Team Test Solutions
Stanford Math Tournament
February 23, 2002

1. Evaluate $\sqrt{i}$ in the form $a + bi$ with $a > 0$, where a and b are real numbers and i is $\sqrt{-1}$.

   Answer: $\boldsymbol{\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i}$

   Solution: Assuming $(a + bi)^2 = (a^2 - b^2) + 2abi = i$, we must have $a^2 - b^2 = 0$ and $2ab = 1$. For the first condition to hold, we must have $b = \pm a$. If $b = -a$, then we get $a^2 = -\frac{1}{2}$, but this implies a is imaginary, contrary to assumption. So $b = a$, and then the second condition implies that $a = \frac{\sqrt{2}}{2} = b$. This gives us $\sqrt{i} = \boldsymbol{\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i}$.

2. Let A be the matrix
$$\begin{bmatrix} 3 & 2 & 1 \\ 2 & 0 & 4 \\ 0 & 6 & 3 \end{bmatrix}$$
   What is $\det(A^{-1})$?

   Answer: $\boldsymbol{-1/72}$

   Solution: First, since $AA^{-1} = I$, we observe that $\det(AA^{-1}) = 1$. Then, the determinant of a product is the product of determinants, so $\det(AA^{-1}) = \det A \det A^{-1}$, which implies that $\det A^{-1} = 1/\det A$. And $\det A = -72$, so $\det A^{-1} = \boldsymbol{-1/72}$.

3. How many positive integers divide the number of positive integers that divide 2002^{2002}?

   Answer: 5

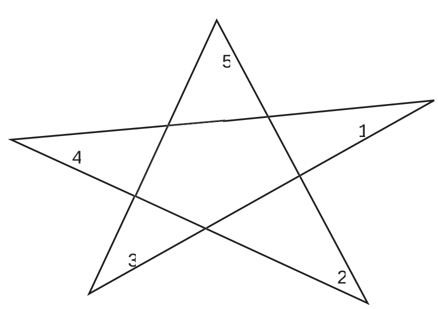
   Solution: The prime factorization of 2002^{2002} is $2^{2002} \cdot 7^{2002} \cdot 11^{2002} \cdot 13^{2002}$, so clearly the number of positive integers that divide 2002^{2002} is 2003^4. Noting that 2003 is prime, we immediately get that the number of positive integers that divide 2003^4 is **5**.

4. In base b, how many $(2n + 1)$-digit numbers are palindromes?

   Answer: $\boldsymbol{(b-1)b^n}$

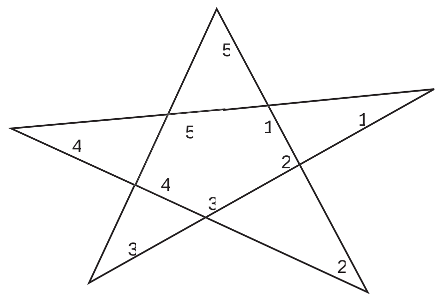
   Solution: If we are constructing a $(2n + 1)$-digit palindrome, we have $(b - 1)$ choices for the first digit (everything except 0), and b choices for each of the next n digits. The remaining n digits are completely determined by these choices, as they must be the first n digits in reverse order. Thus, there are $\boldsymbol{(b-1)b^n}$ ways to construct such a number.

5. In the diagram below, what is $\sum_{n=1}^{5} \theta_n$?



   Answer: $\boldsymbol{\pi}$

   Solution: Label the interior angles of the inner pentagon α_1 through α_5 as pictured here:



Considering the triangle with angles θ_1, θ_4, and α_3, we obtain the relationship $\theta_1 + \theta_4 + \alpha_3$. Similarly, the triangle with angles θ_1, θ_3, and α_5, gives us $\theta_1 + \theta_3 + \alpha_5 = \pi$. Altogether, there are five triangles of this sort, and each one gives us relationship of the same form. Each θ_i appears in exactly two of these triangles, and each α_j appears in exactly one, so if we sum all five equations, we get

$$2\sum_{i=1}^{5}\theta_j + \sum_{j=1}^{5}\alpha_j = 5\pi.$$

Since the α_j are the five interior angles of a pentagon, their sum is 3π. Therefore,

$$\begin{aligned} 2\sum_{i=1}^{5}\theta_j + 3\pi\alpha_j &= 5\pi \\ \sum_{i=1}^{5}\theta_j &= \boldsymbol{\pi}. \end{aligned}$$

6. Let x be the smallest number such that x written out in English without using the word "and" (i.e. 1,942,647 is one million nine hundred forty two thousand six hundred forty seven) has exactly 300 letters. What is the most common digit (0-9) in x?

