Algebra Test Solutions
Stanford Math Tournament
February 23, 2002

1. Completely factor the polynomial $x^4 - x^3 - 5x^2 + 3x + 6$.

   Answer: $\boldsymbol{(x-2)(x+1)(x+\sqrt{3})(x-\sqrt{3})}$

   Solution: We might try a few simple possibilities to come up with the roots 2 and -1. On the other hand, we could also try splitting the given polynomial into two quadratic polynomials with integer coefficients, in which case we would find that it splits into $(x^2 - x - 2)(x^2 - 3)$, from which the full factorization $\boldsymbol{(x-2)(x+1)(x+\sqrt{3})(x-\sqrt{3})}$ easily emerges.

2. Solve for all real x that satisfy the equation $4^x = 2^x + 6$.

   Answer: $\boldsymbol{\log_2 3}$

   Solution: Substituting $y = 2^x$, we obtain the quadratic equation $y^2 - y - 6 = 0$. Factoring this, we see that $y = 3$ or $y = -2$. However, 2^x must be positive, so y must equal 3. Finally, solving $3 = 2^x$, we obtain $x = \boldsymbol{\log_2 3}$.

3. A clockmaker wants to design a clock such that the area swept by each hand (second, minute, and hour) in one minute is the same (all hands move continuously). What is the length of the hour hand divided by the length of the second hand?

   Answer: $\boldsymbol{12\sqrt{5}}$

   Solution: Let r be the length of the second hand and R be the length of the hour hand. For every revolution the hour hand makes, the second hand makes $12 \cdot 60$ revolutions. So, $\pi R^2 = 12 \cdot 60 \cdot \pi r^2$. Thus, $R^2 = 12 \cdot 60 \cdot r^2$ and $R = r\sqrt{12 \cdot 60} = 12r\sqrt{5}$, and $\frac{R}{r} = \boldsymbol{12\sqrt{5}}$.

4. Suppose that $n^2 - 2m^2 = m(n+3) - 3$. Find all integers m such that all corresponding solutions for n will *not* be real.

   Answer: $\boldsymbol{-1, 0}$

   Solution: The given equation is equivalent to $n^2 - nm - 2m^2 - 3m + 3 = 0$. Solving this for n (we treat it as a polynomial in n and apply the quadratic formula), we obtain $n = \frac{m \pm \sqrt{9m^2 + 12m - 12}}{2}$. For n to not be real, the expression under the radical must be negative, so we want to find all integers m such that $9m^2 + 12m - 12 < 0$. We easily calculate that $\boldsymbol{-1}$ and $\boldsymbol{0}$ are the only such numbers.

5. Solve for a, b, and c, given that $a \le b \le c$, and

   $$\begin{aligned} a + b + c &= -1 \\ ab + bc + ac &= -4 \\ abc &= -2. \end{aligned}$$

   Answer: $\boldsymbol{a = -1 - \sqrt{3},\ b = -1 + \sqrt{3},\ c = 1}$

   Solution: Recall that

   $$(x-a)(x-b)(x-c) = x^3 - (a+b+c)x^2 + (ab+bc+ac)x - abc.$$

   Hence, a, b, and c are roots of $x^3 + x^2 - 4x + 2 = 0$. We easily see that $x = 1$ is a root, and factoring this out, we are left with $x^2 + 2x - 2 = 0$, which has roots $-1 \pm \sqrt{3}$. Finally, arranging all three roots in ascending order gives us the answer $\boldsymbol{a = -1 - \sqrt{3},\ b = -1 + \sqrt{3},\ c = 1}$.

6. How many integers x, from 10-99 inclusive, have the property that the remainder of x^2 divided by 100 is equal to the square of the units digit of x?

   Answer: **26**

Solution: Let y be the tens digit and z the units digit. Then $(10y+z)^2 = 100y^2 + 20yz + z^2$. Thus, for the remainder when we divide by 100 to be equal to z^2, we need 100 to divide $20yz$, or equivalently 5 to divide y or z. Therefore, either y is 5, or z is 5 or 0. So, there are 10 numbers whose tens digit is 5 and 18 whose units digit is 0 or 5, but 2 of these are counted twice (50 and 55), so there are $10+18-2=$ **26** in total.

7. Find x satisfying $x = 1 + \frac{1}{x+\frac{1}{x+\cdots}}$.

   Answer: $\mathbf{\frac{3}{2}}$

   Solution: We rewrite the right side of the equation as follows.

$$\begin{aligned} 1+\frac{1}{x+\frac{1}{x+\cdots}} &= 1+\frac{1}{(x-1)+\left(1+\frac{1}{x+\frac{1}{x+\cdots}}\right)} \\ &= 1+\frac{1}{(x-1)+x} \\ &= 1+\frac{1}{2x-1}. \end{aligned}$$

   So x satisfies $x-1=\frac{1}{2x-1}$, and therefore $2x^2-3x=0$. Clearly, $x=0$ is not a possibility, which leaves us with only $x=\mathbf{\frac{3}{2}}$.

8. Let f be a function with $\frac{f(x)f(y)-f(xy)}{3} = x+y+2$. List all possible values for $f(36)$.

   Answer: 39

   Solution: Let $x=y=0$. Then $\frac{f(0)^2-f(0)}{3}=2$. Solving this quadratic equation, we find that $f(0)=-2$ or $f(0)=3$. If $f(0)=-2$ then $\frac{f(x)f(0)-f(0)}{3}=x+2$ or $-\frac{2}{3}[f(x)-1]=x+2$. Thus $f(x)=-\frac{3}{2}x-2$. Plugging this back into the original equation, we find that it does not satisfy $\frac{f(x)f(y)-f(xy)}{3}=x+y+2$. So we must have $f(0)=3$, and as above $\frac{3}{3}[f(x)-1]=x+2$, so $f(x)=x+3$. We see that this does indeed satisfy $\frac{f(x)f(y)-f(xy)}{3}=x+y+2$, and so $f(36)=36+3=\mathbf{39}$.

9. Given three numbers x_1, x_2, x_3, let $a = x_1+x_2+x_3$, $b = x_1x_2+x_2x_3+x_3x_1$, $c = x_1x_2x_3$, and $d = x_1^3+x_2^3+x_3^3$. If $a=3, b=7, c=10$, what is the value of d?

   Answer: −6

   Solution: By expanding out $(x_1+x_2+x_3)^3$, we get the relation that $a^3 = d+3ab-3c$. So, we get that $d=\mathbf{-6}$.

10. Write $\sqrt[3]{2+5\sqrt{3+2\sqrt{2}}}$ in the form $a+b\sqrt{2}$, where a and b are integers.

    Answer: $\mathbf{1+\sqrt{2}}$

    Solution: First, suppose we can write $\sqrt{3+2\sqrt{2}} = c+d\sqrt{2}$, where c and d are integers. Then squaring both sides gives us $3+2\sqrt{2} = (c^2+2d^2)+(2cd)\sqrt{2}$. From this, we easily see that $c=d=1$ will satisfy the desired relation, that is $\sqrt{3+\sqrt{2}} = 1+\sqrt{2}$. (Note that $c=d=-1$ also satisfies the relation, but gives the same result in the end)

    This reduces the given problem to solving $a+b\sqrt{2} = \sqrt[3]{7+5\sqrt{2}}$ for a and b. Cubing both sides yields the equation

$$(a^3+6ab^2)+(3a^2b+2b^3)\sqrt{2} = 7+5\sqrt{2}.$$

    We then observe that $a=b=1$ satisfies this equation, giving us the final answer of $\mathbf{1+\sqrt{2}}$.