General Test Solutions
Stanford Math Tournament
February 23, 2002

1. Sammy the Owl runs a pole measuring business. His wings are injured so he can't just fly up to the top of the pole. The pole does have a rope tied to the top which is 3 feet longer than the pole itself. When the rope is pulled taut, Sammy is 5 feet from the base of the pole. How tall is the pole?

   Answer: $\mathbf{\frac{8}{3}}$

   Solution: When the rope is taut, we have a right triangle with legs of length 5 and h and hypotenuse of length $h+3$. Using the Pythagorean Theorem, we get that $25 + h^2 = h^2 + 6h + 9$ or that $6h = 16$, and so $h = \mathbf{\frac{8}{3}}$.

2. Publicity from his success leads to more business for Sammy. The next pole he has to measure has no rope so Sammy backs away from the pole thinking, until he notices something. His eyes are 2 feet from the ground and when he looks at the top of the pole, the top of a nearby 6 foot tree looks like it is exactly the same height. The tree is 10 feet away and the pole is 20 feet further. How tall is the pole?

   Answer: 14

   Solution: We have two similar triangles here (Sammy's eyes to the top of the tree to 2 feet above the ground and Sammy's eyes to the top of the pole to 2 feet above the ground), so the height h of the pole satisfies $\frac{h-2}{30} = \frac{6-2}{10}$. Therefore, $h = 3 \cdot 4 + 2 = \mathbf{14}$.

3. A perfect number is a number N whose divisors, excluding itself, add up to N. An even perfect number is always of the form $(2^n - 1)2^{n-1}$, where $(2^n - 1)$ is prime. Find the first three even perfect numbers.

   Answer: 6, 28, 496

   Solution: The first three values of n for which $2^n - 1$ is prime are 2, 3, and 5. Plugging these into the formula yields **6, 28,** and **496**, which, as you can check, are indeed perfect numbers.

4. In the xy-plane, the segment with endpoints $(3, 8)$ and $(-5, 2)$ is the diameter of the circle. If the point $(x, 10)$ is also on the circle, what is the value of x?

   Answer: $\mathbf{-1}$

   Solution: Since the segment with endpoints $(3, 8)$ and $(-5, 2)$ is the diameter, then the center of the circle is $(\frac{3-5}{2}, \frac{8+2}{2}) = (-1, 5)$. We see that the radius of the circle is $\sqrt{4^2 + 3^2} = 5$, so the only point on the circle with y-value 10 is the point $(-1, 10)$. Thus, $x = \mathbf{-1}$.

5. Upon cutting a certain rectangle in half, you obtain two rectangles that are scaled down versions of the original. What is the ratio of the longer side length to the shorter side length?

   Answer: $\mathbf{\sqrt{2}}$

   Solution: Let's say the original rectangle has side lengths a and b with $a > b$. Clearly, we have to cut perpendicular to the longer side, so the two halves will each have side lengths $a/2$ and b. To be scaled down versions of the original means that the ratio of the side lengths is the same, so we need to have $\frac{a}{b} = \frac{b}{a/2}$. Thus, $\frac{a^2}{b^2} = 2$, so $\frac{a}{b} = \mathbf{\sqrt{2}}$.

6. Consider a lattice consisting of the points (x, y) where x and y are integers $0 \le x \le 4$, $0 \le y \le 4$. How many possible non-negative slopes (including ∞) can be formed by drawing a line between two points on the lattice?

   Answer: 13

   Solution: We see that all fractional slopes of the form $\frac{a}{b}$ can be attained, where $0 \le a, b \le 4$. So, the slopes are $0, \infty, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, 1, 2, 3, 4, \frac{2}{3}, \frac{3}{2}, \frac{3}{4}, \frac{4}{3}$, and there are **13** of these. Note that if $s(n)$ is the number of such slopes with a lattice from $(0, 0)$ to (n, n), then $s(n) = s(n-1) + 2\phi(n)$, where $\phi(n)$ is the Euler phi function, which is kinda neat.

7. Evaluate $2002^3 - 2002 \cdot 2003 \cdot 2001$

   Answer: 2002

   Solution: Let $a = 2002$, then the expression is $a^3 - a(a+1)(a-1) = a(a^2 - (a+1)(a-1)) = a(a^2 - (a^2 - 1)) = a$, and this is **2002**.