   Answer: 3

   In general, a number will take the form "...XXX billion XXX million XXX thousand XXX", where each XXX is a three-digit number written out in English.

   After a bit of work, we find out that a three-digit number XXX can have at most 24 letters. This gives us an easy way to get an upper bound on the number of digits of x. It turns out that if we start a number with "one octillion" and fill in each successive XXX (there are nine of them) with a 24-letter number, we get exactly 300 letters. From this, we see that every XXX must have *exactly* 24 letters; if one of them didn't, we would have to compensate for it prior to the word "octillion", which would invariably give us a larger number than the one we have already found. Hence, all we have to do now is find the smallest XXX with 24 letters. It turns out that this is 373, so every XXX in x is 373, and **3** is the most common digit.

7. Define $g(x) = \int_x^{x+1} 2^t\,dt$, and $g'(x) = \frac{d}{dx}g(x)$. Compute $g'(10)$.

   Answer: 1024

   Solution: From the Fundamental Theorem of Calculus, $g'(x) = 2^{x+1} - 2^x = 2^x$ and so $g'(10) = 2^{10} =$ **1024**.

8. If $xy = 24$ with x and y real, what is the minimum value that $x^2 + 4y^2$ can attain?

   Answer: 96

   Solution: Since the quadratic mean of any two numbers is greater than their geometric mean, we apply this to x and $2y$ to determine that $\sqrt{\frac{x^2+4y^2}{2}} \geq \sqrt{2xy}$. (An easy way of deriving this is to use the fact that $(x-2y)^2 \geq 0$ and expand it out) So, squaring both sides (since they are both clearly positive), we determine that $\frac{x^2+4y^2}{2} \geq 2xy$, or $x^2+4y^2 \geq 4xy$. Since $4xy = 4 \cdot 24 = 96$, we see that **96** is the minimum (this is attained with $x = 4\sqrt{3}, y = 2\sqrt{3}$.

9. Find the cubic polynomial $f(x)$ such that $f(1) = 1$, $f(2) = 5$, $f(3) = 14$, and $f(4) = 30$.

   Answer: $\boldsymbol{x(x+1)(2x+1)/6}$

   Solution: Let $f(x) = ax^3 + bx^2 + cx + d$. Then, using the four given pieces of information, we end up with the following system of equations:

$$\begin{array}{rcrcrcrcr} a & + & b & + & c & + & d & = & 1 \\ 8a & + & 4b & + & 2c & + & d & = & 5 \\ 27a & + & 9b & + & 3c & + & d & = & 14 \\ 64a & + & 16b & + & 4c & + & d & = & 30 \end{array}$$

   The rest is just algebra, and we get $a = 1/3$, $b = 1/2$, $c = 1/6$, and $d = 0$.

10. What is $1^2 + 2^2 + 3^2 + \cdots + 2002^2$?

    Answer: 2676679005

    The last problem probably seemed like a lot of hassle for one stupid cubic equation. But if you were paying attention to the values we were given, you might have noticed that for a positive integer n, $f(n) = \sum_{k=1}^{n} k^2$; once we have the formula, it's not very hard to prove this fact by induction. So the answer to this question is $f(2002) = \mathbf{2676679005}$.

11. The rth power mean P_r of n numbers $x_1, \ldots, x_n$ is defined as

$$P_r(x_1, \ldots, x_n) = \left(\frac{x_1^r + \cdots + x_n^r}{n}\right)^{1/r}.$$

    for $r \neq 0$, and $P_0 = (x_1 x_2 \cdots x_n)^{1/n}$. The Power Mean Inequality says that if $r > s$, then $P_r \geq P_s$. Using this fact, find out how many ordered pairs of positive integers (x, y) satisfy $48\sqrt{xy} - x^2 - y^2 \geq 289$.

