Advanced Topics Test
Stanford Math Tournament
February 23, 2002

1. A clockmaker wants to design a clock such that the area swept by each hand (second, minute, and hour) in one minute is the same (all hands move continuously). What is the length of the hour hand divided by the length of the second hand?

2. Define a lattice point to be a point (x, y) whose coordinates x and y are integers. Three points are collinear if there is a line that passes through them; for example, the points $(0,0)$, $(1,2)$, and $(2,4)$ are collinear. What is the minimum number n such that, given n lattice points with $0 \leq x \leq 4$ and $0 \leq y \leq 4$, there must be three of these points collinear?

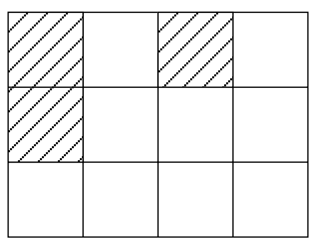
3. Tommy Theta has a bag with 11 marbles. One is black and the rest are white. Every time he picks a marble, the bag is reset and one white marble is exchanged for a black marble. What is the probability that Tommy draws a black marble on or before the third draw?

4. If 2001! were written in base 23, how many trailing zeros would there be?

5. 17 penguins are on an ice floe trying to divide up a booty of red herring amongst them. They find when they divide the fish up evenly, 13 are left over. Fighting for these extra fish causes 2 penguins to fall off the floe. When they redivide up the fish among the remaining 15 penguins, they end up with 7 left over. More fighting ensues and 2 more penguins fall off. Finally, the fish divide evenly for the remaining penguins. What is the smallest possible positive number of red herrings?

6. Suppose the integers from 1 through 100 are written on separate slips of paper and placed in a hat. What is the minimum number of slips that must be drawn to ensure that three consecutive numbers are picked?

7. In the 3×4 rectangle shown below, we can form "inner rectangles" by taking adjacent squares in the shape of a rectangle. How many "inner rectangles" can be chosen that do not use any of the forbidden squares? (shaded in the figure)



8. Evaluate the sum $\frac{1}{2 \cdot 5} + \frac{1}{3 \cdot 6} + \frac{1}{4 \cdot 7} + \cdots + \frac{1}{n(n+3)} + \cdots$

9. Calculus Cola is having a contest. One in six 20 oz bottles gives a free cola. If you buy a six pack what is the expected number of free colas you will win? (counting free colas you win off of free colas already won, etc.) Remember that expected value is the sum of each possible outcome times its probability.

10. Let $x - 1/x = i\sqrt{2}$ where $i = \sqrt{-1}$. Compute $x^{2187} - 1/x^{2187}$.