Calculus Test
Stanford Math Tournament
February 23, 2002

1. Evaluate $\int_0^8 \frac{d}{dx}(4x^2+3x+2)\,dx$.

2. From atop a very tall building, a spherical cow drops a lit firecracker. He hears it explode 10 seconds later. Given that sounds travels at about 300 $\frac{m}{s}$ and acceleration due to gravity is 10 $\frac{m}{s^2}$ Find the length of time it falls before it exploded. Remember, that acceleration is change in velocity over time and that the distance covered by a falling object is the area under the velocity curve.

3. A spherical balloon expands at a rate of 1 cm^3/sec. In terms of the radius r, at what rate is the radius of the balloon increasing?

4. Consider an isosceles trapezoid with vertices at $(0,0),(2,3),(6,3),(8,0)$, and let d_1 and d_2 be the diagonals. If we let $f(t)$ be the distance between d_1 and d_2 along the line $y=t$, then what is the average value of $f(t)$ over the interval $[0,3]$ (the whole trapezoid)?

5. Suppose $f(x)=e^{ax}+e^{bx}$, where $a\neq b$, and that $f''-2f'-15=0$ for all x. Give all possible ordered pairs (a,b).

6. Evaluate $\int_{-1/2}^{1/2}\sqrt{1-x^2}\,dx$.

7. What is the derivative of x^x, assuming $x>0$?

8. Evaluate
$$\lim_{x\to 0}\frac{1}{x^3}\int_0^x \frac{t^2+t^3}{1+\sin t}\,dt.$$

9. Ford has an ivory bathtub in the shape of a cone, apex down, filled with sand. The cone has height 10 ft and maximum radius of 8 ft. He pulls the plug out of the apex and the sand drains out such that the radius decreases at a constant rate of $.5\frac{\text{ft}}{\text{min}}$. The sand drains into a paraboloid shaped swimming pool with a horizontal cross-section of a circle and a vertical cross-section which follows the graph of $h=r^2$, and it is already filled up to a height of 6 ft with water. Assuming the sand sinks quickly, at what rate is the water in the swimming pool rising when the height is 8 ft?

10. What is $e^1\cdot e^{-1/2}\cdot e^{1/3}\cdot e^{-1/4}\ldots$?