8. A clockmaker wants to design a clock such that the area swept by each hand (second, minute, and hour) in one minute is the same (all hands move continuously). What is the length of the hour hand divided by the length of the second hand?

   Answer: $\mathbf{12\sqrt{5}}$

   Solution: Let r be the length of the second hand and R be the length of the hour hand. For every revolution the hour hand makes, the second hand makes $12 \cdot 60$ revolutions. So, $\pi R^2 = 12 \cdot 60 \cdot \pi r^2$. Thus, $R^2 = 12 \cdot 60 \cdot r^2$ and $R = r\sqrt{12 \cdot 60} = 12r\sqrt{5}$, and $\frac{R}{r} = \mathbf{12\sqrt{5}}$.

9. Bill Gates has $1 billion. Every four days he goes out on a spending spree where he spends three times as many as the previous spree. Assuming he spends $1 on the first spree, how many days will it take for him to be bankrupt? His first spree is day 1.

   Answer: 77 days

   Solution: Assume he goes on $n+1$ sprees. Then the total money spent is $1 + 3^1 + 3^2 \cdot + 3^n = \frac{3^{n+1}-1}{3-1}$. We can see that this just exceeds his billion dollars when $n = 19$. This can be seen by multiplication or logarithm estimates. Thus, it takes 20 sprees to go bankrupt, and this occurs on day $4n - 3 = \mathbf{77}$.

10. Sue is 4 years older than Betty. 5 years ago, Sue was twice as old as Betty. How old will Betty be in 10 years?

    Answer: 19

    Solution: Let S equal Sue's age and B equal Betty's age. Then $S = 4 + B$ and $S - 5 = 2 \cdot (B - 5)$, and thus $B = 9$ now. In 10 years, she'll be **19**.

11. In the game Jotto, two players choose 5 letter words. Assuming that words can be any string of 5 letters where order is important, what are the chances that more than 3 letters are the same? By this, we mean the same letter in the same location in the word. (You may leave your answer with an exponent in the denominator)

    Answer: $\mathbf{\frac{126}{26^5}}$

    Solution: Since each player picks their word independently, we can assume that one player has already chosen. The probability of the other player choosing a word with exactly 4 letters being the same is the number of words with exactly 1 letter different divided by the total number of words (which is 26^5). There are 5 possible places for the different letter to go, and 25 possible choices for it. Thus, there are $5 \cdot 25 = 125$ such words, and the chance that one of these is picked is $\frac{125}{26^5}$. The probability of all 5 letters being the same is $\frac{1}{26^5}$. The sum of these two probabilities is the answer, and this is $\mathbf{\frac{126}{26^5}}$.

12. Greg has a 12 foot by 24 foot garden which he is fencing in. Company Alpha's fencing costs a base of $10, but then every 2 feet you have to put fencepost that costs an extra $2, and every other fencepost must be a strong fencepost that costs $3. Company Beta's fencing costs a base of $12, but then every 3 feet you have to put fencepost that costs an extra $3, and every other fencepost must be a strong fencepost that costs $4. What is positive difference in the cost to Greg between the two companies' fencing?

    Answer: $4

    Solution: The perimeter of the garden is 72 feet. Thus using Company Alpha, we need 18 regular fenceposts and 18 strong ones, leading to a total cost of $10 + 18 * 2 + 18 * 3 = \$100$. Using Company Beta, we need 12 regular fenceposts and 12 strong ones, leading to a total cost of $12 + 12 * 3 + 12 * 4 = \$96$. The difference between the costs is $**4**.

13. You have 81 coins that are exactly the same except for one, which is counterfeit and slightly lighter. You have a beam balance upon which you can weigh 2 groups of coins to determine if they weigh the same or which group is heavier. What is the least number of weighings needed to establish the counterfeit coin with certainty?

    Answer: 4

    Solution: For the first weighing, weigh the first 27 coins against the next 27 - if they weigh the same, you know the counterfeit coin is in the group that you did not weigh. If not, then the counterfeit coin is in whichever group is lighter. So, after one weighing, you have it down to a group of 27, and you can continue this process, each time with the group containing the counterfeit coin. After two weighings you will have it down to a group of 9, after three you will have it down to a group of 3, and after four weighings you will know which one the counterfeit coin is.

