General Test
Stanford Math Tournament
February 23, 2002

1. Sammy the Owl runs a pole measuring business. His wings are injured so he can't just fly up to the top of the pole. The pole does have a rope tied to the top which is 3 feet longer than the pole itself. When the rope is pulled taut, Sammy is 5 feet from the base of the pole. How tall is the pole?

2. Publicity from his success leads to more business for Sammy. The next pole he has to measure has no rope so Sammy backs away from the pole thinking, until he notices something. His eyes are 2 feet from the ground and when he looks at the top of the pole, the top of a nearby 6 foot tree looks like it is exactly the same height. The tree is 10 feet away and the pole is 20 feet further. How tall is the pole?

3. A perfect number is a number N whose divisors, excluding itself, add up to N. An even perfect number is always of the form $(2^n - 1)2^{n-1}$, where $(2^n - 1)$ is prime. Find the first three even perfect numbers.

4. In the xy-plane, the segment with endpoints $(3, 8)$ and $(-5, 2)$ is the diameter of the circle. If the point $(x, 10)$ is also on the circle, what is the value of x?

5. Upon cutting a certain rectangle in half, you obtain two rectangles that are scaled down versions of the original. What is the ratio of the longer side length to the shorter side length?

6. Consider a lattice consisting of the points (x, y) where x and y are integers $0 \leq x \leq 4$, $0 \leq y \leq 4$. How many possible non-negative slopes (including ∞) can be formed by drawing a line between two points on the lattice?

7. Evaluate $2002^3 - 2002 \cdot 2003 \cdot 2001$

8. A clockmaker wants to design a clock such that the area swept by each hand (second, minute, and hour) in one minute is the same (all hands move continuously). What is the length of the hour hand divided by the length of the second hand?

9. Bill Gates has \$1 billion dollars. Every four days he goes out on a spending spree where he spends three times as many as the previous spree. Assuming he spends \$1 dollar on the first spree, how many days will it take for him to be bankrupt? His first spree is day 1.

10. Sue is 4 years older than Betty. 5 years ago, Sue was twice as old as Betty. How old will Betty be in 10 years?

11. In the game Jotto, two players choose 5 letter words. Assuming that words can be any string of 5 letters where order is important, what are the chances that more than 3 letters are the same? By this, we mean the same letter in the same location in the word. (You may leave your answer with an exponent in the denominator)

12. Greg has a 12 foot by 24 foot garden which he is fencing in. Company Alpha's fencing costs a base of \$10, but then every 2 feet you have to put fencepost that costs an extra \$2, and every other fencepost must be a strong fencepost that costs \$3. Company Beta's fencing costs a base of \$12, but then every 3 feet you have to put fencepost that costs an extra \$3, and every other fencepost must be a strong fencepost that costs \$4. What is positive difference in the cost to Greg between the two companies' fencing?

13. You have 81 coins that are exactly the same except for one, which is counterfeit and slightly lighter. You have a beam balance upon which you can weigh 2 groups of coins to determine if they weigh the same or which group is heavier. What is the least number of weighings needed to establish the counterfeit coin with certainty?

14. A committee is to be chosen from a group of 10 people. How many distinct committees can be formed from this group, provided that the committee can be of any size, but must contain of at least 1 person?

15. A lattice point is a point on the coordinate plane with integer values for x and y. How many lattice points lie on a circle centered at the origin with radius 25?

16. Suppose that $n^2 - 2m^2 = m(n+3) - 3$. Find all integers m such that all corresponding solutions for n will not be real numbers.

17. Define a lattice point to be a point (x, y) whose coordinates x and y are integers. Three points are collinear if a line passes through them; for example, the points $(0,0)$, $(1,2)$, and $(2,4)$ are collinear. What is the minimum number n such that, given n lattice points with $0 \leq x \leq 4$ and $0 \leq y \leq 4$, there must be three of these points collinear?

18. An equilateral triangle has sides 1 inch long. An ant walks around the triangle, maintaining a distance of 1 inch from the triangle at all times. How far does the ant walk?

19. If $a \star b$ is defined as $(3a)^b - \log_b a - b^2$, what is $3 \star 3$?

20. One day, Maggie the Meticulous decides to write down all the numbers from 1 to 1000, inclusive. How many of the digits he wrote were the numeral 9?

21. Let A be the set of all primes less than 100. Let B be the set of all numbers whose units digit is 7. Let C be the set of all numbers whose leading digit is a multiple of 3. What is the size of $A \cap (B \cup C)$?

22. Factor the polynomial $x^4 - x^3 - 5x^2 + 3x + 6$.

23. What is the value of $.4\overline{7} - .47$ expressed as a reduced fraction?

24. Suppose x, y, z is a geometric series with a common ratio of r such that $x \neq y$. If $x, 3y, 5z$ is an arithmetic sequence, what is the value of r?

25. Barbara Manatee chooses 10 cards without replacement from a standard 52-card deck of cards (without jokers). What is the probability that she does not draw a 3 but that she does choose the 7 of diamonds?

26. How many integers, from 10-99 inclusive, have the property that the remainder of their square divided by 100 is equal to the square of the units digit of the number?

27. Jonathan lost all the stickers to his rubix cube from cheating too much, so now he has to paint all six sides of his cube. He however only has 3 colors and thus plans to use each color on two sides of the cube. How many distinct ways can he paint the cube? (If one painted cube can be rotated to look like another, then they are the same.)

28. Each valve A, B and C, when open, releases water into a tank at its own constant rate. With valve A open alone, the tank fills in 5 hours. With valves B and C open, the tank fills in 1 hour and 12 minutes. How long does it take to fill the tank with all 3 valves open?

29. Lattice paths are paths consisting of one-unit steps in the positive horizontal or vertical directions. How many distinct lattice paths are there from the point (-1,0) to the point (3,5) if we allow up to one diagonal step (a vertical unit and a horizontal unit at once)?

30. Suppose $a_1, a_2, a_3, \ldots$ is a sequence of numbers defined such that $a_1 = 1$ and $a_{n+1} = a_n + n$ for all positive integers n. Find a_{101}.

31. Find the greatest integer x for which $2^{135} > 27^{5x}$.

32. There are 100 people in Whosville that watch baseball. 63 of them watch the Atlanta Alphas. 45 watch the Baltimore Betas. Only 30 actually watch the Georgetown Gammas. Furthermore, 21 watch both (but not necessarily exclusively) the Alphas and the Betas and 15 watch both the Betas and the Gammas. 9 diehard fans watch all three teams. How many fans watch both the Alphas and the Gammas?

33. If $\log_A B + \log_B A^2 = 4$ and $B < A$, find $\log_A B$.