    Answer: 0

    Solution: First, we do some calculations. If x and y satisfy the given inequality, then

$$\begin{aligned} 48\sqrt{xy} - x^2 - y^2 &\geq 289 \\ 48\sqrt{xy} &\geq 289 + x^2 + y^2 \\ 12\sqrt{xy} &\geq \frac{289 + x^2 + y^2)}{4} \\ \sqrt{144xy} &\geq \frac{289 + x^2 + y^2}{4} \\ (144xy)^{1/4} &\geq \sqrt{\frac{289 + x^2 + y^2}{4}}. \end{aligned}$$

    But by the Power Mean Inequality,

$$\begin{aligned} (12 \cdot 12 \cdot x \cdot y)^{1/4} &\leq \sqrt{\frac{12^2 + 12^2 + x^2 + y^2}{4}} \\ (144xy)^{1/4} &\leq \sqrt{\frac{288 + x^2 + y^2}{4}}. \end{aligned}$$

    So there are no solutions.

12. After meeting him in the afterlife, Gauss challenges Fermat to a boxing match. Each mathematician is wearing glasses, and Gauss has a 1/3 probability of knocking off Fermat's glasses during the match, whereas Fermat has a 1/5 chance of knocking off Gauss's glasses. Each mathematician has a 1/2 chance of losing without his glasses and a 1/5 chance of losing anyway with his. Note that it is possible for both Fermat and Gauss to lose (simultaneous knockout) or for neither to lose (the match is a draw). Given that Gauss wins the match (and Fermat loses), what is the probability that Gauss has lost his glasses?

    Answer: 5/37

    Solution: First, let's ignore the fact that Gauss wins. In any given match, the chance that Gauss does not lose his glasses, then does not lose the match, is $\frac{4}{5} \cdot \frac{4}{5} = \frac{16}{25}$. Meanwhile, the chance that Gauss loses his glasses, then does not lose the match, is $\frac{1}{5} \cdot \frac{1}{2} = \frac{1}{10}$. Hence, the probability that Gauss does not lose is $\frac{16}{25} + \frac{1}{10} = \frac{37}{50}$. Given that Gauss does not lose, the chance that he has lost his glasses is the probability that Gauss loses his glasses but does not lose the match, divided by the probability that he does not lose the match. This is

$$\frac{\frac{1}{10}}{\frac{37}{50}} = \frac{5}{37}.$$

    Notice that we didn't even need to know that Fermat loses.

13. Evaluate

$$\frac{1}{-1 + \frac{1}{-1+\frac{1}{-1+\cdots}}}.$$

    Answer: $\mathbf{\frac{1-\sqrt{5}}{2}}$

    Solution: Setting x equal to the continued fraction we wish to evaluate, we see that $x = 1/(-1+x)$. This yields the equation $x^2 - x - 1 = 0$, and solving this tells us that $x = (1 \pm \sqrt{5})/2$. To determine which of these is the solution we want, notice that the continued fraction can be written as

$$\frac{1}{-1 + \frac{1}{-1+\frac{1}{-1+\cdots}}} = -1 \cdot \frac{1}{1 + \frac{1}{1+\frac{1}{1+\cdots}}},$$

    therefore it is negative. Hence, $x < 1$, so we must have $x = \mathbf{(1 - \sqrt{5})/2}$.

14. What is the smallest positive integer x such that $x^2 + x + 41$ is not prime?

    Answer: 40

    Solution: Using modular arithmetic, one can easily realize that the value of $x^2 + x + 41$ cannot ever be divisible by 3, 5, 7, 11, 13, 17, nor 19. For example, if 3 divides x then, 3 cannot divide $x^2 + x + 41$. If x divided by 3 has remainder 1, then x^2 mod 3 has remainder 1 and 3 does not divide 43. Likewise, if $x/3$ has remainder 2 then x^2 mod 3 has remainder 1 as well and 3 does not divide 44. We know that $x = 40$ gives the value 41^2, a composite. One can quickly check now the remaining values from $x = 17$ to 39 are all prime.

15. Let $A(t)$ be an $n \times n$ square matrix whose entries are all functions of t, and suppose that $\det A(t) \neq 0$ for all t. Then $\frac{dA}{dt} = A'$ is simply the matrix formed by differentiating each entry of $A(t)$ with respect to t. Write $\frac{d}{dt}\left(A^{-1}(t)\right)$ in terms of $A(t)$ and A', where the only differentiation occurs in A' itself.

    Answer: $\mathbf{(A^{-1})' = -A^{-1}A'A^{-1}}$

    Solution: After checking that the product rule does indeed work for matrices, notice that $AA^{-1} = I$, where I is the identity matrix, so

$$\begin{aligned} A'A^{-1} + A(A^{-1})' &= 0 \\ (A^{-1})' &= \mathbf{-A^{-1}A'A^{-1}}. \end{aligned}$$