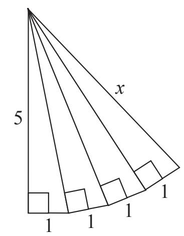
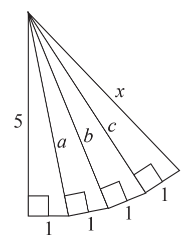
Geometry Test Solutions
Stanford Math Tournament
February 23, 2002

1. In the figure below, what is the length of x?



Answer: $\boldsymbol{\sqrt{29}}$

Solution: See the figure below. Using the Pythagorean Theorem, we see that $a = \sqrt{26}$, and similarly $b = \sqrt{27}$, $c = \sqrt{28}$, and finally $x = \boldsymbol{\sqrt{29}}$.



2. Upon cutting a certain rectangle in half, you obtain two rectangles that are scaled down versions of the original. What is the ratio of the longer side length to the shorter side length?

Answer: $\boldsymbol{\sqrt{2}}$

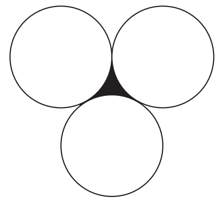
Solution: Let's say the original rectangle has side lengths a and b with $a > b$. Clearly, we have to cut perpendicular to the longer side, so the two halves will each have side lengths $a/2$ and b. To be scaled down versions of the original means that the ratio of the side lengths is the same, so we need to have $\frac{a}{b} = \frac{b}{a/2}$. Thus, $\frac{a^2}{b^2} = 2$, so $\frac{a}{b} = \boldsymbol{\sqrt{2}}$.

3. An equilateral triangle has sides 1 inch long. An ant walks around the triangle, maintaining a distance of 1 inch from the triangle at all times. How far does the ant walk?

Answer: $\mathbf{3 + 2\boldsymbol{\pi}}$

Solution: The ant must trace three sides of length 1 inch each and the three arcs around each corner of arc length $\frac{2\pi}{3}$ for a total distance of $\mathbf{3 + 2\boldsymbol{\pi}}$.

4. Given that the circles below each have radius 1 and are pairwise tangent, what is the area of the shaded region?



Answer: $\boldsymbol{\sqrt{3} - \frac{\pi}{2}}$

Solution: The triangle formed by the centers of the three circles, which contains the entire shaded region, is an equilateral triangle with sides of length 2, which has area $\sqrt{3}$. The unshaded portion of the triangle consists of 3 circular sectors, each of which is one-sixth of a circle of radius 1. Hence, the total unshaded area of the triangle is $3 \cdot \frac{\pi}{6} = \frac{\pi}{2}$. Subtracting this from the area of the triangle gives us $\boldsymbol{\sqrt{3} - \frac{\pi}{2}}$.

5. A lattice point is a point on the coordinate plane with integer values for x and y. How many lattice points lie on a circle centered at the origin with radius 25?

Answer: 20

Solution: We need $x^2 + y^2 = 625$. From our knowledge of all simple pythagorean triangles, we know that 15, 20, 25 and 7, 24, 25 are the only two with 25 as a hypotenuse. Adding in that they may be positive or negative and the four cardinal points gives **20** solutions.

6. A kite is a quadrilateral with sides of length a, a, b, and b, where the sides of length a are adjacent, as are the sides of length b. If such a kite can be inscribed in a circle, then what is its area?

Answer: $\boldsymbol{ab}$

Solution: Let θ be the angle between each pair of intersecting sides of different length. If a kite is inscribed in a circle, then these two angles subtend complementary arcs on the circle. Since each angle subtends an arc whose measure is twice its own, and these arcs sum up to the entire circle, we have $2\theta + 2\theta = 2\pi$, and therefore $\theta = \frac{\pi}{2}$. Hence, the kite is made up of two right triangles, each with area $\frac{1}{2}ab$, so we easily see that its area is $\boldsymbol{ab}$.

