Advanced Topics Solutions
2004 Stanford Mathematics Tournament
February 28, 2004

1. **Answer:** $\begin{pmatrix} \mathbf{1} & \mathbf{2500} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}$

   First note that $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & (a+b) \\ 0 & 1 \end{pmatrix}$. This implies the given product is

   $$\begin{pmatrix} 1 & (1+3+5+...+99) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 50^2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2500 \\ 0 & 1 \end{pmatrix}.$$

2. **Answer:** $\mathbf{.010\overline{13}_6}$.

   $\frac{1}{9} - \frac{1}{10} + \frac{1}{7} - \frac{1}{8} = \frac{73}{9\cdot 8\cdot 7\cdot 5}$. Note this is all base 10. To convert to a base 6 decimal, we get each digit by multiplying the fraction by 6 and taking the integer portion. This is the reverse of converting integers to base 6. Note that the decimal begins to repeat after the first five places. The answer is $.010\overline{13}_6$.

3. **Answer:** $\mathbf{sin(80°)}$

   The expression is equivalent to $\cos(10°)+\cos(10°)+\sin(100°)+\sin(280°+2\cdot 360°)+\sin(280°+27\cdot 360°) = \cos(10°) + \cos(10°) + \sin(100°) + \sin(280°) + \sin(280°) = \cos(10°) + \cos(10°) + \cos(10°) - \cos(10°) - \cos(10°) = \cos(10°) = \sin(80°)$.

4. **Answer:** $\mathbf{3\sqrt{2}}$

   Using the law of cosines in the right triangle, we find $9^2 = 6^2 + 5^2 - 2(5)(6)\cos\theta$, where θ is the angle the triangles have in common. Thus $\cos\theta = -\frac{1}{3}$. Since $\sin^2\theta + \cos^2\theta = 1$, $\sin\theta = \frac{\sqrt{8}}{3}$. Using the law of sines in the left triangle yields $\frac{x}{\sin 30°} = \frac{8}{\sin\theta}$. Thus, $x = 3\sqrt{2}$.

5. **Answer: 1680**

   There are 8! orders for RICEOWLS and each of the 4! orders of WISE are equally likely within them, so $\frac{8!}{4!} = 8 * 7 * 6 * 5 = 1680$ have the correct order.

6. **Answer:** $\mathbf{2F_{2005}}$.

   $F_0 = 1 = F_1$. Then $F_1 + F_2 = F_3$. Then $F_3 + F_4 = F_5$. And so on until we get $F_{2006} + F_{2003}$. This equals $F_{2005} + F_{2004} + F_{2003} = 2F_{2005}$.

7. **Answer:** $\mathbf{\frac{17}{4}}$

   Let X be the number of days until Chris is king if Adam is king and Y be the number of days until Chris is king if Bill is king. Then $X = 1 + \frac{X}{3} + \frac{Y}{3} + \frac{0}{3}$ and $Y = 1 + \frac{X}{2} + \frac{Y}{4} + \frac{0}{4}$. Solving for X we get $\frac{13}{4}$ but we want expected day, not "days until" so we add 1 day to get $\frac{17}{4}$.

8. **Answer: 0**

   $(i+1)^4$ is in the direction of -1. Also, $(i-1)^4$ is in the direction of -1 and they have the same magnitude. Call $(i+1)^4 = n$. Then this is $n^{501} - n^{501} = 0$.

9. **Answer: 30**

   It's best to rewrite it as $\cos x = \frac{x^2}{2004}$ and first to consider only positive values. Clearly, $x < \sqrt{2004}$ in agreement with the range of $\cos x$. We'll definitely have 2 solutions for every interval $[2\pi*(n-1), 2\pi*(n)]$ for $n = 1, 2, \cdot, m$ for some m. It's not hard to see that m is the largest integer that does not exceed $\frac{\sqrt{2004}}{2\pi}$. Since $44^2 < 2004 < 45^2$, $44 < x < 45$. The expression is hence between $\frac{22}{\pi}$ and $\frac{22.5}{\pi}$. Note that $\frac{22}{7} > 3.142 > \pi$, so $\frac{22}{\pi} > 7$ and $m = 7$ (since $\frac{22.5}{\pi} < 8$). But at the end of the seventh inteval, $x = 14\pi$, and $\frac{x^2}{2004} = \frac{49\pi^2}{501} < \frac{\pi^2}{10} < \frac{3.15^2}{10} < 1 = \cos 14\pi$. Hence there must be at least one more solution. There cannot be more than one in $[14\pi, 15\pi]$ since $\cos x$ decreases and $\frac{x^2}{2004}$ increases. Note also that $\frac{(15\pi)^2}{2004} = \frac{225\pi^2}{2004} > \frac{225\pi^2}{2025} = \frac{\pi^2}{9} > 1$, so $x \geq 15\pi$ yields no solutions. There are a total of $2(7) + 1 = 15$ solutions for positive x and hence 30 overall.

10. **Answer:** $\mathbf{\frac{433}{833}}$

Instead of thinking about people picking cards, we will place the aces in the deck. So we assign a number between 1 and 52 to each of the aces. There are $\binom{52}{4}$ ways to do this. Now we examine what happens if the first ace is in an odd numbered slot. If the first ace is number 1, we have $\binom{51}{3}$ possibilities for the other 3 aces. Similarly, for slot 3 we have $\binom{49}{3}$ and so forth. So the probability can be written as $\frac{\binom{51}{3}+\binom{49}{3}+\binom{47}{3}+\ldots+\binom{3}{3}}{\binom{52}{4}}$. We can put the numerator into summation form as $\sum_{n=1}^{\infty} \frac{1}{6}(2n+1)(2n)(2n-1)$. This is $\frac{2*25^2*26^2-25*26}{6} = \frac{844350}{6} = 140725$. Then $\frac{140725}{\binom{52}{4}} = \frac{433}{833}$