Advanced Topics Test Solutions
Stanford Math Tournament
February 23, 2002

1. A clockmaker wants to design a clock such that the area swept by each hand (second, minute, and hour) in one minute is the same (all hands move continuously). What is the length of the hour hand divided by the length of the second hand?

   Answer: $\mathbf{12\sqrt{5}}$

   Solution: Let r be the length of the second hand and R be the length of the hour hand. For every revolution the hour hand makes, the second hand makes $12 \cdot 60$ revolutions. So, $\pi R^2 = 12 \cdot 60 \cdot \pi r^2$. Thus, $R^2 = 12 \cdot 60 \cdot r^2$ and $R = r\sqrt{12 \cdot 60} = 12r\sqrt{5}$, and $\frac{R}{r} = \mathbf{12\sqrt{5}}$.

2. Define a lattice point to be a point (x, y) whose coordinates x and y are integers. Three points are collinear if there is a line that passes through them; for example, the points $(0,0)$, $(1,2)$, and $(2,4)$ are collinear. What is the minimum number n such that, given n lattice points with $0 \leq x \leq 4$ and $0 \leq y \leq 4$, there must be three of these points collinear?

   Answer: 11

   It is possible to demonstrate an arrangement of 10 points such that no three are collinear. One such arrangement:

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   + - + - -
   - - + + -
   + + - - -
   - + - - +
   - - - + +
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   Then the proof for 11 follows simply from the pigeonhole principle: there are 5 rows and 11 points, hence some row must contain at least 3 points. So the answer is **11**.

3. Tommy Theta has a bag with 11 marbles. One is black and the rest are white. Every time he picks a marble, the bag is reset and one white marble is exchanged for a black marble. What is the probability that Tommy draws a black marble on or before the third draw?

   Answer: $\mathbf{\frac{611}{1331}}$

   Solution: The first draw has a $\frac{1}{11}$ chance of succeeding. To succeed exactly on the second draw, he must fail on the first draw and succeed on the second, giving a probability of $\frac{10}{11} \cdot \frac{2}{11}$. The probability of him drawing a black marble exactly on the third draw is similarly $\frac{10}{11} \cdot \frac{9}{11} \cdot \frac{3}{11}$. The sum of these is $\frac{611}{11^3} = \mathbf{\frac{611}{1331}}$, which is the answer.

4. If 2001! were written in base 23, how many trailing zeros would there be?

   Answer: 90

   Solution: Since 23 is prime, the question is equivalent to asking how many times 23 goes into 2001!, that is, the greatest integer n such that 23^n divides 2001!. From 1 to 2001, there are 87 multiples of 23 and 3 multiples of 23^2, and no multiples of 23^3 and above since $23^3 > 2001$. So, altogether 23 goes into 2001! exactly $87 + 3 = \mathbf{90}$ times.

5. 17 penguins are on an ice floe trying to divide up a booty of red herring amongst them. They find when they divide the fish up evenly, 13 are left over. Fighting for these extra fish causes 2 penguins to fall off the floe. When they redivide up the fish among the remaining 15 penguins, they end up with 7 left over. More fighting ensues and 2 more penguins fall off. Finally, the fish divide evenly for the remaining penguins. What is the smallest possible positive number of red herrings?

   Answer: 2002

   Solution: Let f be the number of fish. We know f is a multiple of 13, and it must also have a remainder of 7 when divided by 15. 52 is the smallest (positive) number with this property, so f must have the form $52 + 13 \cdot 15 \cdot n$, where n is any nonnegative integer. And since f has a remainder of 13 when divided by 17, this gives us

$$\begin{aligned} 52 + 13 \cdot 15n &\equiv 13 \pmod{17} \\ 15n + 4 &\equiv 1 \pmod{17} \\ 15n &\equiv 14 \pmod{17}. \end{aligned}$$

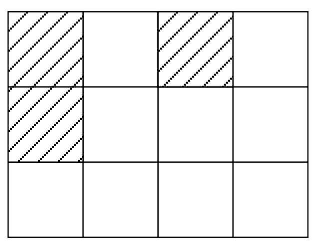
Testing values of n, we see that $n = 10$ is the smallest number that satisfies this equivalence. Thus, $f = \mathbf{2002}$.

6. Suppose the integers from 1 through 100 are written on separate slips of paper and placed in a hat. What is the minimum number of slips that must be drawn to ensure that three consecutive numbers are picked?

   Answer: 68

   Solution: Consider all 98 possible sets of 3 consecutive numbers from 1 to 100: $\{1, 2, 3\}$, $\{2, 3, 4\}$, ..., $\{98, 99, 100\}$. When we draw a number, let us count the number of "hits" to be the how many sets this number lies in. If we draw any number other than 1, 2, 99, or 100, we have three hits, while 2 and 99 yield two hits each, and 1 and 100 each give one. So after drawing n numbers for $n \geq 4$, we have at least $3n - 6$ hits. Since $3 \cdot 68 - 6 = 198$ is greater than twice the number of sets, it follows from the pigeonhole principle that we must have three hits in one of the sets. Thus, drawing **68** numbers will give us three consecutive numbers. Meanwhile, supposing we draw $1, 2, 4, 5, 7, 8, \ldots, 97, 98, 100$ (all integers less than or equal to 100 that are not divisible by 3), it is easy to see that we can pick 67 numbers without any consecutive triples.