7. Consider a regular n-gon with side length s. What is the ratio of the area to the square of the perimeter in terms of n and s? (expressed in fraction form)

Answer: $\boldsymbol{\dfrac{\cot \frac{180}{n}}{4n}}$

Solution: First, we calculate the area. Consider a triangle whose vertices are the center of the polygon and two adjacent vertices. There are n of these triangles, and the area of the polygon is the sum of the area of the triangles. Now, this triangle has an angle of $\frac{360}{n}$ at the center of the polygon, so we can drop a perpendicular to the base of the triangle (the side of the polygon). So the base has a length of s and the height of the triangle is $\frac{\frac{s}{2}}{\tan \frac{180}{n}} = \frac{s}{2} \cot \frac{180}{n}$. So, the area of the triangle is $\frac{1}{2}s \cdot \frac{s}{2} \cot \frac{180}{n} = \frac{1}{4}s^2 \cot \frac{180}{n}$, and the area of the polygon is $n \cdot \frac{1}{4}s^2 \cot \frac{180}{n}$. Thus, the ratio of area to the square of the perimeter is $\frac{n \cdot \frac{1}{4}s^2 \cot \frac{180}{n}}{(ns)^2} = \boldsymbol{\frac{\cot \frac{180}{n}}{4n}}$.

8. Four regular triangular pyramids are in a line. The first three have side lengths of 3, 4, and 5, and the volume of the last pyramid is the sum of the volumes of the first three. What is the side length of the last pyramid?

Answer: 6

Solution: Multiplying the sides of a pyramid by some constant k multiplies the volume by k^3, so the volume of this particular type of pyramid must take the form $V = cx^3$, where x is the side length and c is some constant.

Now, let s be the side length of the fourth pyramid. Then

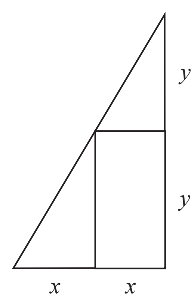
$$cs^3 = c \cdot (3^3 + 4^3 + 5^3) = 216c.$$

Therefore, $s = \boldsymbol{\sqrt[3]{216} = 6}$.

9. A ladder of height h leans against a wall. It starts out flat against the wall, and then the base slides out along the ground until the ladder lies flat. The ladder touches the wall throughout this motion. What is the area underneath the path traced by the midpoint of the ladder?

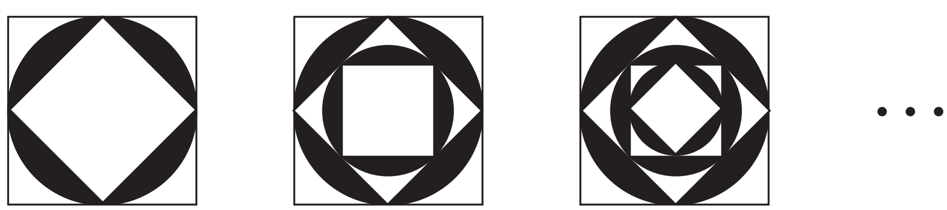
Answer: $\boldsymbol{\frac{1}{16}\pi h^2}$

Solution: Applying the Pythagorean Theorem to either of the smaller right triangles above tells us



that $x^2 + y^2 = \frac{1}{4}h^2$. Meanwhile, we can also use x and y as coordinates for the position of the midpoint relative to the base of the wall. Then the relation we found says that the midpoint traces a quarter circle of radius $\frac{1}{2}h$, centered at the base of the wall, which has area $\boldsymbol{\frac{1}{16}\pi h^2}$.

10. In the diagrams below, each circle is inscribed in the surrounding square, and each square is inscribed in the surrounding circle. Suppose the pattern continues on to infinity. If the outermost square has side length 1, what will the area of the shaded region be?



Answer: $\boldsymbol{\frac{\pi}{2} - 1}$

Solution: Consider the four corner triangles in the first picture in the sequence. Eventually, the entire square will be covered by triangles like these of various sizes. The ratio of shaded area to total area is the same in each of these, so as the pattern goes to infinity, the ratio of the shaded area to total area in the entire square is the same as it is in one of these triangles. Hence, it suffices to calculate this ratio for one of the outmost triangles. The shaded area is $\frac{1}{16}\pi - \frac{1}{8}$, and the total area is $\frac{1}{8}$, so the ratio is $\frac{\pi}{2} - 1$. And since the big square has total area 1, the shaded area is $\boldsymbol{\frac{\pi}{2} - 1}$.