    To show that you need at least four weighings is a little trickier. Let $c(w)$ be the greatest number of coins from which we can always extract the counterfeit in w weighings. On the first weighing, we split coins into three sets, A, B, and C, where A and B have the same size (as these will be weighed against each other). The weighing tells us which set contains the counterfeit.

    Since the counterfeit could potentially be in any of the three sets, each of these must have no more than $c(w-1)$ coins, as we only have $w-1$ more weighings to use. Hence, $c(w) \leq 3c(w-1)$. On the other hand, each set can clearly have up to $c(w-1)$ coins in it, so in fact, $c(w) = 3c(w-1)$. Since $c(0) = 1$ (we can only tell which coin is counterfeit in zero weighings if it's the only coin we have!), this gives us $c(1) = 3$, $c(2) = 9$, and $c(3) = 27$, all of which are less than 81. (And you can see that in general, $c(w) = 3^w$.) Thus, we need at least four weighings, so this is indeed the minimum number.

14. A committee is to be chosen from a group of 10 people. How many distinct committees can be formed from this group, provided that the committee can be of any size, but must contain of at least 1 person?

    Answer: 1023

    Solution: Let's say we're forming a committee. Then for each member of the group, we have two options: we can include them in the committee, or we can exclude them. Since there are 10 people, this gives us $2^1 0 = 1024$ choices. However, one of these is a committee of 0 members, which is disallowed, so there are **1023** allowed committees.

15. A lattice point is a point on the coordinate plane with integer values for x and y. How many lattice points lie on a circle centered at the origin with radius 25?

    Answer: 20

    Solution: We need $x^2 + y^2 = 625$. From our knowledge of all simple pythagorean triangles, we know that 15, 20, 25 and 7, 24, 25 are the only two with 25 as a hypotenuse. Adding in that they may be positive or negative and the four cardinal points gives **20** solutions.

16. Suppose that $n^2 - 2m^2 = m(n+3) - 3$. Find all integers m such that all corresponding solutions for n will not be real numbers.

    Answer: $\mathbf{-1, 0}$

    Solution: The given equation is equivalent to $n^2 - nm - 2m^2 - 3m + 3 = 0$. Solving this for n (we treat it as a polynomial in n and apply the quadratic formula), we obtain $n = \frac{m \pm \sqrt{9m^2+12m-12}}{2}$. For n to not be real, the expression under the radical must be negative, so we want to find all integers m such that $9m^2 + 12m - 12 < 0$. We easily calculate that $\mathbf{-1}$ and $\mathbf{0}$ are the only such numbers.

17. Define a lattice point to be a point (x, y) whose coordinates x and y are integers. Three points are collinear if a line passes through them; for example, the points $(0,0)$, $(1,2)$, and $(2,4)$ are collinear. What is the minimum number n such that, given n lattice points with $0 \leq x \leq 4$ and $0 \leq y \leq 4$, there must be three of these points collinear?

    Answer: 11

It is possible to demonstrate an arrangement of 10 points such that no three are collinear. One such arrangement:

+ − + − −
− − + + −
+ + − − −
− + − − +
− − − + +

Then the proof for 11 follows simply from the pigeonhole principle: there are 5 rows and 11 points, hence some row must contain at least 3 points. So the answer is **11**.

18. An equilateral triangle has sides 1 inch long. An ant walks around the triangle, maintaining a distance of 1 inch from the triangle at all times. How far does the ant walk?

    Answer: $\mathbf{3 + 2\pi}$

    Solution: The ant must trace three sides of length 1 inch each and the three arcs around each corner of arc length $\frac{2\pi}{3}$ for a total distance of $\mathbf{3 + 2\pi}$.

19. If $a \star b$ is defined as $(3a)^b - \log_b a - b^2$, what is $3 \star 3$?

    Answer: 719

    Solution: $9^3 - 1 - 9 = 729 - 10 = \mathbf{719}$

20. One day, Maggie the Meticulous decides to write down all the numbers from 1 to 1000, inclusive. How many of the digits he wrote were the numeral 9?

    Answer: 300

    Solution: Since one out of every 10 numbers will have a 9 in the digits place, Maggie will write a 9 in the digits place 100 times. Similarly, she will write a 9 in the tens place 100 times and a 9 in the hundreds place 100 times, for a total of **300** occurrences of the numeral 9.

21. Let A be the set of all primes less than 100. Let B be the set of all numbers whose units digit is 7. Let C be the set of all numbers whose leading digit is a multiple of 3. What is the size of $A \cap (B \cup C)$?

