Calculus Solutions
2004 Stanford Mathematics Tournament
February 28, 2004

1. **Answer: $\frac{7}{4}$**

   $\lim_{x\to\infty}(\sqrt{4x^2+7x}-2x) = \lim_{x\to\infty}(\sqrt{4x^2+7x}-2x)\cdot\frac{(\sqrt{4x^2+7x}+2x)}{(\sqrt{4x^2+7x}+2x)} = \lim_{x\to\infty}\frac{7x}{(\sqrt{4x^2+7x}+2x)}\cdot\frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to\infty}\frac{7}{(\sqrt{4+\frac{7}{x}}+2)} = \frac{7}{4}$.

2. **Answer: 19**

   The derivative of $f(x)-f(2x)$ is $f'(x)-2f'(2x)$. So $f'(1)-2f'(2)=5$, $f'(2)-2f'(4)=7$. Thus

   $$f'(1)-4f'(4) = (f'(1)-2f'(2)) + 2(f'(2)-2f'(4)) = 5+2\cdot 7 = 19,$$

   the answer.

3. **Answer: [3,4] or (3,4) or from t=3 to t=4**

   The velocity of the object is given by $v(t)=x'(t)=20t^3-5t^4$, and the acceleration function is $a(t)=v'(t)=60t^2-20t^3$. The object is slowing down when the velocity is positive and the acceleration is negative, or vice versa. $v(t)$ is positive from $t=0$ to $t=4$ and is negative after that. $a(t)$ is positive from $t=0$ to $t=3$ and negative afterward. These only differ in sign from $t=3$ to $t=4$.

4. **Answer: 1**

   Let $g(x)=\log f(x)=x\log x$. Then $\frac{f'(x)}{f(x)}=g'(x)=1+\log x$. Therefore $f(x)=f'(x)$ when $1+\log x=1$, that is, when $x=1$.

5. **Answer: $5-\sqrt{3}$ miles**

   Let x be the amount of old road restored. Then the length of the new road is $\sqrt{9+(5-x)^2}$ using the Pythagorean Theorem. Thus the total cost of the plan is $C(x)=200000x+400000\sqrt{x^2-10x+34}$. The minimum cost occurs at one of the critical points which are $x=5\pm\sqrt{3}$. Clearly $5+\sqrt{3}$ is not a valid answer and one can check $5-\sqrt{3}$ is indeed a minimum.

6. **Answer: 2**

   The two graphs intersect at $x^2-2x^2+8x^2=28$ or rather $x=\pm 2$ with $y=\pm 4$. At $x=+2$, $m_1=2$ and $m_2=y'(2)$. Using implicit differentiation on the second graph, we find $y'(x)=\frac{y-2x}{4y-x}$ and plugging in $(2,4)$ gives a slope of 0. If α is the angle between the graphs then $|\tan(\alpha)|=|\frac{m_2-m_1}{1+m_1m_2}|$. Plugging in the values yields the answer 2. $x=-2$ yields the same value.

7. **Answer: $\pi/6$**

   The mouse can wait some amount of time while the table rotates and then spend the remainder of the time moving along that ray at 1 m/s. He can reach any point between the starting point and the furthest reachable point along the ray, $(1-\theta/\pi)$ meters out. So the area is

   $$\int_0^\pi (1/2)(1-\theta/\pi)^2\,d\theta = (1/2)(1/\pi)^2\int_0^\pi \theta^2\,d\theta = \pi/6.$$

8. **Answer: 27500 foot-pounds**

   Let x indicate the distance the cow has yet to travel. Then the work for a distance dx is $(2x+200-\frac{1}{2}(100-x))dx$. Thus the total work is $\int_0^{100}(\frac{5}{2}x+150)dx = 27500$ foot-pounds.

9. **Answer: $\frac{3\sqrt{3}}{2}$**

   The base region is bounded on the left by $x=y^2$ and on the right by $2y^2=3-x$. The intersection points are $(1,1)$ and $(1,-1)$. Each cross-section, say $x=a$, is an equilateral triangle. The length of a side is $2y$ where $y = \sqrt{x}$ for $a \le 1$ but it is $y = \sqrt{\frac{3-x}{2}}$ for $a \ge 1$. The area of an equilateral triangle is $\frac{\sqrt{3}s^2}{4}$ where s is the side length. Thus the volume is $\int_0^1 \frac{\sqrt{3}(2\sqrt{x})^2}{4} dx + \int_1^3 \frac{\sqrt{3}}{4}(2\sqrt{\frac{3-x}{2}})^2 dx = \sqrt{3}\int_0^1 x dx + \frac{\sqrt{3}}{2}\int_1^3 (3-x)dx = \frac{3\sqrt{3}}{2}$.

10. **Answer:** $\boldsymbol{\frac{1}{e}}$

The ratio test tells us that the series converges if

$$\lim_{n\to\infty} \frac{\frac{(n+1)!}{(c(n+1))^{n+1}}}{\frac{n!}{(cn)^n}} = \frac{1}{c} \cdot \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n$$

is less than one and diverges if it is greater than one. But

$$\lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^{-n} = \frac{1}{e}.$$

Then the limit above is just $\frac{1}{ce}$, so the series converges for $c > \frac{1}{e}$ and diverges for $0 < c < \frac{1}{e}$.