7. In the 3×4 rectangle shown below, we can form "inner rectangles" by taking adjacent squares in the shape of a rectangle. How many "inner rectangles" can be chosen that do not use any of the forbidden squares? (shaded in the figure)

   Answer: 28

   Solution: We first count the total number of "inner rectangles" there are - we have $1 + 2 + 3 + 4 = 10$ choices of which columns the rectangle should span (1 choice if it is 4 columns wide, 2 choices if it is 3 columns wide, etc.), and $1 + 2 + 3 = 6$ choices of which rows the rectangle should span, so there are 60 total "inner rectangles". Now we count the "inner rectangles" that use one or both of the two forbidden squares in the leftmost column. Here there are 4 choices of columns to span and 5 choices of rows to span, so 20 use these squares, and similarly 18 of them use the forbidden square in the 3rd column. Also, there are 6 "inner rectangles" that use both the forbidden square in the 3rd column and one or both of the forbidden squares in the first column, so by the Inclusion-Exclusion Principle there are $60 - 20 - 18 + 6 = \mathbf{28}$ allowable "inner rectangles".

8. Evaluate the sum $\frac{1}{2 \cdot 5} + \frac{1}{3 \cdot 6} + \frac{1}{4 \cdot 7} + \cdots + \frac{1}{n(n+3)} + \cdots$

   Answer: $\mathbf{\frac{13}{36}}$

   Solution: Notice that the sum is equal to $\frac{1}{3}((\frac{1}{2} - \frac{1}{5}) + (\frac{1}{3} - \frac{1}{6}) + (\frac{1}{4} - \frac{1}{7}) + \cdots)$. Since all later terms cancel out, the sum of the series is $\frac{1}{3}(\frac{1}{2} + \frac{1}{3} + \frac{1}{4}) = \mathbf{\frac{13}{36}}$.

9. Calculus Cola is having a contest. One in six 20 oz bottles gives a free cola. If you buy a six pack what is the expected number of free colas you will win? (counting free colas you win off of free colas already won, etc.) Remember that expected value is the sum of each possible outcome times its probability.

   Answer: $\mathbf{\frac{6}{5}}$

   Solution: We determine the expected number of free colas starting with one cola and multiply by 6 to get the final answer (since the six colas in the six pack are independent). Let e be the expected value of free colas, starting with one cola. For each integer $n \geq 0$, let $P(n)$ be the probability of getting n free colas. Notice that the chance of getting $n+1$ free colas is the same as the chance of getting 1 free cola times the chance of getting n free colas off of that one, so we have $P(n+1) = \frac{1}{6}P(n)$. The definition of e gives us $e = \sum_{n=0}^{\infty} n \cdot P(n) = \sum_{n=1}^{\infty} n \cdot P(n)$, so

$$\begin{aligned} e &= \sum_{n=0}^{\infty}(n+1) \cdot P(n+1) \\ &= \frac{1}{6}\sum_{n=0}^{\infty}(n+1) \cdot P(n) \\ &= \frac{1}{6}\left(\sum_{n=0}^{\infty} n \cdot P(n) + \sum_{n=0}^{\infty} P(n)\right) \\ &= \frac{1}{6}(e+1), \end{aligned}$$

   since $\sum_{n=0}^{\infty} P(n)$ is the sum of all possible probabilities, and therefore equals 1. Solving $e = \frac{1}{6}(e+1)$ gives us $e = \frac{1}{5}$. Hence, the expected number of free colas from a six pack is $6e = \frac{6}{5}$.

10. Let $x - 1/x = i\sqrt{2}$ where $i = \sqrt{-1}$. Compute $x^{2187} - 1/x^{2187}$.

   Answer: $\boldsymbol{i\sqrt{2}}$

   Let A_n denote $x^n - 1/x^n$. Then

$$\begin{aligned} (A_n)^3 &= x^{3n} - 3x^n + 3x^{-n} - x^{-3n} \\ &= A_{3n} - 3A_n \end{aligned}$$

   and so

$$\begin{aligned} A_{3n} &= (A_n)^3 + 3A_n \\ &= A_n((A_n)^2 + 3). \end{aligned}$$

   Now since $(A_1)^2 + 3 = 1$, we have $A_1 = A_3 = A_9 = \ldots = A_{2187}$, so $A_{2187} = \boldsymbol{i\sqrt{2}}$.

   Alternatively, it is possible to use the quadratic formula and DeMoivre's law to obtain the answer.