    Answer: 9

    Solution: We need numbers that either start with 3, 6 or 9 or end with 7 for $(B \cup C)$. Remember that we are then intersecting with A, the set of primes less than 100. The only primes that start with 3, 6, or 9 are 3, 31, 37, 61, 67, and 97. The only primes ending with a 7 are 7, 17, 37, 47, 67, and 97. There are $6 + 6 - 3 = \mathbf{9}$.

22. Factor the polynomial $x^4 - x^3 - 5x^2 + 3x + 6$.

    Answer: $\mathbf{(x-2)(x+1)(x+\sqrt{3})(x-\sqrt{3})}$

    Solution: We might try a few simple possibilities to come up with the roots 2 and −1. On the other hand, we could also try splitting the given polynomial into two quadratic polynomials with integer coefficients, in which case we would find that it splits into $(x^2 - x - 2)(x^2 - 3)$, from which the full factorization $\mathbf{(x-2)(x+1)(x+\sqrt{3})(x-\sqrt{3})}$ easily emerges.

23. What is the value of $.4\overline{7} - .47$ expressed as a reduced fraction?

    Answer: $\mathbf{\frac{7}{900}}$

    Solution: If $x = .4\overline{7}$ then $10x = 4.\overline{7}$ and $100x = 47.\overline{7}$, so $90x = 43$, which gives us $x = \frac{43}{90}$. Thus, the answer is $\frac{43}{90} - \frac{47}{100} = \mathbf{\frac{7}{900}}$.

24. Suppose x, y, z is a geometric series with a common ratio of r such that $x \neq y$. If $x, 3y, 5z$ is an arithmetic sequence, what is the value of r?

    Answer: $\mathbf{\frac{1}{5}}$

Solution: Since x, y, z is a geometric series with common ratio r, then we have $y = xr$ and $z = xr^2$. So, since $x, 3y, 5z$ is an arithmetic sequence, then $3y - x = 5z - 3y$, or $3xr - x = 5xr^2 - 3xr$. Simplifying, we get that $5r^2 - 6r + 1 = 0$, which means that $r = 1$ or $r = \frac{1}{5}$. Since $x \neq y$, then $r \neq 1$, so $r = \mathbf{\frac{1}{5}}$.

25. Barbara Manatee chooses 10 cards without replacement from a standard 52-card deck of cards (without jokers). What is the probability that she does not draw a 3 but that she does choose the 7 of diamonds?

Answer: $\mathbf{\frac{41}{476}}$

Solution: She must choose the 7 of diamonds, so she only has a choice in 9 of her cards. And for these nine cards, she cannot choose the 7 of diamonds or a 3, so she is choosing from only 47 cards. Thus, she has $\binom{47}{9}$ ways to choose. The total number of ways to pick a hand of 10 cards is $\binom{52}{10}$. Thus the probability is $\frac{\binom{47}{9}}{\binom{52}{10}} = \mathbf{\frac{41}{476}}$.

26. How many integers, from 10-99 inclusive, have the property that the remainder of their square divided by 100 is equal to the square of the units digit of the number?

Answer: 26

Solution: Let x be the tens digit and y the units digit. Then $(10x + y)^2 = 100x^2 + 20xy + y^2$. Thus, for the remainder when we divide by 100 to be equal to y^2, we need 100 to divide $20xy$, or equivalently 5 to divide x or y. Therefore, either x is 5, or y is 5 or 0. There are 10 numbers whose tens digit is 5 and 18 whose units digit is 0 or 5, but 2 of these are counted twice (50 and 55), so there are $10 + 18 - 2 = \mathbf{26}$ altogether.

27. Jonathan lost all the stickers to his rubix cube from cheating too much, so now he has to paint all six sides of his cube. However, he only has 3 colors and thus plans to use each color on two sides of the cube. How many distinct ways can he paint the cube? (If one painted cube can be rotated to look like another, then they are the same.)

Answer: 6

Solution: Assume the colors are red, blue, and green, and that Jonathan paints them on in that order. First, he paint one face red (it doesn't matter which one). If the second red face is opposite the first one, then there are two options for completing the painting - either the blue faces are opposite, or they are adjacent.

