Calculus Test Solutions
Stanford Math Tournament
February 23, 2002

1. Evaluate $\int_0^8 \frac{d}{dx}(4x^2 + 3x + 2)\, dx$.

   Answer: 280

   Solution:

$$\begin{aligned} \int_0^8 \frac{\mathrm{d}}{\mathrm{dx}}(4x^2 + 3x + 2)\mathrm{dx} &= \int_0^8 (8x + 3)\mathrm{dx} \\ &= 4x^2 + 3x|_0^8 \\ &= 4 \cdot 8^2 + 3 \cdot 8 \\ &= \mathbf{280} \end{aligned}$$

2. From atop a very tall building, a spherical cow drops a lit firecracker. He hears it explode 10 seconds later. Given that sounds travels at about 300 $\frac{m}{s}$ and acceleration due to gravity is 10 $\frac{m}{s^2}$ Find the length of time it falls before it exploded. Remember, that acceleration is change in velocity over time and that the distance covered by a falling object is the area under the velocity curve.

   Answer: $\mathbf{-30 + 10\sqrt{15}}$

   Solution: Let t_0 be the amount of time the object spent falling. The distance it travels in this time is $5t_0^2$ The sound travels for $10 - t_0$ seconds, and it goes the same distance up as the object fell. Thus, $300(10 - t) = 5t^2$, which we can solve using the quadratic equation, obtaining $\mathbf{-30 + 10\sqrt{15}}$.

3. A spherical balloon expands at a rate of 1 cm^3/sec. In terms of the radius r, at what rate is the radius of the balloon increasing?

   Answer: $\mathbf{\frac{1}{4\pi r^2}}$ cm/sec

   Solution: As a function of the radius, the volume $V(r)$ of the balloon is given by $V(r) = \frac{4}{3}\pi r^3$, so $\frac{dV}{dr} = 4\pi r^2$. Meanwhile, the chain rule tells us that $\frac{dV}{dt} = \frac{dV}{dr}\frac{dr}{dt}$. And we are given $\frac{dV}{dt} = 1$, so $\frac{dr}{dt} = \left(\frac{dV}{dr}\right)^{-1} = \mathbf{\frac{1}{4\pi r^2}}$.

4. Consider an isosceles trapezoid with vertices at $(0,0), (2,3), (6,3), (8,0)$, and let d_1 and d_2 be the diagonals. If we let $f(t)$ be the distance between d_1 and d_2 along the line $y = t$, then what is the average value of $f(t)$ over the interval $[0,3]$ (the whole trapezoid)?

   Answer: $\mathbf{\frac{10}{3}}$

   Solution: The average value of $f(y)$ can be graphically represented as the sum of the areas of the two triangles formed between the intersection of d_1 and d_2 and the two bases of the trapezoid divided by the range of the interval. So, since the intersection of d_1 and d_2 is at $(4,2)$, the area of the top triangle is 2 and the area of the bottom triangle is 8, and the total area is 10, so the average value of $f(y)$ is $\mathbf{\frac{10}{3}}$.

5. Suppose $f(x) = e^{ax} + e^{bx}$, where $a \neq b$, and that $f'' - 2f' - 15 = 0$ for all x. Give all possible ordered pairs (a, b).

   Answer: $\mathbf{(3, -5)}$ or $\mathbf{(-5, 3)}$

   Solution: First, we calculate $f'(x) = ae^{ax} + be^{bx}$ and $f''(x) = a^2e^{ax} + b^2e^{bx}$. Substituting these into the equation we are given and grouping terms, we obtain $(a^2 - 2a - 15)e^{ax} + (b^2 - 2b - 15)e^{bx} = 0$. This can only be zero for all x if the coefficient of each exponential is zero, so 3 and -5 are the only possible values for both a and b. The condition $a \neq b$ makes $\mathbf{(3, -5)}$ and $\mathbf{(-5, 3)}$ the only possible answers.

6. Evaluate $\int_{-1/2}^{1/2} \sqrt{1-x^2}\,dx$.

   Answer: $\boldsymbol{\pi/6+\sqrt{3}/4}$

   Solution: Draw the unit circle. The area to be evaluated is the region between $x=-\frac{1}{2}$ and $x=\frac{1}{2}$, which is composed of one-sixth of the unit circle and two triangles with base $1/2$ and height $\sqrt{3}/2$. The total area is then clearly $\boldsymbol{\pi/6+\sqrt{3}/4}$.

