first room contains 1000 people and the remaining rooms are empty. Each minute, the following happens: for each room containing more than one person, someone in that room decides it is too crowded and moves to the next room. All these movements are simultaneous (so nobody moves more than once within a minute). After one hour, how many different rooms will have people in them?

7. What is the largest whole number that is equal to the product of its digits?

8. Suppose f is a function that assigns to each real number x a value $f(x)$, and suppose the equation
$$f(x_1+x_2+x_3+x_4+x_5)=f(x_1)+f(x_2)+f(x_3)+f(x_4)+f(x_5)-8$$
holds for all real numbers x_1,x_2,x_3,x_4,x_5. What is $f(0)$?

9. How many ways can you mark 8 squares of an 8×8 chessboard so that no two marked squares are in the same row or column, and none of the four corner squares is marked? (Rotations and reflections are considered different.)

10. A rectangle has perimeter 10 and diagonal $\sqrt{15}$. What is its area?

11. Find the ordered quadruple of digits (A,B,C,D), with $A>B>C>D$, such that
$$\begin{array}{rl} & ABCD \\ - & \underline{DCBA} \\ = & BDAC. \end{array}$$

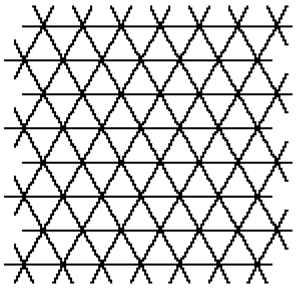
12. Let ACE be a triangle with a point B on segment AC and a point D on segment CE such that BD is parallel to AE. A point Y is chosen on segment AE, and segment CY is drawn. Let X be the intersection of CY and BD. If $CX = 5$, $XY = 3$, what is the ratio of the area of trapezoid $ABDE$ to the area of triangle BCD?

13. You have a 10×10 grid of squares. You write a number in each square as follows: you write $1, 2, 3, \ldots, 10$ from left to right across the top row, then $11, 12, \ldots, 20$ across the second row, and so on, ending with a 100 in the bottom right square. You then write a second number in each square, writing $1, 2, \ldots, 10$ in the first column (from top to bottom), then $11, 12, \ldots, 20$ in the second column, and so forth.

    When this process is finished, how many squares will have the property that their two numbers sum to 101?

14. Urn A contains 4 white balls and 2 red balls. Urn B contains 3 red balls and 3 black balls. An urn is randomly selected, and then a ball inside of that urn is removed. We then repeat the process of selecting an urn and drawing out a ball, without returning the first ball. What is the probability that the first ball drawn was red, given that the second ball drawn was black?

15. A floor is tiled with equilateral triangles of side length 1, as shown. If you drop a needle of length 2 somewhere on the floor, what is the largest number of triangles it could end up intersecting? (Only count the triangles whose interiors are met by the needle — touching along edges or at corners doesn't qualify.)



16. Find the largest number n such that $(2004!)!$ is divisible by $((n!)!)!$.

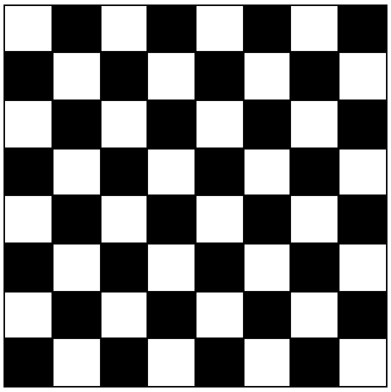
17. Andrea flips a fair coin repeatedly, continuing until she either flips two heads in a row (the sequence HH) or flips tails followed by heads (the sequence TH). What is the probability that she will stop after flipping HH?

18. How many ordered pairs of integers (a, b) satisfy all of the following inequalities?

$$\begin{aligned} a^2 + b^2 &< 16 \\ a^2 + b^2 &< 8a \\ a^2 + b^2 &< 8b \end{aligned}$$

19. A horse stands at the corner of a chessboard, a white square. With each jump, the horse can move either two squares horizontally and one vertically or two vertically and one horizontally

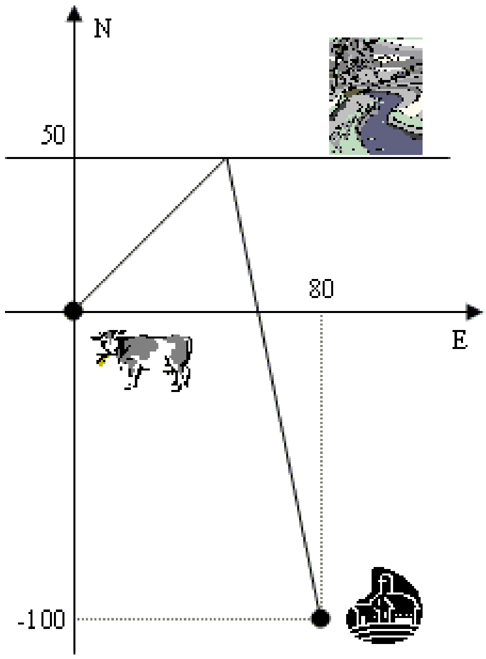
(like a knight moves). The horse earns two carrots every time it lands on a black square, but it must pay a carrot in rent to rabbit who owns the chessboard for every move it makes. When the horse reaches the square on which it began, it can leave. What is the maximum number of carrots the horse can earn without touching any square more than twice?



20. Eight strangers are preparing to play bridge. How many ways can they be grouped into two bridge games — that is, into unordered pairs of unordered pairs of people?

21. a and b are positive integers. When written in binary, a has 2004 1's, and b has 2005 1's (not necessarily consecutive). What is the smallest number of 1's $a+b$ could possibly have?

22. Farmer John is grazing his cows at the origin. There is a river that runs east to west 50 feet north of the origin. The barn is 100 feet to the south and 80 feet to the east of the origin. Farmer John leads his cows to the river to take a swim, then the cows leave the river from the same place they entered and Farmer John leads them to the barn. He does this using the shortest path possible, and the total distance he travels is d feet. Find the value of d.



23. A freight train leaves the town of Jenkinsville at 1:00 PM traveling due east at constant speed. Jim, a hobo, sneaks onto the train and falls asleep. At the same time, Julie leaves Jenkinsville on her bicycle, traveling along a straight road in a northeasterly direction (but not due northeast) at 10 miles per hour. At 1:12 PM, Jim rolls over in his sleep and falls from the train onto the side of the tracks. He wakes up and immediately begins walking at 3.5 miles per hour directly towards the road on which Julie is riding. Jim reaches the road at 2:12 PM, just as Julie is riding by. What is the speed of the train in miles per hour?

24. Given is a regular tetrahedron of volume 1. We obtain a second regular tetrahedron by reflecting the given one through its center. What is the volume of their intersection?

25. A *lattice point* is a point whose coordinates are both integers. Suppose Johann walks in a line from the point $(0, 2004)$ to a random lattice point in the interior (not on the boundary) of the square with vertices $(0, 0)$, $(0, 99)$, $(99, 99)$, $(99, 0)$. What is the probability that his path, including the endpoints, contains an even number of lattice points?