Algebra Test
2004 Stanford Mathematics Tournament
February 28, 2004

1. How many ordered pairs of integers (a,b) satisfy all of the following inequalities?

$$\begin{aligned} a^2+b^2 &< 16 \\ a^2+b^2 &< 8a \\ a^2+b^2 &< 8b \end{aligned}$$

2. Find the largest number n such that (2004!)! is divisible by $((n!)!)!$.

3. Compute:

$$\lfloor \frac{2005^3}{2003 \cdot 2004} - \frac{2003^3}{2004 \cdot 2005} \rfloor.$$

Where $\lfloor x \rfloor$ denotes the greatest integer less than x.

4. Given that:

$$x > 0,\ y > 0,\ z > 0,$$

$$xy - 3x - 7y + 15 = 0,$$

$$xz - 2x - 7z + 8 = 0,$$

$$yz - 2y - 3z + 2 = 0.$$

Solve for x, y, and z.

5. There exists a positive real number x such that $\cos(\tan^{-1}(x)) = x$. Find the value of x^2.

6. Dan, Shravan, and Jake attempt to clean their room for the first time in six months. It would take Dan 3 hours to clean it by himself. It would take Shravan 10 hours and Jake 15 hours to clean the room. How long does it take the three of them together to clean the room?

7. Find all real x such that

$$\frac{x(x-2)(x-4)}{(x-1)(x-3)(x-5)} < 0.$$

8. Let x be a real number such that $x^3 + 4x = 8$. Determine the value of $x^7 + 64x^2$.

9. A sequence of positive integers is defined by $a_0 = 1$ and $a_{n+1} = a_n^2 + 1$ for each $n \geq 0$. Find $\gcd(a_{999}, a_{2004})$.

10. There exists a polynomial P of degree 5 with the following property: if z is a complex number such that $z^5 + 2004z = 1$, then $P(z^2) = 0$. Calculate the quotient $\frac{P(1)}{P(-1)}$.