7. What is the derivative of x^x, assuming $x>0$?

   Answer: $\boldsymbol{x^x(\ln x+1)}$

   Solution: First, we write $x^x = e^{\ln x^x} = e^{x\ln x}$. Differentiating this expression via the chain rule, we obtain $\frac{d}{dx}(e^{x\ln x}) = e^{x\ln x}(\ln x+1) = \boldsymbol{x^x(\ln x+1)}$.

8. Evaluate

$$\lim_{x\to 0}\frac{1}{x^3}\int_0^x \frac{t^2+t^3}{1+\sin t}\,dt.$$

   Answer: 1/3

   Solution: Applying L'Hospital's rule, we see that

$$\begin{aligned}
\lim_{x\to 0}\frac{\int_0^x \frac{t^2+t^3}{1+\sin t}\,dt}{x^3} &= \lim_{x\to 0}\frac{\frac{x^2+x^3}{1+\sin x}}{3x^2}\\
&= \lim_{x\to 0}\frac{1+x}{3(1+\sin x)}\\
&= \boldsymbol{\frac{1}{3}}.
\end{aligned}$$

9. Ford has an ivory bathtub in the shape of a cone, apex down, filled with sand. The cone has height 10 ft and maximum radius of 8 ft. He pulls the plug out of the apex and the sand drains out such that the radius decreases at a constant rate of $.5\frac{\text{ft}}{\text{min}}$. The sand drains into a paraboloid shaped swimming pool with a horizontal cross-section of a circle and a vertical cross-section which follows the graph of $h=r^2$, and it is already filled up to a height of 6 ft with water. Assuming the sand sinks quickly, at what rate is the water in the swimming pool rising when the height is 8 ft?

   Answer: $\boldsymbol{\frac{1}{16}5^{1/3}\cdot 299^{2/3}}$ ft/min

   Solution: Denote the volume of sand in the bathtub by V_b, its height by h_b, and the radius of the circle at the top of the sand by r_b. Then $V_b=\frac{1}{3}\pi r_b^2 h_b = \frac{5}{12}\pi r_b^3$ since $h_b=\frac{5}{4}r_b$ by similar triangles. Therefore, using the notation $f'=\frac{df}{dt}$ for functions f, we have $V_b' = \frac{5}{4}\pi r_b^2 r_b' = \frac{5}{8}\pi r_b^2$ since we are given $r_b'=0.5$.

   Now, let V_p denote the volume of water in the pool, and let h_p denote the height of the water. Using volume of rotation, $V_p=\frac{1}{2}\pi h^2$, and $V_p'=\pi h_b h_b'$ (keep in mind we want to solve for h_b'). When $h_b=8$, we have $V_p=32\pi$, and 18π of that is water, so 14π is sand. Thus, $V_b=\frac{640\pi}{3}-14\pi=\frac{598\pi}{3}$, and $r_b=2\sqrt[3]{\frac{299}{5}}$. The bathtub drains at the same rate the pool fills, so $V_p'=V_b'$, which gives us

$$\begin{aligned}
\pi h_b h_b' &= \frac{5}{8}\pi r_b^2\\
8h_b' &= \frac{1}{2}5^{1/3}\cdot 299^{2/3}\\
h_b' &= \boldsymbol{\frac{1}{16}5^{1/3}\cdot 299^{2/3}}.
\end{aligned}$$

10. What is $e^1\cdot e^{-1/2}\cdot e^{1/3}\cdot e^{-1/4}\cdots$?

    Answer: 2

Solution: Let $P = e^1 \cdot e^{-1/2} \cdot e^{1/3} \cdot e^{-1/4} \cdots$. Then

$$\begin{aligned} \ln P &= \ln e + \ln e^{-1/2} + \ln e^{1/3} + \ln e^{-1/4} + \cdots \\ &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \end{aligned}$$

Now recall from the study of power series that for $|x| < 1$,

$$\ln(x+1) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} x^n.$$

This formula works for $x = 1$ as well, and using it, we obtain $\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$. Hence, $\ln P = \ln 2$, so $P = \mathbf{2}$.