Team Test
Stanford Math Tournament
February 23, 2002

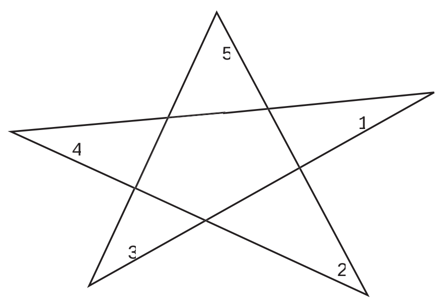
1. Evaluate $\sqrt{i}$ in the form $a + bi$ with $a > 0$, where a and b are real numbers and i is $\sqrt{-1}$.

2. Let A be the matrix
$$\begin{bmatrix} 3 & 2 & 1 \\ 2 & 0 & 4 \\ 0 & 6 & 3 \end{bmatrix}$$
What is $\det(A^{-1})$?

3. How many positive integers divide the number of positive integers that divide 2002^{2002}?

4. In base b, how many $(2n+1)$-digit numbers are palindromes?

5. In the diagram below, what is $\sum_{n=1}^{5} \theta_n$?



6. Let x be the smallest number such that x written out in English (i.e. 1,647 is one thousand six hundred forty seven) has exactly 300 letters. What is the most common digit (0-9) in x?

7. Define $g(x) = \int_x^{x+1} 2^t \, dt$, and $g'(x) = \frac{d}{dx} g(x)$. Compute $g'(10)$.

8. If $xy = 24$ with x and y real, what is the minimum value that $x^2 + 4y^2$ can attain?

9. Find the cubic polynomial $f(x)$ such that $f(1) = 1$, $f(2) = 5$, $f(3) = 14$, and $f(4) = 30$.

10. What is $1^2 + 2^2 + 3^2 + \cdots + 2002^2$?

11. The rth power mean P_r of n numbers $x_1, \ldots, x_n$ is defined as
$$P_r(x_1, \ldots, x_n) = \left(\frac{x_1^r + \cdots + x_n^r}{n} \right)^{1/r}.$$
for $r \neq 0$, and $P_0 = (x_1 x_2 \cdots x_n)^{1/n}$. The Power Mean Inequality says that if $r > s$, then $P_r \geq P_s$. Using this fact, find out how many ordered pairs of positive integers (x, y) satisfy $48\sqrt{xy} - x^2 - y^2 \geq 289$.

12. After meeting him in the afterlife, Gauss challenges Fermat to a boxing match. Each mathematician is wearing glasses, and Gauss has a 1/3 probability of knocking off Fermat's glasses during the match, whereas Fermat has a 1/5 chance of knocking off Gauss's glasses. Each mathematician has a 1/2 chance of losing without his glasses and a 1/5 chance of losing anyway with his. Note that it is possible for both Fermat and Gauss to lose (simultaneous knockout) or for neither to lose (the match is a draw). Given that Gauss wins the match (and Fermat loses), what is the probability that Gauss has lost his glasses?

13. Evaluate
$$\frac{1}{-1+\frac{1}{-1+\frac{1}{-1+\cdots}}}.$$

14. What is the smallest positive integer x such that x^2+x+41 is not prime?

15. Let $A(t)$ be an $n \times n$ square matrix whose entries are all functions of t, and suppose that $\det A(t) \neq 0$ for all t. Then $\frac{dA}{dt} = A'$ is simply the matrix formed by differentiating each entry of $A(t)$ with respect to t. Write $\frac{d}{dt}\left(A^{-1}(t)\right)$ in terms of $A(t)$ and A', where the only differentiation occurs in A' itself.