Next, suppose the second red face is adjacent to the first. Then we have $\binom{4}{2} = 6$ ways to pick the blue sides, leaving the last two for green. Notice that if we rotate the cube to swap the two red sides, two of the orientations are the same, so we obtain only 4 distinct colorings this way. Thus there $2 + 6 - 2 = \mathbf{6}$ ways to color the cube.

28. Each valve A, B and C, when open, releases water into a tank at its own constant rate. With valve A open alone, the tank fills in 5 hours. With valves B and C open, the tank fills in 1 hour and 12 minutes. How long does it take to fill the tank with all 3 valves open?

Answer: $\mathbf{\frac{30}{31}}$ **hours**

Solution: Let r_A, r_B, r_C be the rates of valves A, B, and C in tanks per hour, respectively. So $r_A = \frac{1}{5}$ and $\frac{6}{5}(r_B + r_C) = 1$. Thus, $(r_A + r_B + r_C) = \frac{1}{5} + \frac{5}{6} = \frac{31}{30}$, so it takes $\mathbf{\frac{30}{31}}$ **hours** to fill the tanks.

29. Lattice paths are paths consisting of one-unit steps in the positive horizontal or vertical directions. How many distinct lattice paths are there from the point (-1,0) to the point (3,5) if we allow up to one diagonal step (a vertical unit and a horizontal unit at once)?

Answer: 406

Solution: The trip involves 5 vertical steps and 4 horizontal steps (disregarding the diagonal step). We simply need to choose when in the 9 moves to put the 5 vertical ones, giving us (in the case of no diagonal move) $\binom{9}{4}$ as our answer. If we do a diagonal step, then we have 4 vertical, 3 horizontal and 1 diagonal. We have $\binom{7}{3}$ ways of arrange the vertical and horizontal moves and then 8 choices of where to put in the diagonal move. That is $126 + 35 \cdot 8 = \mathbf{406}$ possible paths.

30. Suppose $a_1, a_2, a_3, \ldots$ is a sequence of numbers defined such that $a_1 = 1$ and $a_{n+1} = a_n + n$ for all positive integers n. Find a_{101}.

Answer: 5051

Solution: We need a "non-recursive" definition for the sequence. Note that $a_n = a_{n-1} + n - 1 = a_{n-2} + n - 2 + n - 1 = a_{n-3} + n - 3 + n - 2 + n - 1$. If we continue back to the original, we get that $a_n = a_1 + 1 + 2 + ... + n - 1$. The sum of the series $1 + 2 + ... + n - 1$ is $\frac{n(n-1)}{2}$, thus we have a general solution $a_n = \frac{n(n-1)}{2} + 1$ which is easily proven by induction. So $a_{101} = \frac{101 \cdot 100}{2} + 1 = \mathbf{5051}$.

31. Find the greatest integer x for which $2^{135} > 27^{5x}$.

Answer: 5

Solution: $27^{5x} = 3^{15x}$. Also, $2^{135} > 3^{15x}$ if and only if $2^9 > 3^x$. $2^9 = 512$, and thus $3^{\mathbf{5}} = 243$ is the highest power under 512.

32. There are 100 people in Whosville that watch baseball. 63 of them watch the Atlanta Alphas. 45 watch the Baltimore Betas. Only 30 actually watch the Georgetown Gammas. Furthermore, 21 watch both (but not necessarily exclusively) the Alphas and the Betas and 15 watch both the Betas and the Gammas. 9 diehard fans watch all three teams. How many fans watch both the Alphas and the Gammas?

Answer: 11

Solution: Using the principle of inclusion-exclusion, the following equation must be true, where x is the number of fans who watch both the Alphas and Gammas:

$$(63 + 45 + 30) - (21 + 15 + x) + 9 = 100.$$

This simplifies to $x = \mathbf{11}$.

33. If $\log_A B + \log_B A^2 = 4$ and $B < A$, find $\log_A B$.

Answer: $\mathbf{2 - \sqrt{2}}$

Solution: Let $x = \log_A B$. Then $A^x = B$. This implies $A^{2x} = B^2$ and then $A^2 = B^{2/x}$. Thus, $\log_B A^2 = \frac{2}{x}$. We need to solve $x + \frac{2}{x} = 4$. Multiplying all by x, yields the quadratic $x^2 - 4x + 2 = 0$ which has roots $2 \pm \sqrt{2}$. However, since $B < A$ then $x = \log_A B < 1$ and thus only $\mathbf{2 - \sqrt{2}}$ is the answer.